BKL and spike dynamics near spacelike singularities

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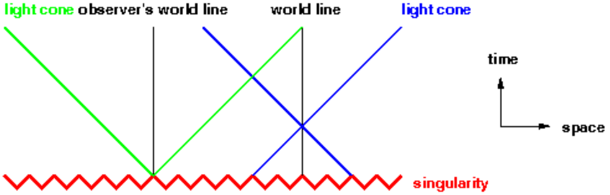
Abstract

It is widely known that, near spacelike singularities, most worldlines would undergo the so-called Belinski-Khalatnikov-Lifshitz (BKL) singularity dynamics, which is an infinite sequence of Kasner saddle states connected by Taub transitions. Some worldlines however would undergo highly inhomogeneous spike dynamics, which is an infinite sequence of Kasner saddle states connected by spike transitions. I will describe these dynamics using the Hubble-normalised orthonormal frame formulation of Einstein's field equations, which is very useful for spatially homogeneous models (we get a system of ordinary differential equations), and also suitable for inhomogeneous models. My contributions in this area are the discovery of the exact solutions which describe the spike transitions, and the development of a numerical zooming technique specially designed for spike simulations. I will illustrate the spike transitions with snapshots of the Hubble-normalised state space trajectories, and point out interesting features.

The BKL Conjecture

How does a general cosmological solution of Einstein's field equations (EFE) behave near a spacelike initial singularity?



Conjecture by Vladimir Belinski, Isaak Khalatnikov and Evgeny Lifshitz: [LK 1963, BKL 1970,1982]

A generic singularity is

1. Vacuum-dominated ("matter doesn't matter")
2. Local (spatial derivatives are negligible)
3. Oscillatory (an infinite sequence of Kasner states)

Background: BKL dynamics in spatially homogeneous spacetimes

Building blocks of the BKL dynamics are made of

1. the Kasner saddle states (Bianchi type I) [Kasner 1925]
2. the Taub transitions (Bianchi type II) [Taub 1951]

They are the two simplest vacuum, anisotropic, spatially homogeneous solutions of the EFE.

Q: How do these solutions behave?

The EFE for spatially homogeneous spacetimes can be written as a system of first order ODEs.
We will explain the evolution of the solution in the language of dynamical systems (state space, equilibrium points, orbits, attractor).

The evolution of a solution is represented by an orbit in the state space. The majority of the orbits may approach a special subset of the state space asymptotically. We call that subset the attractor.
Self-similar solutions can be represented by equilibrium points if you use the right state space variables.

Orthonormal frame formalism of EFE

[Ehlers 1961, Ellis 1971, MacCallum 1973, van Elst & Uggla 1997]
An appropriate choice for the state space variables is the Hubble-normalized scale-invariant variables of the orthonormal frame formalism.
Timelike congruence $\mathbf{u}$. Decompose $u_{a;b}$ into irreducible parts.

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\Theta(g_{ab} + u_a u_b) - \dot{u}_a u_b$$

Θ rate of expansion scalar.
σ_{ab} rate of shear tensor. Symmetric traceless.
ω_{ab} vorticity tensor. Antisymmetric.
$\dot{u}_a$ acceleration vector.

In cosmological context, Hubble scalar $H = \frac{1}{3}\Theta$ is used.
We usually use $\mathbf{u}$ with zero vorticity.

Construct an orthonormal frame $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_0 = \mathbf{u}$ and 3 spatial frame vector $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Orthonormal frame vector fields (assuming $\mathbf{e}_0$ has zero vorticity)

$$\text{timelike} \qquad \mathbf{e}_0 = \frac{1}{N}\frac{\partial}{\partial t}$$

$$\text{spacelike} \qquad \mathbf{e}_\alpha = e_\alpha{}^i \frac{\partial}{\partial x^i}, \quad \alpha = 1,2,3, \quad i = 1,2,3.$$

N is the lapse function, and $e_\alpha{}^i$ the frame variables.
Orthonormality: $e_\alpha{}^i e_\beta{}^j g_{ij} = \delta_{\alpha\beta}$
$e_\alpha{}^i$ has inverse $e^\alpha{}_i$.

Related to the metric as follows

$$ds^2 = -N^2 dt^2 + \delta_{\alpha\beta} e^\alpha{}_i e^\beta{}_j \, dx^i dx^j.$$

Further reduce the degrees of freedom by aligning the orthonormal frame vectors to the coordinate vectors as much as possible. Later, for Gowdy spacetimes, we align $\mathbf{e}_2$ with ∂_y (no x and z components), and align $\mathbf{e}_3$ with a linear combination of ∂_y and ∂_z (no x component).
The orthonormal frame vector fields are linear combinations of partial differential operators with non-constant coefficients, and they do not commute.

$$[\mathbf{e}_0, \mathbf{e}_\alpha] \neq 0, \quad [\mathbf{e}_\alpha, \mathbf{e}_\beta] \neq 0.$$

Frame commutator coefficients

The orthonormal frame vectors do not commute as operators. The commutator coefficients are

$$[\mathbf{e}_0, \mathbf{e}_\alpha] = \dot{u}_\alpha \mathbf{e}_0 - (H\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta - \epsilon_{\alpha\gamma}{}^\beta \Omega^\gamma)\mathbf{e}_\beta$$
$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = -(2a_{[\alpha}\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta} n^{\delta\gamma})\mathbf{e}_\gamma$$

Ω^α is the angular velocity of the spatial frame $\mathbf{e}_\alpha$, and is determined by the alignment of the frame vectors with the coordinate vectors.

a_α and $n_{\alpha\beta}$ (symmetric) determine the curvature of the spacelike hypersurface $t = \text{const.}$

The commutator coefficients are essentially partial derivatives of the lapse N and the frame variables $e_\alpha{}^i$.

