New solutions to Petrov Type D equation

with U(1)-symmetry

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Plan of the talk

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1 NUT spacetimes

- Plebański-Demiański spacetimes
- Kerr-NUT-adS and its problems
- Orbits of Killing vector fields and the bundle structure

2 Isolated Horizons and Type D equation

- Bundle structure of Isolated Horizons
- Petrov Type D equation

3 New solutions to Type D equations and their embeddings

- Horizons with bundle action transversal to the null direction
- Non-trivial horizons over higher genus surfaces

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NUT spacetimes

(with U(1)-principle bundle structure)

Plebański-Demiański spacetimes

- Most general Petrov Type D solution - Weyl tensor has 2 double principle directions
- Lambda-electro-vaccum solution to EEs in 4D, with EM field aligned with principal null directions
- Generalisation of Schwarzschild, Kerr etc ...
- Parameters:

$$\underbrace{M}_{\text{"mass"}},\ \underbrace{a}_{\text{Kerr}},\ \underbrace{\alpha}_{\text{acceleration}},\ \underbrace{e,\ g}_{\text{e.m. charges}},$$

$$\underbrace{\Lambda}_{\text{cosmological constant}},\ \underbrace{l}_{\text{N(ewmn)-U(unti)-T(amburino)}}$$

,

- No topology restriction. Useful to distinguish 2D surfaces with curvature

$$\epsilon > \text{ or } = \text{ or } < 0$$

- 2 commuting Killing Vector fields $\approx$ time translation, rotation symmetry

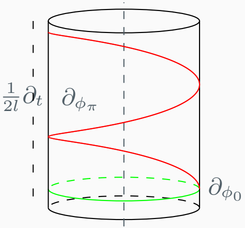
On conical singularity

Axial and cyclic KVF

1. Cyclic - closed orbits
2. Axial - vanishes at axis ($\leftrightarrow$ def of axis) $\implies$ $\frac{\text{circumference}}{\text{radius}}$ test makes sense

Taub-NUT: $g = -f(r)(\mathrm{d}t - 2l(\cos\theta - 1)\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)g^{S^2}$

$\partial_{\phi_0} = \partial_\phi$ for $\theta = 0$ Axial and cyclic.
$\partial_{\phi_\pi} = \partial_\phi + \frac{1}{2l}\partial_t$ for $\theta = \pi$ Axial but not cyclic!



How to calculate conical defect if KVF is cyclic?

1. Stop integrating when above (along $\frac{1}{2l}\partial_t$) starting point
2. Project ∂_{ϕ_π} = to space of orbits of $\frac{1}{2l}\partial_t$

$$\mathrm{d}s^2 = -\frac{1}{F^2}\left[\frac{\mathcal{Q}}{\Sigma}(\mathrm{d}t - A\mathrm{d}\phi)^2 + \frac{\Sigma}{\mathcal{Q}}\mathrm{d}r^2 + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}t - \rho\mathrm{d}\phi)^2\right]$$

$\mathcal{Q} = \mathcal{Q}(r)$ – zeros define Killing Horizon,

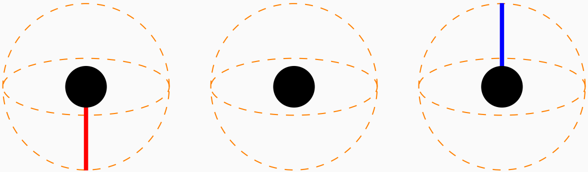
$\mathcal{P} = \mathcal{P}(\theta) > 0, \quad \Sigma = \Sigma(r,\theta) \neq> 0, \quad \rho = \rho(r), \quad F = F(r,\theta) \neq 0$

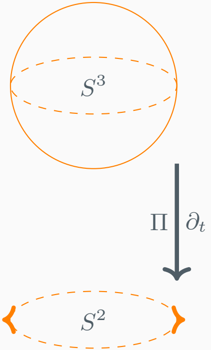
$A = a\sin^2\theta - 2l\left(\cos\theta - 1\right), \quad A(\theta = 0) = 0, \quad A(\theta = \pi) = 4l \neq 0$

$\omega := \mathrm{d}t - A\mathrm{d}\phi$ not continuos at $\theta = \pi \implies$ singular half-axis.

