On symmetries and charges at spatial infinity

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Overview

- Bondi, Metzner, Sachs(BMS) in 60s: asymptotically flat spacetime at null infinity invariant under infinite dimensional BMS group
- Regge&Teitelboim(R&T) 1974: asymptotic symmetry group of asymptotically flat spacetimes is Poincaré in Hamiltonian(ADM) formalism at spatial infinity, BMS charges vanishing due to parity conditions

Overview

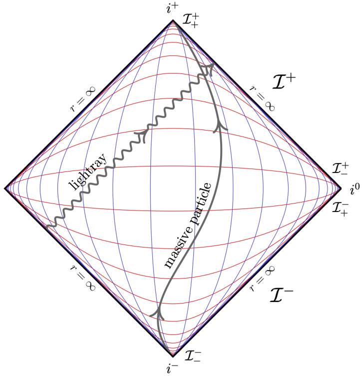

Figure: Penrose diagram of Minkowski space

- spatial infinity i^0 : $t = const., r \to \infty$
- null infinity $\mathcal{I}^+$: $u = t - r = const., r \to \infty$

Overview

- Henneaux&Troessaert(H& T)[1] 2018: New parity conditions, BMS charges finite in ADM formulation $\rightarrow$ BMS at i^0
- Their starting point: asymptotic expansion of spatial metric and conjugate momenta
- We revisited this analysis: 3+1 decomposition of Bondi-type spacetime metric
- Results suggest presence of larger-than BMS symmetry at i^0

[1] M. Henneaux and C. Troessaert. JHEP 2018.3 (2018): 147.

BMS symmetry

- BMS consider on-shell "Bondi metric"

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \quad (\text{ Minkowski }) \\ &+ \frac{2m}{r}du^2 + rC_{AB}dx^A dx^B + D^B C_{AB} dudx^A \\ &+ \text{subleading} \end{aligned}$$

- Determinant condition

$$\det g_{AB} = r^4 \det \gamma_{AB} \rightarrow \gamma^{AB} C_{AB} = 0$$

- Mass loss formula, with Bondi News $N_{AB}(u, x^A) = \partial_u C_{AB}$

$$\frac{d}{du}\int d^2\Omega m(u, x^A) = -\int d^2\Omega N_{AB} N^{AB}$$

BMS symmetry

$$x^\mu \to x^\mu + \xi^\mu, \quad \delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$$

- Which transformations leave Bondi metric asymptotically invariant?

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}\left(r^{-1}\right), \quad \mathcal{L}_\xi g_{AB} = \mathcal{O}\left(r\right), \ldots$$

- Obtain asymptotic Killing vectors which generate "supertranslations" parametrized by $f(x^A)$

$$\xi_f = f\partial_u - \frac{1}{r} D^A f \partial_A + \frac{1}{2} D_A D^A f \partial_r + \text{subleading}$$

- For $\mathcal{L}_\xi g_{uu} = 0$ etc. obtain only Poincaré transformations, i.e. f contains only $l = 0, 1$ modes

BMS symmetry

- And Lorentz transformations parametrized by R^A

$$\xi_R = \frac{u}{2} D_A R^A \partial_u + R^A \partial_A - \frac{r}{2} D_A R^A \partial_r$$

- Form BMS algebra under Lie bracket

$$[\xi_{f_1,R_1}, \xi_{f_2,R_2}] = \xi_{\hat{f},\hat{R}}$$

$$\hat{f} = R_1^A D_A f_2 - \frac{1}{2} f_2 D_A R_1^A - (1 \leftrightarrow 2),$$
$$\hat{R}^A = R_1^B D_B R_2^A - (1 \leftrightarrow 2)$$

Asymptotic symmetries in ADM formalism

- ADM: foliation of spacetime into spacelike surfaces
- Canonical fields are h_{ij}, π^{ij} and gauge transformations are generated by

$$G_\xi = \int d^3x(\xi\mathcal{H} + \xi^i\mathcal{H}_i) \approx 0$$

- via Poisson bracket

$$\delta_\xi h_{ij} = \{h_{ij}, G_\xi\}$$

Asymptotic symmetries in ADM formalism

- For non-compact spaces: surface terms appear in variation of G

$$\begin{aligned}\delta G &= \int d^3x(A^{ij}\delta h_{ij} + B_{ij}\delta\pi^{ij}) + \int d^2\Omega C[\xi, \xi_i] \\ &= \delta G_0 - \delta K\end{aligned}$$

- R&T: redefine Hamiltonian so that functional derivative well-defined

$$\delta G' = \delta(G - K) = \delta G_0$$

- ξ, ξ_i arbitrary $\rightarrow$ value of surface terms arbitrary? choose them to be zero?

Asymptotic symmetries in ADM formalism

- First: define boundary conditions, e.g. asymptotic flat

$$h_{ij} = \delta_{ij} + O(r^{-1})$$

- Only allow ξ, ξ_i which preserve these conditions via

$$\delta_\xi h_{ij} = \{h_{ij}, \xi\mathcal{H} + \xi^i\mathcal{H}_i\}$$

- If corresponding surface terms vanish they are "proper gauge transformations"
- Finite surface terms indicate "improper gauge transformations" $\rightarrow$ asymptotic symmetry(finite charges)!