Choosing the right state space variables

Divide these variables by H to give the Hubble-normalized variables.

$$\frac{1}{\mathcal{N}^H} = \frac{1}{H}\frac{1}{N}, \quad (E_\alpha{}^i)^H = \frac{1}{H} e_\alpha{}^i,$$
$$\Sigma^H_{\alpha\beta} = \frac{1}{H}\sigma_{\alpha\beta}, \quad \dot{U}^H_\alpha = \frac{1}{H}\dot{u}_\alpha, \quad R^H_\alpha = \frac{1}{H}\Omega_\alpha,$$
$$A^H_\alpha = \frac{1}{H} a_\alpha, \quad N^H_{\alpha\beta} = \frac{1}{H} n_{\alpha\beta}$$

The Hubble-normalized variables have the nice property of being constant for self-similar solutions. (A self-similar solution is a scaled version of itself at anytime.)
Gauge choice is to choose timelike congruence orthogonal to the spatially homogeneous slices, and $\mathcal{N}^H = 1$. This leads to $\dot{u}_\alpha$=0. Orthonormal frame alignment earlier gives $R^H_1 = -\Sigma^H_{23}$, $R^H_2 = -\Sigma^H_{31}$, $R^H_3 = \Sigma^H_{12}$.

$(E_\alpha{}^i)^H$ decouple from the ODEs. H also decouples.
Reduced state space variables are $\Sigma^H_{\alpha\beta}, A^H_\alpha, N^H_{\alpha\beta}$.
This means that the self-similar solutions are represented by equilibrium points in the reduced state space of Hubble-normalized variables.

Spatially homogeneous solutions

These are models with (at least) 3 spacelike Killing vector fields acting on 3D manifolds. As a result, the state space variables $\Sigma^H_{\alpha\beta}, A^H_\alpha, N^H_{\alpha\beta}$ are functions of time only.

Spatially homogeneous solutions play an important role in describing asympotically local dynamics.

Spatially homogeneous solutions are classified into different Bianchi types

In 1898, Luigi Bianchi classified three-dimensional Lie groups of isometries of a Riemannian manifold.

Bianchi class	Bianchi type	Eigenvalues of $n_{\alpha\beta}$		
A	I	0	0	0
(a_α is zero)	II	0	0	$+$
	VI_0	0	$+$	$-$
	VII_0	0	$+$	$+$
	VIII	$-$	$+$	$+$
	IX	$+$	$+$	$+$
B	V	0	0	0
(a_α is nonzero)	IV	0	0	$+$
	VI_h	0	$+$	$-$
	VII_h	0	$+$	$+$

Bianchi VI_h has $h < 0$ while Bianchi VII_h has $h > 0$. Bianchi III is Bianchi VI_{-1}. Bianchi $\mathrm{VI}_{-1/9}$ has an exceptional class denoted Bianchi $\mathrm{VI}^*_{-1/9}$.

The Kasner solution is a self-similar solution

For the Kasner solution,

$$\Sigma^H_{\alpha\beta} = \text{diag}(-2\Sigma^H_+, \Sigma^H_+ + \sqrt{3}\Sigma^H_-, \Sigma^H_+ - \sqrt{3}\Sigma^H_-).$$

where (Σ^H_+, Σ^H_-) are constant and satisfy ${\Sigma^H_+}^2 + {\Sigma^H_-}^2 = 1$. All other Hubble-normalized variables are zero.

In the Hubble-normalized state space, each Kasner solution is represented by an equilibrium point on the unit circle ${\Sigma^H_+}^2 + {\Sigma^H_-}^2 = 1$.

The Taub solution is asymptotic to Kasner solutions

For the Taub solution,

$$\Sigma^H_{\alpha\beta} = \text{diag}(-2\Sigma^H_+, \Sigma^H_+ + \sqrt{3}\Sigma^H_-, \Sigma^H_+ - \sqrt{3}\Sigma^H_-),$$
$$N^H_{\alpha\beta} = \text{diag}(N^H_{11}, 0, 0).$$

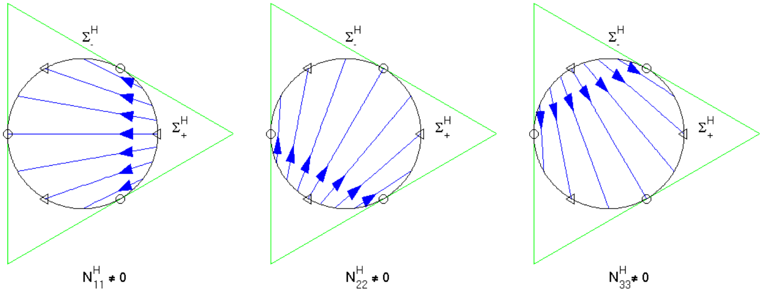
where $(\Sigma^H_+, \Sigma^H_-, N^H_{11})$ satisfy ${\Sigma^H_+}^2 + {\Sigma^H_-}^2 + \frac{1}{12}(N^H_{11})^2 = 1$. All other Hubble-normalized variables are zero. The explicit expression for $(\Sigma^H_+, \Sigma^H_-, N^H_{11})$ is not needed here.