Misner interpretation

$t' := t - 4l\phi \implies \omega := \mathrm{d}t - A'\mathrm{d}\phi, \quad A' = a\sin^2\theta - 2l\left(\cos\theta + 1\right)$
t is cyclic with period $8\pi l$! ∂_t, ∂_{ϕ_π} - cyclic





$\times\mathbb{R}$, metric g

$\times\mathbb{R}$, metric q

$U(1)$-principle bundle

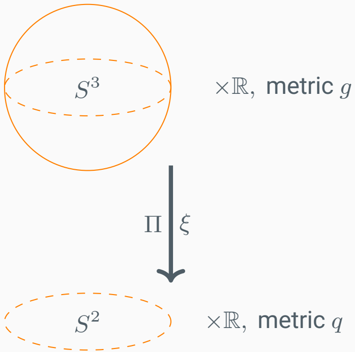
- Base space: $S^2 \times \mathbb{R}$
- Fibres: group $U(1)$
- Total space:

  $S^3 \times \mathbb{R} \overset{\text{locally}}{\approx} S^2 \times U(1) \times \mathbb{R}$
- $U(1)$ preserves fibres and acts on them:
  - Freely: no non-trivial fixed points
  - Transitively: every point is reachable from any other

$$g = -\underbrace{(\Pi^* f)}_{\text{lapse function}}\,\omega \otimes \underbrace{\omega}_{\text{connection}} + \Pi^* \underbrace{q}_{\text{orbit-space metric}}, \qquad (\partial_t)^\mu (\partial_t)_\mu = -(\Pi^* f)$$

Residual conical singularity on the space of the orbits!

Orbits of Killing vector fields done right



Non-singular spacetimes

- Generically: $\xi = \partial_t + b\partial_\phi, \quad b \in \mathbb{R}$
- Exactly 2 (equivalent) choices of ξ with no conical singularity

$$\mathcal{P}(0) = \frac{\mathcal{P}(\pi)}{|(1-4bl)|}$$

- Generates horizon if $\xi = \ell = \partial_t + \frac{a}{\rho(r_H)\partial_\phi}$ & constraint on parameters.
- $\omega := \frac{g(\xi,\cdot)}{g(\xi,\xi)}$

$$g = -\underbrace{(\Pi^* f)}_{\text{lapse function}}\,\omega\otimes\underbrace{\omega}_{\text{connection}} + \Pi^*\underbrace{q}_{\text{orbit-space metric}}, \qquad \xi^\mu\xi_\mu = -(\Pi^* f)$$

Every component is well defined $\implies$ the spacetime is non-singular, including horizon as a 3D manifold.

Generically ($\xi \neq \ell$) space of the null orbits is not smooth - quotient by two different vector fields.

Isolated Horizons and Type D equation

(with U(1)-principle bundle structure)

Intrinsic structure of Isolated Horizons

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Definition: *Non-expanding horizon* (H, g, ∇)

- $H =$ 3D null surface with q of signature $(0++)$ and topology of bundle $\Pi : H \to S$. S compact 2D surface with Riemannian ${}^{(2)}q$ (generalisation of $H = S^2 \times \mathbb{R}$)
  - $q = \Pi^* \, {}^{(2)}q$.
- Non-expanding: $\ell \in \Gamma(TH)$ s.t. $q(\ell, \cdot) = 0 \implies \ell$ is Killing, $\mathcal{L}_\ell q = 0$
  - ℓ symmetry $\implies \ell' = f\ell$ also.
- H-null $\implies \nabla$ is not unique, instead given externally s.t. : ${}^{(2)}\nabla \; {}^{(2)}q = 0$

Definition: *Isolated Horizon* $(H, g, [\ell], \nabla)$

- "Stationary to the second order": $[\mathcal{L}_\ell, \nabla] = 0$
- Now $\ell \mapsto c_0\ell$, $c_0 \in \mathbb{R}$
- Rotation 1-form $\omega^{(\ell)}$: $\nabla\ell =: \omega^{(\ell)} \otimes \ell$, $\mathcal{L}_\ell\omega^{(\ell)} = 0$
  - Surface gravity: $\nabla_\ell \ell = \kappa^{(\ell)}\ell$, assume its constant on H
  - Pseudo-scalar invariant: ${}^{(2)}\mathrm{d}\omega^{(\ell)} = \Omega \; {}^{(2)}\eta$

- For IH: ω - unique, ℓ and κ up to a real const $\implies$ either $\underbrace{\text{extremal}}_{\kappa^{(\ell)}=0}$ or $\underbrace{\text{not}}_{\kappa^{(\ell)}\neq 0}$

