Towards semi-classical cosmology in LQG

- via Symmetry Restriction and the Quantum Speed Limit

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Based on works with Wojciech Kamiński and Łukasz Rudnicki

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Motivation

- Status Quo: Quantum Gravity is conjectured to explain the breakdown of General Relativity at singularities as well as to establish the rigorous definition of interacting 4-dim QFT

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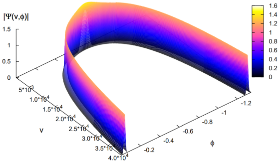
- Meanwhile: Without experiments, it's paramount to not only study some conjectured quantum gravity models, but to develop general tools (with potential applications outside of QG)
- Goal for today: We will present two frameworks on a fairly general level and - to showcase their strength - apply them to quantum cosmology

Motivation

- Loop Quantum Cosmology originates as canonical quantisation of a polymerised version of isotropic FLRW cosmology
- Isotropy = Invariance under symmetry group (translations & rotations)
- Classical only degree of freedom scale factor $p = a^2$ gives a phase space with symplectic structure $\omega_{FRW} = dp \wedge dc$
- Physical Hilbert space $\mathcal{H}^{phys}$ of LQC are solutions to a "polymerized" constraint operator
- Construct coherent states, sharply peaked $\Psi_{(h,P)} \in \mathcal{H}^{phys}$ describing *semiclassical* geometry

$$\begin{tikzcd}
\text{GR} \arrow[d] & \\
\text{Cosm} \ (\sqrt{p}\, c^2) \arrow[r] \arrow[d] & \text{WDW} \\
\text{polymerized Cosm} \ (\sqrt{p}\, \sin(\mu_0 c)^2) \arrow[r] & \text{LQC} \arrow[d] \\
& \text{semiclassi Limit}
\end{tikzcd}$$



Overview

1 Motivation

2 Classical: Symmetry Restriction

Determining special solutions wrt given symmetry group

3 Quantum: Quantum Speed Limit

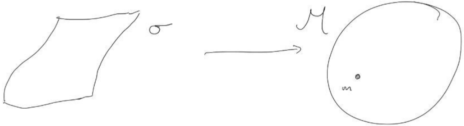
Coherent states as proposal for semiclassical quantum cosmology

4 Conclusion & Outlook

1) Symmetry Restriction

Kinematical Setup

Let $(\mathcal{M}, \omega)$ be a phase space characerised by coordinates and momementa of a field theory over a spatial slice σ.



Let Φ be a group, which allows a representation on $\mathcal{M}$ in terms of symplectomorphisms, that is:

- For each $\phi \in \Phi$ there exist $\phi : \mathcal{M} \mapsto \mathcal{M}$ (extendable to act on functions $f \in C^\infty(\mathcal{M})$ via $\phi_* f := f \circ \phi^{-1}$ etc.)
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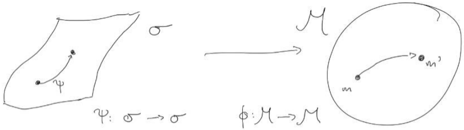

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Invariant subspace

Our interest are phase space points invariant under Φ, i.e.

$$\overline{\mathcal{M}} := \{m \in \mathcal{M} \ : \ \phi(m) = m \quad \forall \phi \in \Phi\} \tag{1}$$

It is possible to restrict functions f and n-forms ω to $\overline{\mathcal{M}}$:

$$f|_{\overline{\mathcal{M}}} := f(m) \qquad m \in \overline{\mathcal{M}}$$
$$\overline{\omega}(X_1, ...X_n) := \omega(X_1, ...X_n)|_{\overline{\mathcal{M}}}$$

It is possible to define a restricted Poisson bracket on $\overline{\mathcal{M}}$:

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Question: In what cases can we analyse only the restricted system without loosing information?

One can show the following theorem to hold for finite-dim as well as for infinte-dim phase spaces:

Symmetry restriction of dynamics [Kamiński, KL, '20]

Let $(\overline{\mathcal{M}}, \overline{\omega})$ be the symmetry restriction of $(\mathcal{M}, \omega)$ wrt Φ and let H be a Φ-invariant function. Then, the flow generated by $H|_{\mathcal{M}}$ agrees with the flow of H on $\overline{\mathcal{M}}$.