In the Hubble-normalized state space, each Taub solution is represented by an orbit connecting two Kasner equilibrium points (one source and one sink).

There are two other orientations $\text{diag}(0, N^H_{22}, 0)$ and $\text{diag}(0, 0, N^H_{33})$.

Kasner points and Taub orbits

3 orientations of Taub orbits projected onto the (Σ_+^H, Σ_-^H) plane.
Arrows indicate evolution towards singularity.

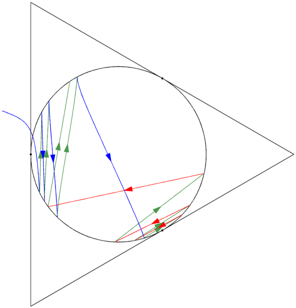


The Kasner points represent Kasner epochs and the Taub orbits represent the transitions between two Kasner epochs.

Vacuum Bianchi VIII and IX – the original BKL archetype

For Bianchi VIII and IX spatially homogeneous solutions, $n_{\alpha\beta}$ has 3 non-zero eigenvalues and their eigenvectors do not rotate.

All Kasner equilibrium points are saddle points.



A Bianchi VIII or IX BKL orbit is built from consecutive Taub orbits connecting the Kasner points.

The BKL regime consists of quick Taub transitions between long Kasner epochs.

A sequence of Kasner epochs with alternatingly active pair of $n_{\alpha\beta}$ eigenvalues defines a Kasner era.

The BKL index u tracks the Kasner epochs within each Kasner era

BKL introduced an index u that characterize a Kasner solution that is independent of spatial frame orientation. It can be defined implicitly through

$$\det \Sigma^H_{\alpha\beta} = 2 - \frac{27(1+u)(1+\frac{1}{u})}{(1+u+\frac{1}{u})^3}.$$

u satisfies $u \geq 1$.

The Kasner map for each Taub transition is $u \to u - 1$ if $u \geq 2$.

A Kasner era ends when $1 \leq u < 2$.

Then the next Kasner era begins with the map $u \to \dfrac{1}{u-1}$.

Vacuum Bianchi VI and VII – a single Kasner era

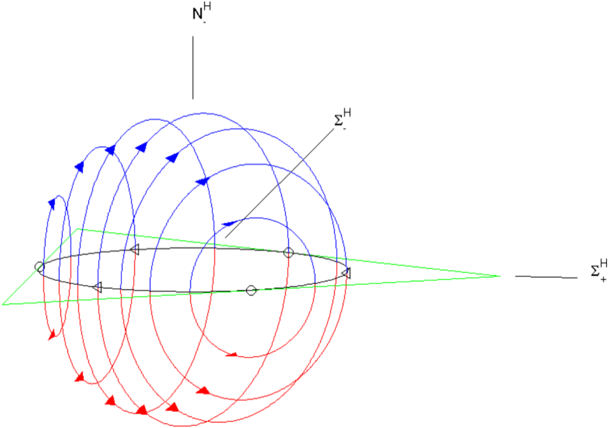
Vacuum Bianchi VI and VII can only sustain the BKL oscillation for a single Kasner era, terminating at a final Kasner epoch.

$n_{\alpha\beta}$ has 2 non-zero eigenvalues and their eigenvectors rotates on one axis every Kasner epoch.

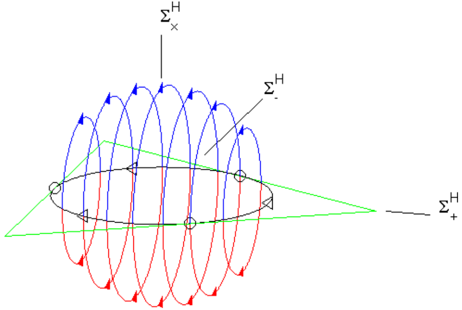
The spatial frame has to rotate with the eigenvectors, so that the circle of Kasner points remain on the (Σ_+^H, Σ_-^H) plane.

This introduces a frame transition, whose orbit is a vertical line when projected onto the (Σ_+^H, Σ_-^H) plane.