Principle bundle structure

- So far: fibre bundle $\Pi : H \to S$
- More: principle bundle $G \hookrightarrow H \xrightarrow{\Pi} S$
  - G acts via the flow of ℓ
  - Connected group: $G = \mathbb{R}$ or $U(1)$. Classification by S and G.
- Bundle connection (choice of horizontal v.f. via ker $\tilde{\omega}$) $A = \tilde{\omega} \otimes \ell^*$ if
  - $\mathcal{L}_\ell \tilde{\omega} = 0$ ✓
  - $\tilde{\omega}(\ell) = 1 \implies \tilde{\omega} = \frac{\omega}{\kappa}$
- Bundle curvature $F = \frac{1}{\kappa} \mathrm{d}\omega \otimes \ell^*$
- $\tilde{\omega}$ and shorthand for principal bundle connection
- Characterisation of $U(1)$-bundles
  - $\int_S K^{(2)}\eta = 2\pi\chi_E(S) = 2 - 2$genus $\in \mathbb{N}$ - Euler characteristic
    K Gausian curvature of $(S, {}^{(2)}q)$
  - $\int_S \Omega^{(2)}\eta = 2\pi\chi_C(S) \in \mathbb{N}$ - Chern number

Petrov Type D Equation

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- For embedded IH: Type D $\Leftrightarrow$ Weyl tensor is Type D at the horizon $C^{\alpha}{}_{\beta\gamma\delta}\big|_{\text{Horizon}}$
- For un-embedded IH: Type D $\Leftrightarrow K + \mathrm{i}\Omega - \frac{\Lambda}{3} \neq 0$
- Null co-frame on S: m_A $(A, B \in 1, 2)$
- Einstein Eqs $\big|_{\text{horizon}} \implies$ Petrov Type D eq:
  $\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K + \mathrm{i}\Omega - \frac{\Lambda}{3}\right)^{-1/3} = 0$
- Known solutions:
  - $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0 [1]
  - $H = S^2 \times \mathbb{R}$ with axial symmetry[2]
  - $H = S^3 \to S^2$, Hopf bundle with axial symmetry [3]
  - $H = S^3 \to S^2$, Hopf bundle with axial symmetry and conical singularities [4]
  - $H \to S$, with structure of $U(1)-$non -trivial bundle over S, smooth Riemann surface with genus > 0 [5]
- Spherical and non-axially symmetric solution - open problem!

[1] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski

[2] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski

[3] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

[4] Phys. Rev. D 108 (2023), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and MO

[5] In preparation, Jerzy Lewandowski, and MO

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New solutions to Type D equations and their embeddings

(mostly with U(1)-principle bundle structure)

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Definition: Conical singularity

Discontinuity of the metric tensor at the pole of rotational symmetry due to wrong choice of the range of angular coordinate, $\phi \in [0; 2\pi\beta[$, $\beta \neq 1$.

May be different at different poles
However, always to regularize one of them!
Idea: Construct a smooth horizon at the cost of $U(1)$-bundle structure.
Take 2 horizons

- $\mathcal{H} = \left(S^2 \backslash \{p_\pi\}\right) \times S^1$ with
  - coordinates (θ, ϕ, v) and
  - metric $q = U^2(\theta)\left[\mathrm{d}\theta^2 + f^2(\theta)\sin^2\theta\left(\frac{1}{f(0)}\mathrm{d}\phi + \left(\frac{1}{f(\pi)} - \frac{1}{f(0)}\right)\mathrm{d}v\right)^2\right]$
- $\mathcal{H}' = \left(S^2 \backslash \{p_0\}\right) \times S^1$ with
  - coordinates (θ', ϕ', v') and
  - metric $q' = U^2(\theta')\left[\mathrm{d}\theta'^2 + f^2(\theta')\sin^2\theta'\left(\frac{1}{f(\pi)}\mathrm{d}\phi' + \left(\frac{1}{f(0)} - \frac{1}{f(\pi)}\right)\mathrm{d}v'\right)^2\right]$

and transition map

$$\theta = \theta', \quad \phi = -\phi', \quad v = v' - \phi', \quad \text{for} \quad \theta, \theta' \neq 0, \pi.$$

Horizons with bundle structure transversal to ℓ

Modelled after S^3 with 3 left and right invariant vector fields.
However only 1 right invariant v.f. - ∂_v - and 1 left - $\partial_\phi + \frac{1}{2}\partial_v$ - are Killing:

- $\ell = f(\pi)\partial_v + \Big(f(\pi) - f(0)\Big)\partial_\phi$ - null direction (generates the horizon)
- ∂_v - generates $U(1)$-action

To be a rotation 1-form ω has to be well defined in both maps.
By inspection: always can find κ such that it is.