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In other words: the evolution wrt H of any observable $O : \mathcal{M} \to \mathbb{R}$, when restricted to $\overline{\mathcal{M}}$, agrees with the evolution of $O|_{\overline{\mathcal{M}}}$ with respect to $H|_{\overline{\mathcal{M}}}$ computed via $\{.,.\}_{\overline{\mathcal{M}}}$:

$$\{H, O\}|_{\overline{\mathcal{M}}} = \{H|_{\overline{\mathcal{M}}}, O|_{\overline{\mathcal{M}}}\}_{\overline{\mathcal{M}}} \tag{3}$$

Symplectic Reduction vs Symmetry restriction

Symmetry restriction is closely related to the notion of constraint systems.
A gauge group G describes symplectomorphisms, whose generators are *constraints* $J = \{C_I\}_I$, giving rise to the constraint surface $J^{-1}(0)$. In nice situations,

$$\mathcal{M}//G = \{[m] \ : \ m \in J^{-1}(0)\} \tag{4}$$

is manifold (the physical interesting part of phase space) and allows reduction of the symplectic form (aka *symplectic reduction*).
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Symmetry restriction is complementary to the symplectic reduction: first, we will symmetry restrict to $\overline{\mathcal{M}}$ on which a non-degenerate symplectic form exists – thereby taking care of the singular points – and afterwards we can perform a symplectic reduction with respect to the remaining symplectomorphisms in G.

Symmetry restriction applied: Symmetry group in GR

In GR, we are in the special situation that we are interested in symmetry groups $\Psi \subset \mathrm{Diff}(\sigma)$, a gauge group of GR.
Connection formulation of GR: $(q^{ab} = E^a_I E^b_J \delta_{IJ}/|\det(E)|)$

$$\mathcal{M}^{O(3)}_{AB} = \{(E_I, A^J) \in ({}^1C^\infty(\sigma), {}_1C^\infty(\sigma))\,,\ \det(E) \neq 0\} \qquad (6)$$

There is also the $O(3)$ gauge group $\mathbb{G} := \mathbb{G}^+ \times \{\mathbb{1}, -\mathbb{1}\}$

$$\mathbb{G}(O)(E,A) := (O^J_I E^a_J, O^I_J A^J_a + \frac{1}{2}\epsilon_J{}^{IK}(\partial_a O^J_L){O^{-1}}^L{}_K) \qquad (7)$$

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How to lift Ψ-action on σ to $\mathcal{M}^{O(3)}_{AB}$?
Clearly, diff-part acts in natural way, but we have a freedom in defining the action of the $O(3)$-part.
Lemma: Given some E_o such that $q_o = E^I_o E^J_o \delta_{IJ}$ is Ψ-invariant, choosing

$$\Phi_{AB,E_o} = \{(\mathbb{D}(\psi), E^a_{oJ}\psi_* E^I_{oa}) \; : \; \psi \in \Psi\} \subset \mathbb{D}(\mathrm{Diff}(\sigma)) \ltimes \mathbb{G} \tag{9}$$

ensures that $O(\psi)^I_J \circ \psi_* E_{oI}(x) = E_{oJ}(x)$.

First: on non-compact topolgy restriction of ω is not well deifned:

$$\omega = \frac{2}{\kappa\beta} \int_\sigma \mathrm{d}^3x \; dE^a_I(x) \wedge dA^I_a(x) \tag{10}$$

because $\Psi := SO(3) \rtimes \mathbb{R}^3$ has invariant subspace:

$$\overline{\mathcal{M}} = \{(E^a_I = p\delta^a_I,\; A^I_a = c\delta^I_a) \;:\; p, c \in \mathbb{R}\} \tag{11}$$

Therefore: restriction to compact spatial slice (Torus $\sigma = (\mathbb{R}/T)^3$)

Symmetry restriction applied: FLRW cosmology

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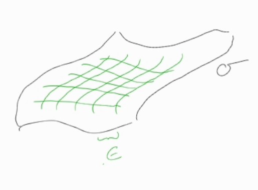
Constraints: One can show that symmetry restriction reduce the constraints in the following way $\vec{D}[\vec{N}] \to$ the set of all $\vec{N}$ invariant under Ψ. $C[N]$ the set of all N invariant under Ψ.
Now, $\Psi := SO(3) \rtimes \mathbb{R}^3$ acts transitively ($\Rightarrow N = 1$) and has no invariant vector field ($\vec{N} = 0$). Thus, we are left with one constraint, explicitly

$$C[1]|_{\overline{\mathcal{M}}} := -\frac{6}{\kappa\beta^2} c^2 \sqrt{|p|} + C_{\mathrm{matter}}|_{\overline{\mathcal{M}_{AB}}} \tag{12}$$

What is lattice GR ?