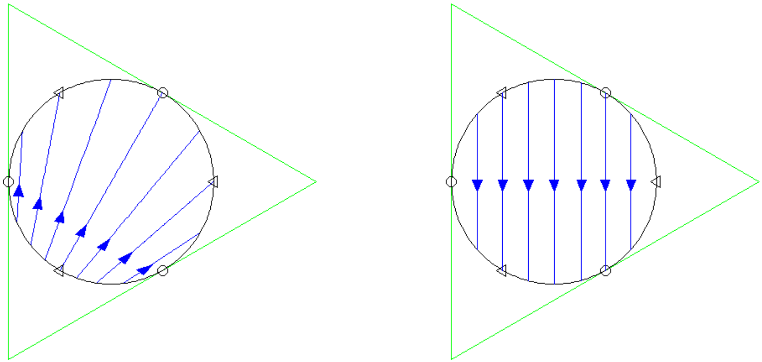
Bianchi VI_0's Taub transition orbits



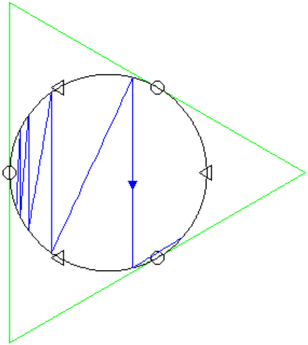
Bianchi VI_0's frame transition orbits



Bianchi VI_0's Taub and frame transition orbits projected



Bianchi VI_0 orbit during a Kasner era



Vacuum Bianchi $\mathrm{VI}^*_{-1/9}$ – the other BKL archetype

[Hewitt, Horwood & Wainwright 2003]
Vacuum Bianchi $\mathrm{VI}^*_{-1/9}$ can sustain the BKL oscillation indefinitely.

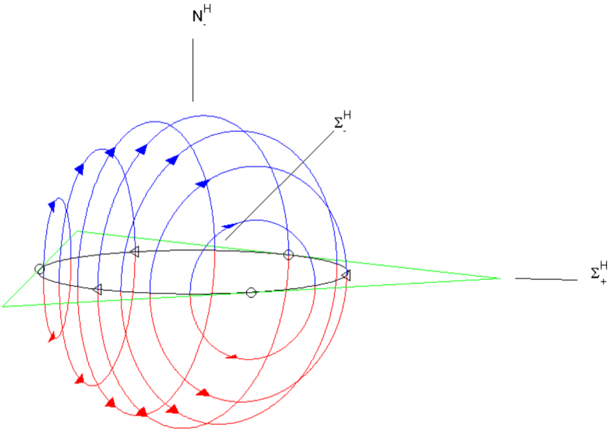
$n_{\alpha\beta}$ has 2 non-zero eigenvalues and their eigenvectors rotates on 2 axes – one rotation occurs once every Kasner epoch, and the other rotation occurs only once between two Kasner eras.

The spatial frame has to rotate with the eigenvectors to maintain the circle of Kasner points on the (Σ_+^H, Σ_-^H) plane.

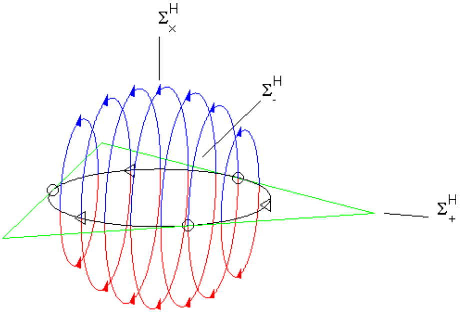
This introduces two frame transitions.

Both BKL archetypes differ in details but the Kasner map is the same: $u \to u - 1$.

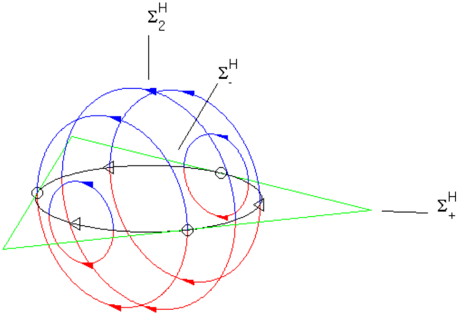
Bianchi $VI^*_{-1/9}$'s Taub transition orbits



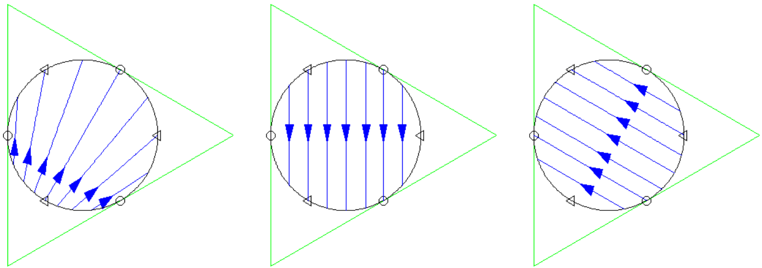
Bianchi $\mathrm{VI}^*_{-1/9}$'s frame transition orbits



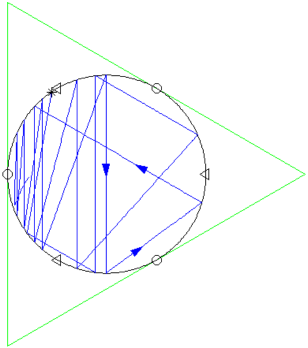
Bianchi $\text{VI}^*_{-1/9}$'s second frame transition orbits



Bianchi $\mathrm{VI}^*_{-1/9}$'s Taub and frame transition orbits projected



Bianchi $\text{VI}^*_{-1/9}$'s BKL orbit



Having understood BKL dynamics in spatially homogeneous models, we now turn to BKL dynamics in spatially inhomogeneous models.