Solutions Type D Equation with conical singularity

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- Complex frame adapted to axial symmetry:
  $$m_A dx^A = \frac{R}{\sqrt{2}}\left(\frac{1}{P(x)}dx - iP(x)d\varphi\right).$$
  - $\implies g_{AB}dx^A dx^B = R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\varphi^2\right)$
- Type D Equation reduces to ODE for $K + \mathrm{i}\Omega - \frac{\Lambda}{3}$
- $x = \pm 1 \sim$ poles of the sphere
  - Assume $P^2(\pm 1) = 0$, behaves like $\sin^2\theta$
    Geometric characterization of generic horizon of accelerated KNadS
  - Do not assume continuity $\partial_x P^2(\pm 1) = \mp 2 \implies$ conical singularity
- Classes of solutions
  - Ω=const, embeddable in Taub-NUT-adS, $\partial_x P^2(\pm 1) = \mp 2$ always satisfied
  - $\Omega = \Omega(x)$, $P^2 = R^2\frac{-\frac{\Lambda}{3}x^2 + Cx + D}{x^2 + Ax + B}(x^2 - 1)$, embeddable in accelerated-Kerr-NUT-(anti-) de Sitter
- Option 1: Use $P(x)$ and $\Omega(x)$ construct a non-singular horizon with a Hopf bundle structure with action transversal to null direction or ...
- Option 2: ... simply construct $H = S \times \mathbb{R}$, where S is 2-sphere with conical singularities

Higher genus horizon: trivial case

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- S - smooth, oriented, compact, Riemann surface $\implies$ topologically characterized only by genus
- $K=$const, $\Omega=$const (actually $\Omega=0$)
- Toroidal case: $\psi, \phi \in [0, 2\pi[$
  - ${}^{(2)}g = \frac{1}{P_0^2}(a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\psi \mathrm{d}\phi + (1+b^2)\mathrm{d}\psi^2)\ a, b, P_0^2 \in \mathbb{R},\ a > 0$ - flat metric
  - ${}^{(2)}\omega = A\mathrm{d}\phi + B\mathrm{d}\psi$, $A, B \in \mathbb{R}$ - essentially ${}^{(2)}\omega$ closed
- Genus > 1: $K = \frac{4\pi(1-\text{genus})}{\text{Area}}$, except $K = \frac{\Lambda}{3}$

Higher genus horizons: non-trivial case

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- K =const, Ω =const still hold! (but now $\Omega \neq 0$)
- Prin$_{U(1)}(S) \cong H^2(S;\mathbb{Z})$ ($\sim$ first Chern class)
  - If S is oriented, compact manifold without boundary (✓), than $H^2(S;\mathbb{Z}) \cong \mathbb{Z}$
  - $Prin_{U(1)}(S) \cong \mathbb{Z}$,
- ${}^{(2)}\omega$ defined up to
  - $U(1)$ gauge
  - 1-forms $\alpha_1, \ldots, \alpha_{2g}$ generating the first de Rham cohomology group $H^1(S,\mathbb{R}) \cong \mathbb{R}^{2\text{genus}}$

Toroidal horizons

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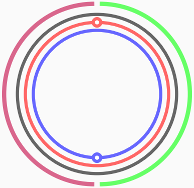

Again ${}^{(2)}g = \frac{1}{P_0^2}(a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\psi \mathrm{d}\phi + (1+b^2)\mathrm{d}\psi^2)$
- flat metric Looking
for two 1-forms on $\mathbb{T}^2$ satisfying the curvature condition

$$\kappa\, {}^{(2)}\mathrm{d}A_{(1/2)} = {}^{(2)}\mathrm{d}\omega_{(1/2)} = \Omega\, {}^{(2)}\eta, \quad 1 = \text{red} \times S^1 \quad 2 = \text{blue} \times S^1$$

and $U(1)$- gauge transformation

$$a_{12}(\psi,\phi) := \begin{cases} 1 & \text{at purple} \times S^1 \\ \exp(im\phi) & \text{at green} \times S^1 \end{cases}$$