Asymptotic symmetries in ADM formalism

- R&T boundary conditions in cartesian coordinates

$$h_{ij} = \delta_{ij} + \frac{\bar{h}_{ij}}{r} O(r^{-2}), \quad \pi^{ij} = \frac{\bar{\pi}^{ij}}{r^2} + O(r^{-3})$$

- preserved by Poincaré transformations and *angle-dependent translations*
- Additional parity conditions are assumed

$$\bar{h}_{ij}(-\mathbf{n}) = \bar{h}_{ij}(\mathbf{n}), \quad \bar{\pi}^{ij}(-\mathbf{n}) = -\bar{\pi}^{ij}(\mathbf{n})$$

- to cancel divergences in charges of the form

$$\int d^2\Omega \bar{\pi}^{ij} \delta \bar{h}_{ij}$$

Asymptotic symmetries in ADM formalism

- Parity conditions let charges associated with angle-dependent translations vanish
- Remaining charges are shown to form Poincaré algebra under Poisson bracket

Asymptotic symmetries in ADM formalism

- H&T 2018: find new parity conditions to enlarge asymptotic symmetry, use spherical coordinates
- Fall-off conditions are defined as

$$
\begin{aligned}
h_{rr} &= 1 + \frac{1}{r}\bar{h}_{rr} + O\left(r^{-2}\right) \\
h_{rA} &= \frac{1}{r}\bar{h}_{rA} + O\left(r^{-1}\right) \\
h_{AB} &= r^2\bar{\gamma}_{AB} + \bar{h}_{AB} + O(1) \\
\pi^{rr} &= \bar{\pi}^{rr} + O\left(r^{-1}\right) \\
\pi^{rA} &= \frac{1}{r}\bar{\pi}^{rA} + O\left(r^{-2}\right) \\
\pi^{AB} &= \frac{1}{r^2}\bar{\pi}^{AB} + O\left(r^{-3}\right)
\end{aligned}
$$

Asymptotic symmetries in ADM formalism

- New parity conditions formulated in spherical coordinates, e.g.

$$\bar{h}_{rr} = even, \quad \bar{\pi}^{rr} = odd$$

- Not only charges, but also kinetic term in action should be finite

$$\int d^3x \pi^{ij} \dot{h}_{ij} = \int \frac{dr}{r} d\theta d\phi \left(\bar{\pi}^{rr} \dot{\bar{h}}_{rr} + ... \right)$$

- Divergences in charges canceled by strengthening of constraints(so parity not needed here), e.g.

$$\mathcal{H} = O(r^{-1}) \Rightarrow \mathcal{H} = O(r^{-2})$$

Asymptotic symmetries in ADM formalism

- Fall-off conditions invariant under Lorentz transf. and supertranslations parametrized by $T(x^A), W(x^A)$

$$\xi = T, \xi^r = W, \xi^A = \frac{D^A W}{r}$$

- Preserving parity conditions requires $W = odd, T = even$
- T, W combine to single f parametrizing BMS supertranslations
- Supertranslation charges in general finite(were vanishing for R&T)

$$Q_{T,W} = \oint d^2x \left(\bar{h}_{rr} T + W(\bar{\pi}^{rr} - \bar{\pi}^A_A) \right)$$

$\Rightarrow$ Asymptotic symmetry is BMS

Our analysis

We expressed $h_{ij}, \pi^{ij}, \dot{h}_{ij}$ etc. in terms of spacetime metric components by using 3+1 decomposition

- What do boundary conditions imply for spacetime metric?
- What are consequences for asymptotic symmetry?
- What do charges look like?

Bondi-type metric as starting point

$$
\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu = & -\left(1 - \frac{2M}{r} + O(r^{-2})\right) du^2 - 2\left(1 - \frac{\bar{g}_{ur}}{r} + O(r^{-2})\right) du dr \\
& + \left(\psi_A + \frac{1}{r} F_A + O(r^{-2})\right) du dx^A \\
& + \left(r^2 \gamma_{AB} + r C_{AB} + D_{AB} + O(r^{-1})\right) dx^A dx^B,
\end{aligned}
$$

- More general than Bondi metric: off-shell, no determinant condition
- Condition $\gamma^{AB} C_{AB} = 0$ too rigid at i^0

3+1 Decomposition

- Define foliation into spacelike surfaces Σ_t by

$$t = u + r + f(x^A) + \frac{g(x^A)}{r} = const.$$

- Transform metric $u \to t$ and compare with decomposition

$$ds^2 = -N^2 dt^2 + h_{ij}(dy^i + N^i dt)(dy^j + N^j dt)$$

$\Rightarrow$ read off h_{ij}, N, N_i

- t is the most general choice such that h_{ij} asymptotically flat

3+1 Decomposition

- For h_{ij} one finds, e.g.