OT G_2 class of inhomogeneous spacetimes

OT G_2 spacetimes

- are vacuum models
- admit 2 Abelian Killing vector fields (spacelike, G_2 group action is orthogonally transitive)
- contain Bianchi VI_0 as spatially homogeneous case.

This class can sustain the BKL dynamics for only a single Kasner era.

Surprise inhomogeneous feature: spikes.

Discovery of spikes in OT G_2 spacetimes

Spikes are small-scale spatial structures that form and then either remain there (permanent spikes) or disappear (transient spikes).
Permanent spikes were discovered incidentally in numerical simulations of OT G_2 models by Berger and Moncrief 1993, whose original goal was to understand the nature of generic singularities.
There are isolated worldlines along which the BKL index u tends to $u > 2$ for their final Kasner epoch, while along other typical worldlines u tends to $u < 2$.
i.e. u tends to its limit pointwise but not uniformly.

Transient spikes in OT G_2 spacetimes

Furthermore, before the final approach, at these same isolated worldlines, recurring transient spikes form, and they smooth out after two Kasner epochs, and then they form again at the next Kasner epoch.

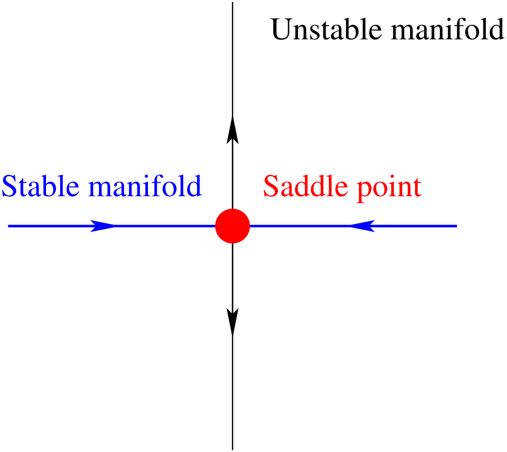
Q: What is so special about these isolated worldlines?

Two possible explanations:

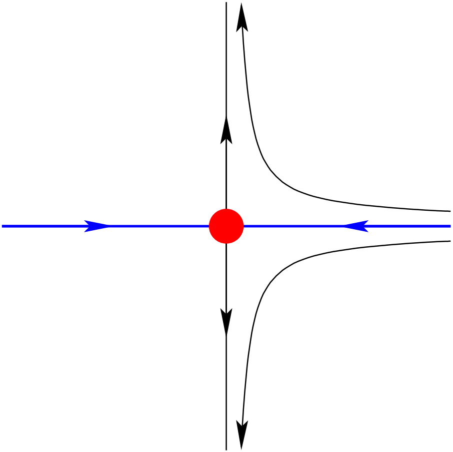
1. The active eigenvalue of $n_{\alpha\beta}$ changes sign here.

2. The trace of $n_{\alpha\beta}$ changes sign here.

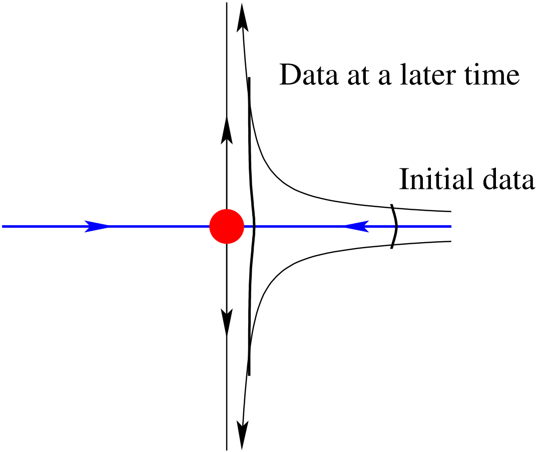
The dynamical reason for spike formation is that the initial data straddle the stable manifold of a saddle point.



As the solution evolves, the data on either side of the stable manifold first approach the saddle point and then leave the neighbourhood of the saddle point, and diverge.



The one datum that lies on the stable manifold also approaches the saddle point, but never leaves.



As the data evolve, the one datum that lies on the stable manifold gets ever close to the saddle point, while its neighbouring data points leave the saddle point.
This means that the data become very inhomogeneous in a short time.
In other words, the $t = const$ profile of the data develops a spike or a step-like structure.

Transient spikes in general (non-OT) G_2 spacetimes

In general (non-OT) G_2 spacetimes, an additional shear component is non-zero. This provides sufficient degree of freedom to the rotation of the $n_{\alpha\beta}$ eigenvector to sustain an infinite sequence of BKL dynamics.

The BKL orbits are similar to those in Bianchi $\text{VI}^*_{-1/9}$.

All spikes are transient here. The permanent OT G_2 spike is a result of unfinished spike transition.