Truncation from continuum to discrete

$$\begin{array}{ccc} (A,E)\in\mathcal{M}_{AB} & \stackrel{\mathbb{G}_{AB}(O(3))}{\longrightarrow} & \mathbb{G}_{AB}(O)(A,e)\in\mathcal{M}_{AB} \\ \downarrow & & \downarrow \\ (h,P)\in\mathcal{M}_{\gamma} & \stackrel{\mathbb{G}^{\gamma}(O(3))}{\longrightarrow} & \mathbb{G}^{\gamma}(g)(h,P)\in\mathcal{M}_{\gamma} \end{array}$$

with

$$\mathcal{M}_{\gamma}=\{(h,P)\ :\ h:\gamma_e\to SU(2)\ ,\ P:\gamma_e\to su(2)\} \tag{13}$$

$$h(e):=\mathcal{P}\exp\left(\int_0^1 \mathrm{d}t\, A_a^J(e_k(t))\tau_J\dot{e}_k^a(t)\right) \tag{14}$$

$$P(e):=h(e_{[0,1/2]})\left[\int_{S_e}\mathrm{d}x h(\rho_x)*E(x)h(\rho_x)^{\dagger}\right]h(e_{[0,1/2]})^{\dagger} \tag{15}$$

Symmetry restriction applied: Lattice cosmology II

- The symplectic form of the lattice GR reads:

$$\omega_\gamma = \sum_{[e]\in\gamma_{[e]}} \omega_e \tag{16}$$

- The symplectic form ω_e is a symplectic form on $T^*SU(2)$ of a given edge that is

$$\omega_e = d\xi_e, \quad \xi_e = \frac{2}{\kappa\beta} P^I(e)\Omega_I(e), \quad \Omega_I(e) := -2tr(\tau_I dh(e)\, h^{-1}(e))$$

i.e. $\Omega_I(e)$ is the right invariant Maurer-Cartan form on $SU(2)$.

- Lastly, truncations of the constraints:

$$G[\Lambda] = \sum_{v\in\gamma} \Lambda_I(v) G^I(v), \qquad G^I(v) = \sum_{e\in\gamma\,:\,e(0)=v} P^I(e) \tag{17}$$

and scalar constraint a function such that $\lim_\epsilon C^\epsilon(v) = C(x = v)$ *(next slide)*

Intermezzo

$$C^\epsilon[N] := \sum_v C^\epsilon(v)\, N(v) \tag{18}$$

Based on regularisation of Thiemann, we introduce the following discretisation (one of many possible) specialised to cubic lattices

$$C^\epsilon(v) := C_E(v) + C_L(v) + C^\epsilon_{\rm matter}(v), \tag{19}$$

$$C_E(v) = \frac{-1}{2\kappa^2\beta} \sum_{i,j,k\in L} \epsilon(i,j,k) \mathrm{Tr}\left([h(\Box_{v,ij}) - h^\dagger(\Box_{v,ij})] h(k,v) \{h^\dagger(k,v), V^\epsilon[\sigma]\}\right)$$

$$C_E[\sigma] = \sum_v C_E(v), \quad V^\epsilon[\sigma] = \sum_v \sqrt{\sum_{ijk\in L} \frac{\epsilon_{IJK}}{48} \epsilon(i,j,k) P^I(i,v) P^J(j,v) P^K(k,v)},$$

$$C_L(v) = \frac{8(1+\beta^2)}{\kappa^4\beta^7} \sum_{ijk\in L} \epsilon(i,j,k) \tag{20}$$

$$\times \,\mathrm{Tr}\left(h(i,v)^\dagger \{h(i,v), K\} h(j,v)^\dagger \{h(j,v), K\} h(k,v)^\dagger \{h(k,v), V\}\right)$$

where $\Box_{v,ij}$ denotes a plaquette starting at v in direction i and returning along direction j and $\epsilon(i,j,k) := \mathrm{sgn}\det(i,j,k) = \mathrm{sgn}(ijk)\epsilon_{|i||j||k|}$.