$$\bar{h}_{rr} = 2(M - \bar{g}_{ur}), \quad \bar{h}_{AB} = C_{AB} \quad \ldots$$

- And for lapse and shift

$$N = 1 - \frac{M}{r} + O(r^{-2}), \quad N_r = \frac{\bar{g}_{ur} - 2M}{r} + O(r^{-2}),$$
$$N_A = \partial_A f + \frac{1}{2}\psi_A + O(r^{-1})$$

3+1 Decomposition

$$t = u + r + f(x^A) + \frac{g(x^A)}{r} = const.$$

- On h_{ij} coordinates are (r, x^A) but $M = M(u, x^A)$?
- u is not independent coordinate, $M = M(u(r, x^A), x^A)$
- In large r limit: $u \to -\infty$, so always implicit limit

$$M = \lim_{u \to -\infty} M(u, x^A)$$

3+1 Decomposition

- From extrinsic curvature

$$K_{ij} = n_{\mu;\nu} e_i^\mu e_j^\nu, \quad n_\nu = -N \partial_\nu t, \quad e_i^\mu = \frac{\partial x^\mu}{\partial y^i}$$

- obtain momenta and "velocity" in terms of $g_{\mu\nu}$

$$\pi^{ij} = \sqrt{h}(K^{ij} - K h^{ij}), \quad \dot{h}_{ij} = 2NK_{ij} + N_{(i|j)}$$

- In particular one finds leading order

$$\pi^{rr} \propto r \gamma^{AB} \partial_u C_{AB}, \quad \pi^{AB} \propto \frac{1}{r} \left(\gamma^{AB} \partial_u M + \partial_u C^{AB} \right)$$

Restricting allowed spacetimes

$$\pi^{rr} \propto r\gamma^{AB}\partial_u C_{AB}, \quad \pi^{AB} \propto \frac{1}{r}\left(\gamma^{AB}\partial_u M + \partial_u C^{AB}\right)$$

- H&T boundary conditions for momenta

$$\pi^{rr} = \bar{\pi}^{rr} + O(r^{-1}), \quad \pi^{AB} = \frac{1}{r^2}\bar{\pi}^{AB} + O(r^{-3})$$

- translate into conditions on $\partial_u C_{AB}$, $\partial_u M$ in limit $u \to -\infty$

Restricting allowed spacetimes

- Natural requirement, on-shell describe rate of gravitational radiation
- Finite amount of radiated energy if

$$\partial_u M \propto u^{-(1+\epsilon)}, \quad \partial_u C_{AB} \propto u^{-(1+\epsilon)}, \quad \epsilon > 0$$

- Since $u = t - r +$ subleading, we can write in the large r limit

$$\partial_u M = \frac{\hat{M}(x^A)}{r^{1+\epsilon}}, \quad \partial_u C_{AB} = \frac{\hat{C}_{AB}(x^A)}{r^{1+\epsilon}}$$

Finiteness of kinetic term

$$\dot{h}_{ij} = 2NK_{ij} + N_{(i|j)}$$

$$\int d^3x \pi^{ij} \dot{h}_{ij} = \int \frac{dr}{r} d\theta d\phi (\bar{\pi}^{rr} \partial_u M + \bar{\pi}^{AB} \partial_u C_{AB})$$

- Extra damping factors $\Rightarrow$ no parity conditions needed to cancel logarithmic divergence for allowed spacetimes
- No more restriction of T, W to preserve parity conditions $\Rightarrow$ potentially larger asymptotic symmetry if charges finite
- If vanishing $\Rightarrow$ proper gauge transformations

Supertranslation charge

$$Q_{T,W} = \oint d^2x \left[T\sqrt{\gamma}\bar{h}_{rr} + 2W\left(\bar{\pi}^{rr} - \bar{\pi}^A_A\right)\right]$$

- Expressed in terms of components of $g_{\mu\nu}$

$$\begin{aligned} Q_{T,W} = \oint d^2x \sqrt{\gamma}\Big[& 4T(M - \bar{g}_{ur}) \\ & + W\left(\left(D^2 + 4\right)(\bar{g}_{ur} - 2M)) - \gamma^{AB}\partial_u D_{AB}\right)\Big] \end{aligned}$$

- Finite for all modes of $T, W \Rightarrow$ supertranslation sector of asymptotic symmetry larger than BMS

Determinant condition and reduction of symmetry

- Asymptotic transformations would also have to preserve $\gamma^{AB}\bar{h}_{AB} = \gamma^{AB}C_{AB} = 0$

$$\gamma^{AB}\delta_\xi \bar{h}_{AB} = 0$$

- Leads to condition of the form

$$(D^2 + 2)W = f(\pi^{ab}).$$

- In general no solution since $D^2 Y_{l,m} = -l(l+1)Y_{l,m}$ and the lhs can not produce $l = 1$ harmonics which are present on rhs.
- Spatial translation do not leave $\gamma^{AB}\bar{h}_{AB} = 0$ invariant

Determinant condition and reduction of symmetry

- At $\mathcal{I}^+$: $\gamma^{AB}\delta C_{AB} = 0$ is fulfilled identically
- Why is it harmless at $\mathcal{I}^+$ but strongly reduces symmetry at i^0?