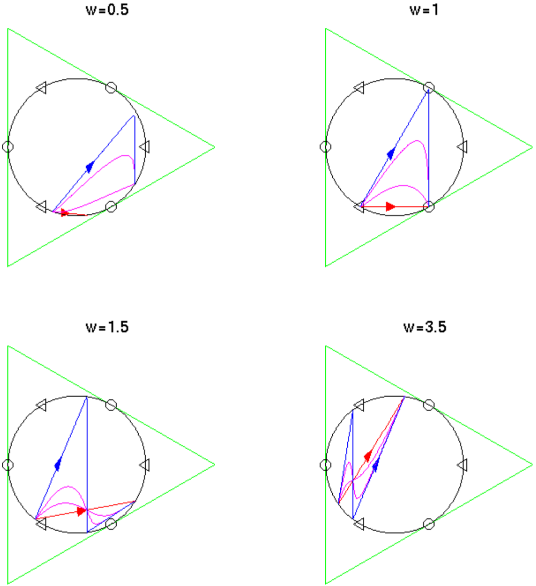
The non-local nature of recurring transient spikes brings the the local nature of the BKL conjecture into doubt.

Exact spike solutions

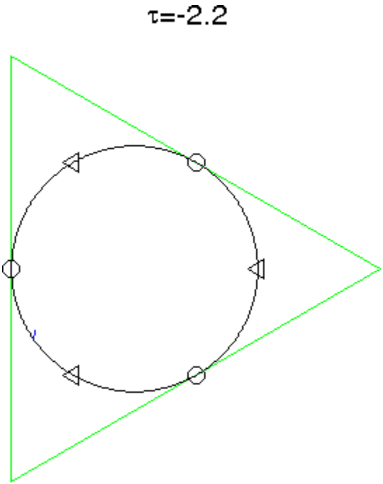
The spike transitions are described by the exact spike solutions, which were discovered in two stages.
The OT G_2 spike solution [Lim 2008] was found by applying the Rendall-Weaver transformation on a Kasner seed solution.
The non-OT G_2 spike solution [Lim 2015] was found by applying the Geroch transformation on a Kasner seed solution.

Numerical confirmation:
OT G_2 spike, zooming technique [Lim, Andersson, Garfinkle, Pretorius 2009]
non-OT G_2 spike [in progress]