Cosmology of Lattice GR, where $\sigma = (\mathbb{R}/T)^3$ gets restricted lattice with M-many points: $\epsilon = \mu_o = T/M$ with $T = 1$ as period of the torus.

We perform symmetry restriction via symmetry group:

$$\Psi^\gamma = \mathbb{Z}_M^3 \times \{R_{e_k}(n\,\pi/2) \ : \ k = 1, 2, 3, n = 0, 1, 2, 3\} \tag{21}$$

The points $\overline{\mathcal{M}}_\gamma^{FRW} \subset \mathcal{M}_\gamma$ invariant under group $\Phi_{AB,E_o}(\Psi^\gamma)$ are

$$(P(e), h(e)) = (\mu_o^2\, p\, \tau_I,\ e^{i\mu_o\, c\, \tau_I}) \tag{22}$$

and the symplectic structure reduces

$$(\sum_e \omega_e)|_{\overline{\mathcal{M}}_\gamma^{FRW}} = \frac{6}{\kappa\beta} dp \wedge dc \tag{23}$$

Finally restriction of the scalar constraints give:

$$C^\epsilon[1]|_{\overline{\mathcal{M}}_\gamma^{FRW}} = \frac{6\mathcal{N}}{\kappa}\sqrt{p}\left[\frac{\sin(c\mu_o)^2}{\mu_o^2} - \frac{1+\beta^2}{\beta^2}\frac{\sin(2c\mu_o)^2}{4\mu_o^2}\right] \tag{24}$$

Symmetry restriction applied: Lattice cosmology III

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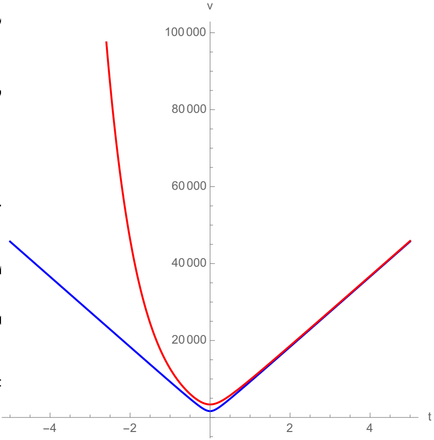

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Symmetry restriction: Lessons learned

Symmetry restriction is a framework to identify symmetric solutions that will keep their symmetry at all times. In applications, it allows to transform complex computations on an infinite-dimensional phase space to one with finitely many degrees of freedom.

Comments:

- Challenge in applications is to find suitable symmetry group
- Obtained insights on how gauge groups (e.g. SU(2), SO(3)) change under truncation to lattice and symmetry restriction
- Correctly reduce the phase space of gauge-covariant fluxes instead of postulating it
- Sym. Res. is applicable to many systems [Kamiński, KL '20] ($k = \pm 1$ cosmologies, Bianchi I, Bianchi I on lattice etc.)
- Big Bounce happens solely due to discretisation artefacts and is not *genuine quantum* prediction

2) Quantum Speed Limit

We have convinced us that *classically* a restriction of the Hamiltonian dynamics of full General Relativity to cosmology is possible. How does this translate to the *quantum* level?

- We introduce the kinematical Hilbert space of Loop Quantum Gravity restricted to a finite lattice
- Coherent state can be sharply peaked on a semiclassical configuration mimicking cosmology
- We develop all necessary tools in order to compute exectation values of polynomial operators including first order quantum corrections
- Via the quantum speed limit, we can draw information on the dynamical behaviour of said coherent states

The basic Hilbert space $L_2(SU(2), d\mu_H)$

- Classically: on every edge of the lattice lives a $SU(2)$-valued holonomy
- We denote the space of square-integrable functions over $SU(2)$ by $\mathcal{H}_e := L_2(SU(2), \mu_H)$ with Haar measure $d\mu_H$
- The Peter-Weyl theorem tells us that these functions have a basis with the irreducible representations of $SU(2)$

$$f(g) = \sum_{j,m,n} c(j,m,n)\; D^{(j)}_{mn}(g) \quad \in \mathcal{H}_e \tag{29}$$