$${}^{(2)}A_{(2)} = {}^{(2)}A_{(1)} + \mathrm{i}a_{12}^{-1}\mathrm{d}a_{12},$$

For example

$${}^{(2)}A_{(1)} = \frac{\Omega a}{\kappa P_0^2}\psi \mathrm{d}\phi + A\mathrm{d}\phi + B\mathrm{d}\psi \text{ at red} \times S^1,$$

$${}^{(2)}A_{(2)} = \begin{cases} \frac{\Omega a}{\kappa P_0^2}\psi \mathrm{d}\phi + A\mathrm{d}\phi + B\mathrm{d}\psi, & \text{at purple} \cup \{0\} \times S^1, \\ \frac{\Omega a}{\kappa P_0^2}\big(\psi - 2\pi\big)\mathrm{d}\phi + A\mathrm{d}\phi + B\mathrm{d}\psi & \text{at green} \cup \{2\pi\} \times S^1. \end{cases}$$

Higher genus horizons

- No practical (easy?) coordinate system
- As in trivial case $K = \frac{4\pi(1-\text{genus})}{\text{Area}}$
  - Now $\Omega \neq 0$ so $K = \frac{\Lambda}{3}$ ok
- $\Omega = \frac{2\pi\chi_C\kappa}{\text{Area}}$ - gauge and cohomology invariant

Generalised Taub-NUT-(anti-) de Sitter (M, l, Λ)

$$g = -f(r)\left(\mathrm{d}t + l\frac{i(\zeta \mathrm{d}\bar{\zeta} - \bar{\zeta}\mathrm{d}\zeta)}{1 + \frac{1}{2}\epsilon\zeta\bar{\zeta}}\right)^2 + f(r)^{-1}\mathrm{d}r^2 + (r^2 + l^2)\frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{(1 + \frac{1}{2}\epsilon\zeta\bar{\zeta})^2},$$

$$f(r) = \frac{\epsilon(r^2 - l^2) - 2Mr - \Lambda(\frac{1}{3}r^4 + 2l^2r^2 - l^4)}{r^2 + l^2}, \quad \epsilon = 0, \pm 1$$

Interpreted a'la Misner: t cyclic.
Limits:

- Spherical: $\epsilon = 1$, $\zeta = \sqrt{2}\tan\frac{1}{2}\theta\exp(\mathrm{i}\phi)$
  - $l = 0$ Schwarzschild-(anti-) de Sitter
- Planar: $\epsilon = 0$, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- Hyperbolic: $\epsilon = -1$, S with genus > 1, $S = \mathbb{H}/\Gamma$, $\Gamma \subset$ PSL$(2, \mathbb{R})$

Embedding spacetime

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Take:

- S compact Riemann surface with Riemannian g^ϵ and curvature $\epsilon = 0, \pm 1$
- $U(1)$-bundle $\pi : P \to S$ with
  - Connection A satisfying $\mathrm{d}A = \tilde{\Omega}\pi^*\eta^\epsilon$, $\tilde{\Omega} \in \mathbb{R}$
  - $U(1)$-action generated by ℓ null w.r.t. $\tilde{\pi}^* g^\epsilon$, $A(\ell) = 1$
- $g = -h(r)A^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)g^\epsilon$, $N \in \mathbb{R}$

Solving E.E. $\implies$ $h(r) = N^2 f(r)$, $N \in \mathbb{R}$, $f(r)$ as in generalised Taub-NUT-(a)dS. Finally Eddington-Finkelstein-like co-frame transformation

$$NA' := NA + \frac{\mathrm{d}r}{f(r)}.$$

Well defined on horizon

$$g = -f(r)N^2 A'^2 + 2NA'\mathrm{d}r + (r^2 + l^2)g^\epsilon.$$

Embedding

To embed IH with given Ω and K choose l and κ so that ($f(r_H) = 0$)

$$\Omega = \frac{\kappa}{r_H^2 + l^2} \frac{2\pi\chi_C}{\text{Area}(g^S)} = \frac{\epsilon - (r_H^2 + l^2)\Lambda}{r_H(r_H^2 + l^2)}$$

Bundle structure extends to entire spacetimes

$$U(1) \hookrightarrow P \times \mathbb{R} \xrightarrow{\Pi} S \times \mathbb{R}.$$

with connection

$$A_{\text{ST}} := \frac{g(\ell, \cdot)}{g(\ell, \ell)} = A$$

(A defined up to $H^1(S, \mathbb{R})$ $\implies$ Gravitational Aharonov-Bohm)

The End

Thank you for your attention!