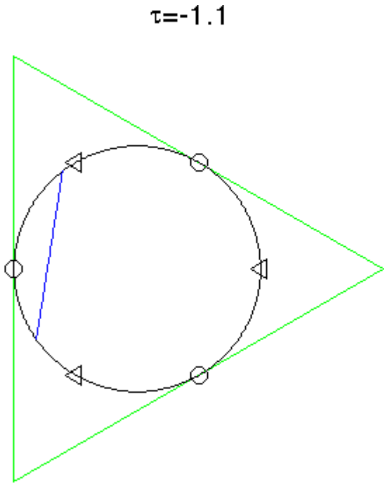
OT G_2 spike orbits



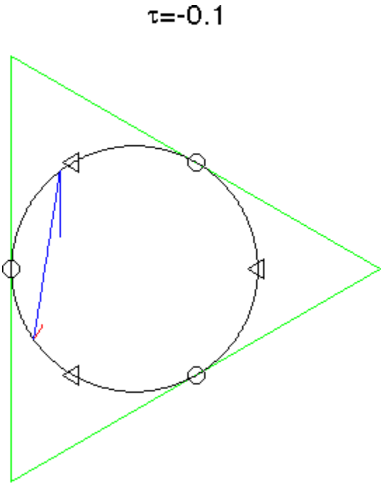
OT G_2 spike orbits



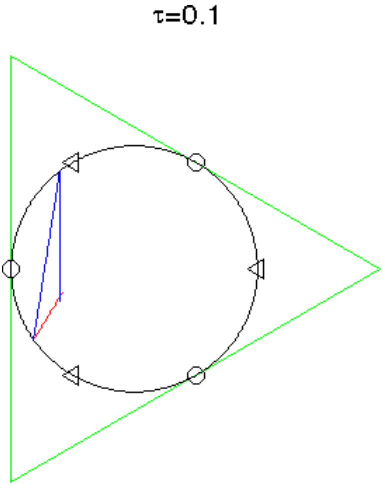
OT G_2 spike orbits



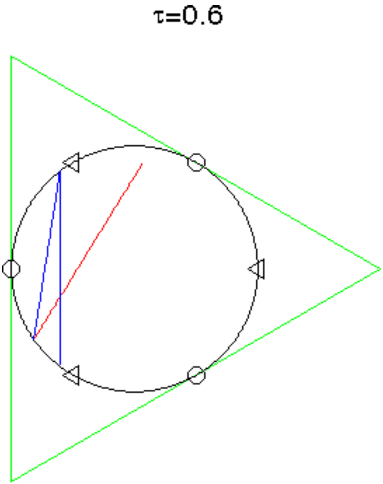
OT G_2 spike orbits



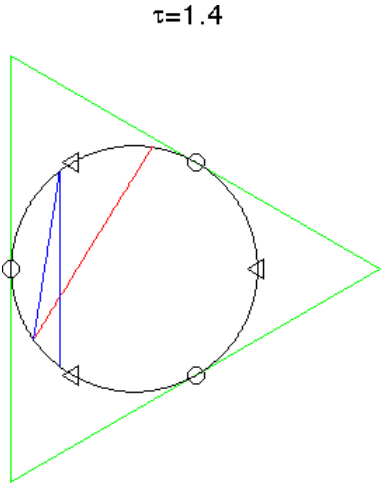
OT G_2 spike orbits



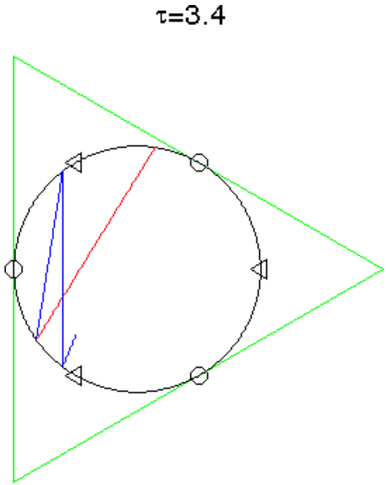
OT G_2 spike orbits



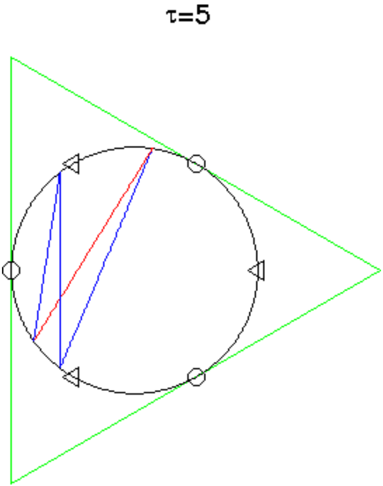
OT G_2 spike orbits



OT G_2 spike orbits

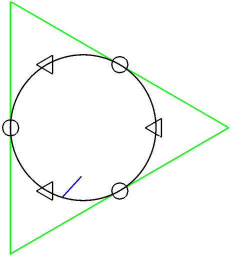


OT G_2 spike orbits



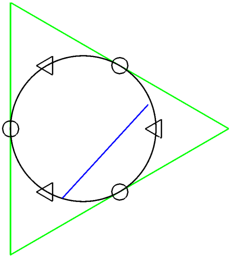
non-OT G_2 spike orbits

$\tau = -3.7$



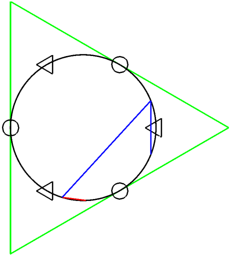
non-OT G_2 spike orbits

$\tau = -3.5$



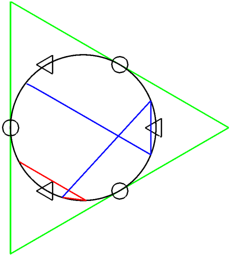
non-OT G_2 spike orbits

$\tau = -1.5$



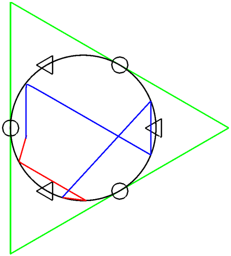
non-OT G_2 spike orbits

$\tau = -0.4$



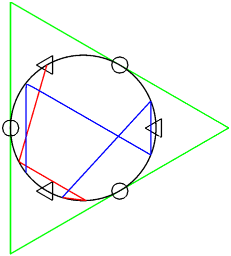
non-OT G_2 spike orbits

$\tau = 0.1$



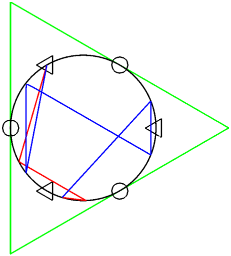
non-OT G_2 spike orbits

$\tau = 5$



non-OT G_2 spike orbits

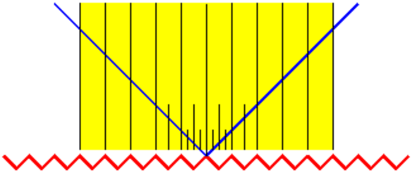
$\tau = 10$

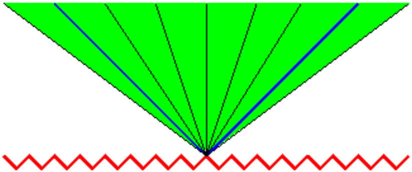


Spike numerics: obstacle and strategy

Goal: Develop a new algorithm to explore spikes numerically.

• Towards the singularity, subsequent spikes become so much narrower that even numerical mesh refinement technique cannot provide reasonable resolution.
• When resolution is inadequate, numerical simulations give erroneous information about spikes.
• Idea from the exact spike solution: zoom in.