- A *irrep* $D^{(j)}$ associates with every $g \in SU(2)$ a d_j-dim matrix $D^{(j)}(g)$ and its matrix elements are thus $D^{(j)}_{mn}(g) \in \mathbb{C}$
- The *irreps* form an orthogonal basis on $\mathcal{H}_e$: $(d_j = 2j+1)$

$$\langle D^{(j)}_{mn}, D^{(j')}_{m'n'} \rangle = \frac{1}{d_j} \delta_{jj'} \delta_{mm'} \delta_{nn'} \tag{30}$$

Coherent states for $SU(2)$

- Complexifier coherent states are for each $H \in SL(2,\mathbb{C})$ [Hall, Thiemann, Winkler]

$$\psi_H^t := \sum_{j\geq 1/2} d_j\, e^{-j(j+1)t} \sum_{m=-j}^{j} D^{(j)}_{mm}(H^\dagger\, g) \tag{31}$$

where $t > 0$ controls the spread of the state.

- A useful decomposition of H is ($\tau_3 = -i\sigma_3/2$)

$$H = n e^{-(\xi - i\eta)\tau_3}\, \tilde{n} \tag{32}$$

- Terminology "semiclassical" is fitting beccause for $t << 1$:

$$\langle \hat{h}_{ab} \rangle = n e^{-\xi\tau_3}\, \tilde{n} \;\; + \mathcal{O}(t), \quad \langle \hat{P}^I \rangle = i\hbar\kappa \frac{\eta}{t} D^{(1)}_{-I0}(n) \;\; + \mathcal{O}(t)$$

with the basic operators

$$(\hat{h}_{ab} f)(g) := D^{(1/2)}_{ab}(g) f(g), \qquad (\hat{P}^I f)(g) = i\hbar\kappa \frac{d}{ds} f(e^{s\tau_I} g)|_{s=0}$$

Coherent states for the whole lattice γ:

$$\Psi_z := \otimes_{e\in\gamma}\psi^t_{H_e(z)} \tag{33}$$

where $H_e(z) \in \mathrm{S}L(2,\mathbb{C})$ captures the discretised geometry on edge e of $z = (\xi, \eta)$, i.e. for every edge in direction a:

$$H_{e(a)}(z) = n_a e^{i(\xi+i\eta)\tau_a} n_a^\dagger, \quad n_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\; n_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},\; n_3 = \mathrm{id}$$

These states have that for some polynomial function F:

$$\langle \Psi_z, F(\hat{h}, t\hat{P})\Psi_z\rangle = F(h, P) + \mathcal{O}(t) \tag{34}$$

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Question: What are explicit formulas for the expectation values of polynomial operators when including quantum corrections?

Explicit formulas for coherent states expectation values

The basic operators for $SU(2)$ are

$$(\hat{h}^{(k)}_{ab} f)(g) := D^{(k)}_{ab}(g) f(g), \qquad (\hat{P}^I f)(g) = i\hbar\kappa \frac{d}{ds} f(e^{s\tau_I} g)|_{s=0}$$

together with the *left-invariant* vector field

$$(\hat{L}^I f)(g) = i\hbar\kappa \frac{d}{ds} f(g e^{s\tau_I})|_{s=0} \tag{35}$$

Further, we make use of the following identity: [Dapor, KL '17]

$$\langle \psi_H, \ ...\hat{L}^J \psi_H \rangle = D^{(1)}_{JM}(n) e^{-izM} D^{(1)}_{MS}(n') \langle \psi_H, \ ...\hat{P}^S \psi_H \rangle \tag{36}$$

and standard $SU(2)$ recoupling theory

$$\hat{h}^{(k_1)}_{ab} \hat{h}^{(k_2)}_{cd} = \sum_{K=|k_1-k_2|}^{k_1+k_2} d_K \begin{pmatrix} k_1 & k_2 & K \\ a & c & M \end{pmatrix} \begin{pmatrix} k_1 & k_2 & K \\ b & d & N \end{pmatrix} (-1)^{M-N} \ \hat{h}^{(K)}_{-M-N}$$

to see that we have full control $\Leftrightarrow$ we know the exp. value of

$$\hat{h}^{(K)}_{MN} \hat{P}^{I_1} ... \hat{P}^{I_N} \tag{37}$$

Explicit formulas for coherent states expectation values

The basic operators for $SU(2)$ are

$$(\hat{h}^{(k)}_{ab} f)(g) := D^{(k)}_{ab}(g) f(g), \qquad (\hat{P}^I f)(g) = i\hbar\kappa \frac{d}{ds} f(e^{s\tau_I} g)|_{s=0}$$

The general formula including everything up to $\mathcal{O}(t^2)$ corrections reads: [KL, Zwicknagel 2020]

$$\begin{aligned}
\langle \hat{h}^{(j)}_{ab} P^{K_1} \dots P^{K_N} \rangle_{\psi^t_{H_I}} &= \langle 1 \rangle \left(\frac{i\eta}{t}\right)^N D^{(1)}_{-K_1 - S_1}(n_I) \dots D^{(1)}_{-K_N - S_N}(n_I) \sum_{c=-j}^{j} \Bigg(\delta^{S_1}_0 \dots \delta^{S_N}_0 \delta_{aa'} + \\
&+ \frac{t}{2\eta}\Bigg[\delta_{aa'} \delta^{S_1}_0 \dots \delta^{S_N}_0 N \left(\frac{N+1}{2\eta} - \coth(\eta)\right) + i \sum_{A=1}^{N} \delta^{S_1}_0 \dots \cancel{\delta}^{S_A}_0 \dots \delta^{S_N}_0 \left(1 - s_A \tanh\left(\frac{\eta}{2}\right)\right) D^{(j)}_{-s_A - L}(n_I^\dagger)[\tau^L]^{(j)}_{aa'} \\
&- \frac{\delta_{aa'}}{\sinh(\eta)} \sum_{A<B=1}^{N} \delta^{S_1}_0 \dots \cancel{\delta}^{S_A}_0 \dots \cancel{\delta}^{S_B}_0 \dots \delta^{S_N}_0 (\delta^{S_A}_{+1}\delta^{S_B}_{-1} + \delta^{S_A}_{-1}\delta^{S_B}_{+1}) e^{S_A \eta}\Bigg]\Bigg) D^{(j)}_{aa'}(n_I) e^{-i\xi c} \gamma^j_c D^{(j)}_{cb}(n_I^\dagger) + \mathcal{O}(t^2),
\end{aligned} \tag{38}$$

with

$$\gamma^j_c = 1 - t\frac{1}{4}\left[(j^2 + j - c^2)\frac{\tanh(\eta/2)}{\eta/2} + c^2\right], \qquad \langle 1 \rangle := \langle \hat{h}^{(0)}_{00} \rangle_{\psi^t_{H_I}} = \sqrt{\frac{\pi}{t^3}} \frac{2\eta\, e^{\eta^2/t}}{\sinh(\eta)} e^{t/4}. \tag{39}$$

Algorithm / code for polynomial operators

We developed a *Mathematica* code, to execute the expectation values of complicated operators over the whole lattice, including $\mathcal{O}(t)$ -corrections. [KL, Rudnicki '20]

Input: A polynomial Operator built out of $\hat{h}_{ab}$, $\hat{P}^K$ and $\hat{Q} = Q(\hat{P})$ (to approximate the Ashtekar-Lewandowksi volume operators à la [Giesel, Thiemann '06]), possibly involving commutators.

Applications

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Applications $\to$ **Quantum Gravity:** Dynamics is driven by the *scalar constraint* of which a version for LQG is known [Thiemann '96]. E.g. its Euclidean part reads

$$\hat{C}_E[1] \sim \sum_{v} \sum_{e_i \cap e_j \cap e_k = v} \epsilon(e_i, e_j, e_k) \times$$

$$tr((\hat{h}(i)\hat{h}(v+e_i, j)\hat{h}(v+e_i+e_j, -i)\hat{h}(v+e_i, -j) - c.c)\hat{h}^\dagger(k)[\hat{h}(k), \hat{V}])$$

$$\langle \hat{C}^\epsilon_E[1] \rangle = \frac{6}{\kappa\epsilon^2}\sqrt{\eta}\sin(\xi)^2 \left[1 + t\left(-\frac{1}{4} - \frac{13\coth(\eta)}{8\eta} + \frac{11}{8\eta\sinh(\eta)} + \frac{9}{32\eta^2}\right) + t\frac{3i}{8\eta}\frac{\sin(\xi/2)^2}{\sin(\xi)}\right] + \mathcal{O}(t^2).$$

Application in Quantum Gravity

- Effective theory in the $t \to 0$ limit $(\{\langle\hat{C}\rangle, \langle\hat{O}\rangle\})$ [Ashtekar, Singh, Pawłowski, Taveras, Assanioussi, Dapor, KL,...]