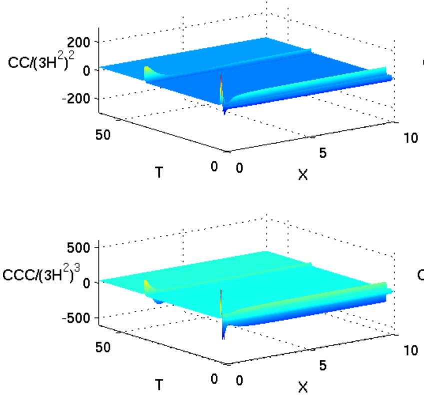


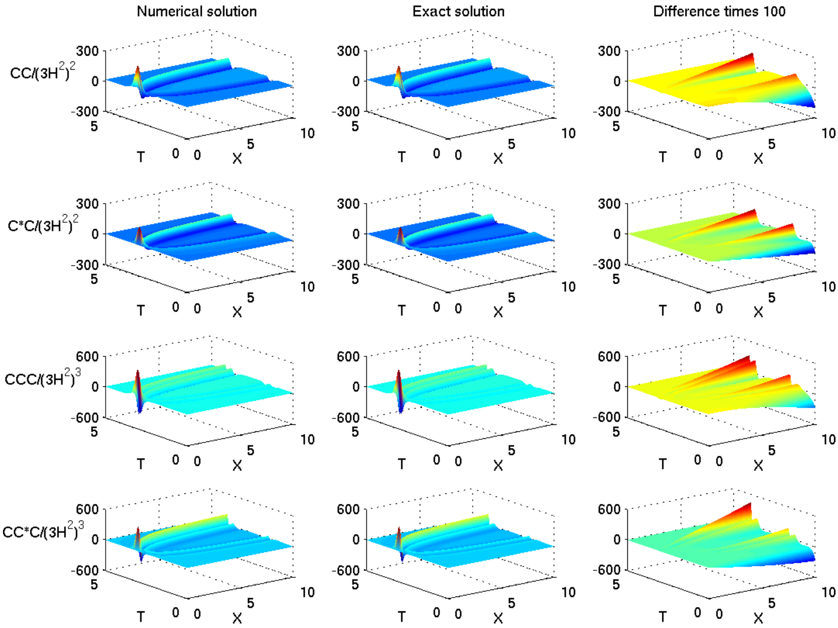


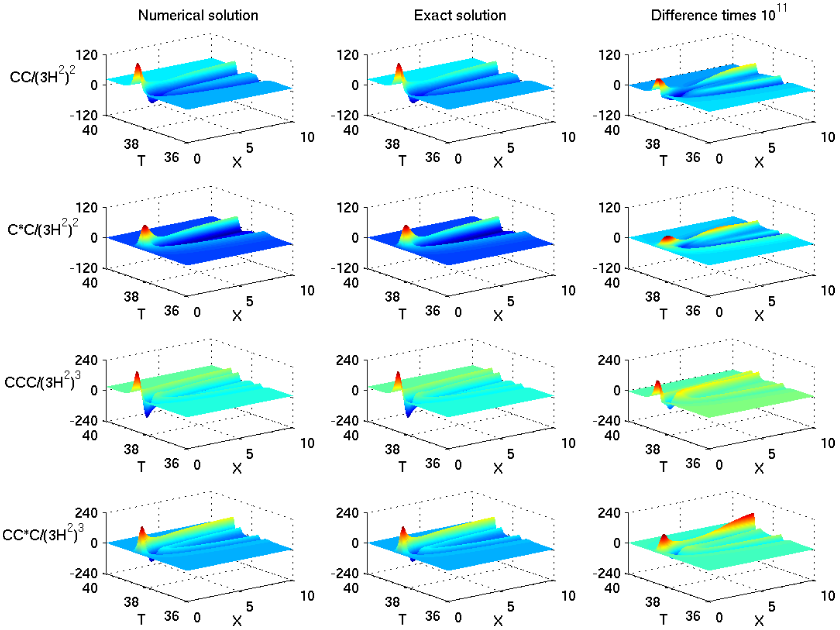
No need for mesh refinement. No need to specify boundary conditions.
Use the classical 4th order Runge-Kutta method for accuracy.

Numerical confirmation: recurring spikes and matching each of them to an exact solution

[Lim, Andersson, Garfinkle & Pretorius 2009]




Numerical solution
Exact solution
Difference times 100
CC/(3H^2)^2
C*C/(3H^2)^2
CCC/(3H^2)^3
CC*C/(3H^2)^3
300
0
-300
600
-600
T
X
0
5
10


Numerical solution
Exact solution
Difference times 10^{11}
$CC/(3H^2)^2$
$C^*C/(3H^2)^2$
$CCC/(3H^2)^3$
$CC^*C/(3H^2)^3$
120
0
-120
240
-240
40
38
36
T
0
5
10
X

Summary

The BKL conjecture:
A generic singularity is
1. Vacuum-dominated ("matter doesn't matter")
2. Local (spatial derivatives are negligible)
3. Oscillatory (an infinite sequence of Kasner states)

- BKL2 modified to account for transient spikes.
- The transitions that connect Kasner states in BKL3 now consist of the Taub solution and the spike solutions.
- New numerical algorithm with zooming technique to simulate spike transitions.

Further developments:
- Spikes with electromagnetic field [Nungesser & Lim 2013]
- Spikes with matter [Coley, Lim, 3 papers 2012–2016]
- G_1 spikes [Coley, Gregoris & Lim, 2 papers 2016–2017]
- Cylindrical spikes [Moughal & Lim 2021]

Quantization?

Towards spacelike singularities, as the spacetime reaches the quantum regime,

most common: Kasner saddle states

common: Taub transitions

rare: transient OT G_2 (same-Kasner-era) spike transitions

rarer: non-OT G_2 (inter-Kasner-era) spike transitions

Try to quantize these solutions.