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- Including $\mathcal{O}(t)$-corrections, one can show that the effective description breaks down $(\{\langle \hat{C}_E \rangle, \langle \hat{O} \rangle\} \neq \langle [\hat{C}_E, \hat{O}] \rangle)$ [Rudnicki, KL '20]

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- **Here:** We investigate the dynamical properties of the coherent state family Ψ_z [Rudnicki, KL '21]

Intermezzo

The *quantum speed limit* originates from the Heisenberg equality:

$$\Delta_s \hat{O} \Delta_s \hat{H} \geq \frac{1}{2} |\langle \Phi_s, [\hat{O}, \hat{H}] \Phi_s \rangle| \tag{40}$$

with some family $\{\Phi_s\}_s$. If $\hat{O} = |\Upsilon\rangle\langle\Upsilon|$ then one can rewrite:

$$\tau \cdot \Delta_o \hat{H} / \hbar \geq |A(0) - A(\tau)| \tag{41}$$

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Many prominent applications:

- Quantum information theory [Okuyama, Ohzeki, Mondal, Datta, Sazim,...]
- Quantum optimal control [Wang, Allegra, Jacobs, Lloyd, Lupo, Mohseni,...]
- Quantum batteries [Mohan, Pati, Campaioli, Pollock, Binder, Céleri, Gold, Modi,...]
- Quantum Gravity **[Now!]**

Quantum Speed Limit in Quantum Gravity I

Given a *classical* trajectory on phase space $z(s) = (\xi(s), \eta(s))$, $s \in \mathbb{R}$ (the evolution by some H in lattice GR), we ask whether there is a family of *quantum* states mimicking the same evolution:

$$\{\Psi_{z(\tau)}\}_{\tau\in\mathbb{R}} \tag{43}$$

and what is the relation to the real evolution by the quantised $\hat{H}$:

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Thanks to our tools we can compute (for short times)

$$(\Delta_{\Psi_{z(0)}} \hat{C}_E)^2 = \frac{3}{2^7}\frac{M^3\beta\hbar}{\eta\kappa}\sin(\xi)^2\left(17 + 256\eta^2 + (256\eta^2 - 17)\cos(2\xi)\right) + \mathcal{O}(t).$$

$$A(0) = \tau(\Delta_{\Psi_{z(0)}} \hat{C}_E)^2/\hbar^2 - \frac{3}{2^6}\frac{M^3\beta}{\eta\kappa\hbar}\sin(\xi)^4 + \mathcal{O}(t),$$

It follows that $0 > |A(0)| - \tau \cdot \Delta_0 \hat{H}/\hbar$ on the whole parameter space $\xi, \eta \geq 0$! $\Rightarrow$ QSL check is passed!

Quantum Speed Limit in QG: Lessons learned

Quantum Speed Limit puts tight bounds on the evolution of a given quantum system. Moreover, it can serve in Quantum Gravity as a necessary consistency check for any proposal of a stable semiclassical system.

Comments:

- QSL is easily violated: Given $z \neq z'$ for $t \to 0$, then whenever $\Delta\hat{H} < \infty$ the transition cannot occur.
- In the limit $t \to 0$ and huge fluxes, we can show that for $\hat{C}_E$ given fixed τ there is $t << 1$ such that the transition is $z \to z(\tau)$ occurs with controlled error.
- Generalisation of the above statement might be possible

[Kamiński, KL, *to appear*]

3) Conclusion and Outlook

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- The quantum speed limit serves as a powerful consistency check in QG that any proposal for a stable family of states needs to pass
- Both tools may have further applications when investigating semiclassical limits of Loop Quantum Gravity

THANK YOU!