Primordial Fields in Quantum Cosmological Spacetimes

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Outline

Discussion of the general framework

Classical model (fluid-driven universe with gravity-waves)

Quantization

Semiclassical universe

WKB universe

Conclusions

Hamiltonian theory of cosmological perturbations

We work with the ADM formalism. We split the geometric and other variables:

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij},\ \delta\pi^{ij} = \pi^{ij} - \bar{\pi}^{ij},\ N \mapsto N + \delta N,\ N^i \mapsto N^i + \delta N^i.$$

We expand the Hamiltonian constraint:

$$\mathbf{H} = N\mathbb{H}_0^{(0)} + \int_\Sigma \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathbb{H}_0 + \delta N^i \delta\mathbb{H}_i\right) \mathrm{d}^3x.$$

We reduce before quantizing:

$$\delta\mathbb{H}_0 = 0 = \delta\mathbb{H}_i \quad \text{and} \quad \delta\mathcal{C}_0 = 0 = \delta\mathcal{C}_i \quad \text{s.t.} \quad Det|\{\delta\mathbb{H}_\mu, \delta\mathcal{C}_\mu\}| \neq 0.$$

In the end we obtain the physical Hamiltonian in terms of δD,

$$\mathbf{H}_{phys} = N\mathbb{H}_0^{(0)} + \int_{\mathbb{T}^3} N\mathcal{H}_{phys}^{(2)}(\delta D)\ \mathrm{d}^3x.$$

(1) When forming Hamilton's equations we neglect the backreaction. (2) δD satisfy first-order constraints. (3) **No backreaction**.

Dynamical law

Issue: How to impose the no-backreaction condition at the quantum level. Forming the Schrödinger equation with

$$\mathbf{H}_{phys} \mapsto \widehat{\mathbf{H}}_{phys} = \widehat{\mathbf{H}}^{(0)} + \widehat{\mathbf{H}}^{(2)}$$

would break that condition.

Let us assume:

$$|\psi\rangle = |\psi_B\rangle \cdot |\psi_P\rangle \in \mathcal{H}_{hom} \otimes \mathcal{H}_{inhom}.$$

We introduce the quantum action

$$S_Q(\psi) := \int \langle\psi| \; i\hbar\frac{\partial}{\partial t} - \widehat{\mathbf{H}}_{phys} \; |\psi\rangle \, \mathrm{d}t.$$

We fix $|\psi_B\rangle$,

$$i\hbar\frac{\partial}{\partial t}|\psi_B\rangle = \widehat{\mathbf{H}}^{(0)}|\psi_B\rangle,$$

and vary $S_Q(\psi)$ wrt $\delta|\psi_P\rangle$,

$$i\hbar\frac{\partial}{\partial t}|\psi_P\rangle = \langle\psi_B|\widehat{\mathbf{H}}^{(2)}|\psi_B\rangle \cdot |\psi_P\rangle.$$

Dynamical law

Given a state,

$$|\psi\rangle = |\psi_B^{(1)}\rangle \cdot |\psi_P^{(1)}\rangle + |\psi_B^{(2)}\rangle \cdot |\psi_P^{(2)}\rangle + \cdots \subset \mathcal{H}_{hom} \otimes \mathcal{H}_{inhom},$$

the variational method yields

$$i\hbar\partial_t \begin{bmatrix} |\psi_P^{(1)}\rangle \\ \vdots \\ |\psi_P^{(n)}\rangle \end{bmatrix} = \begin{bmatrix} \langle\psi_B^{(1)}|\widehat{\mathbf{H}}^{(2)}|\psi_B^{(1)}\rangle & \cdots & \langle\psi_B^{(1)}|\widehat{\mathbf{H}}^{(2)}|\psi_B^{(n)}\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi_B^{(n)}|\widehat{\mathbf{H}}^{(2)}|\psi_B^{(1)}\rangle & \cdots & \langle\psi_B^{(n)}|\widehat{\mathbf{H}}^{(2)}|\psi_B^{(n)}\rangle \end{bmatrix} \begin{bmatrix} |\psi_P^{(1)}\rangle \\ \vdots \\ |\psi_P^{(n)}\rangle \end{bmatrix},$$

where

$$i\hbar\frac{\partial}{\partial t}|\psi_B^{(n)}\rangle = \widehat{\mathbf{H}}^{(0)}|\psi_B^{(n)}\rangle.$$

The states need to satisfy $\langle\psi_B^{(n)}|\psi_B^{(m)}\rangle = \delta_{nm}$.

Model

The FLRW universe filled with perfect fluid and gravitational waves:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a^2(\delta_{ab} + h_{ab})\mathrm{d}x^a\mathrm{d}x^b, \quad \mathcal{M} = \mathbb{T}^3 \times \mathbb{R}.$$

Given $\check{h}_{ab} \to \check{h}_{\pm}$ decomposed into two polarization modes, the Hamiltonian in $(q, p, \check{h}_{\pm}, \check{\pi}_{\pm})$ reads:

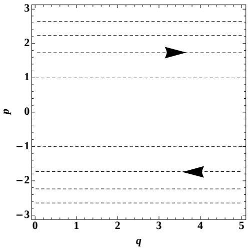
$$\mathbf{H}_{phys} = \mathbf{H}^{(0)} - \sum_{\vec{k}} \mathbf{H}^{(2)}_{\vec{k}}, \quad \mathbf{H}^{(0)} = \frac{1}{2}p^2,$$

$$\mathbf{H}^{(2)}_{\vec{k}} = \frac{1}{2}q^{-2}|\check{\pi}_{\pm}(\vec{k})|^2 + \frac{k^2}{2}q^{\frac{6w+2}{3-3w}}|\check{h}_{\pm}(\vec{k})|^2,$$

where $w = \frac{\mathrm{pressure}}{\mathrm{energy\ density}}$, k comoving wave number. The scale factor $a = q^{\frac{2}{3-3w}}$ and $p \propto$ the Hubble rate H. The dynamics occurs in the fluid time,

$$\omega\Big|_{C=0} = \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}T\mathrm{d}p_T\Big|_{p_T=-\mathbf{H}^{(0)}}.$$

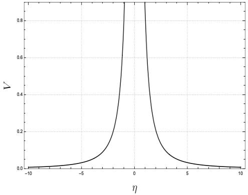
Classical motion



The Hamilton equations yield the background dynamics:

$$p = \sqrt{4\mathbf{H}^{(0)}}, \quad q = p(T - T_s),$$

and the gravity-wave propagation equation in **internal** conformal time, $\eta = \int q^{\frac{6w-2}{3-3w}} \mathrm{d}T$:



$$\boxed{\mu''_{\pm,\vec{k}} + \left(k^2 - \frac{(q^{\frac{2}{3-3w}})''}{q^{\frac{2}{3-3w}}}\right)\mu_{\pm,\vec{k}} = 0,}$$

where $\mu_{\pm,k} = q^{\frac{2}{3-3w}} h_{\pm,k}$ and $a = q^{\frac{2}{3-3w}}$.

Q: How does quantization modify the above dynamics?

Quantization

The phase space:

$$(q \in \mathbb{R}_+, p, \check{h}_{\pm,\vec{k}}, \check{\pi}_{\pm,\vec{k}}).$$

- The canonical prescription $q \mapsto \widehat{Q}$ and $p \mapsto \widehat{P}$ does not work as $\widehat{P}$ is nonself-adjoint. **The dilation operator**, $\widehat{D} = \frac{1}{2}(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q})$, is self-adjoint and $\frac{1}{i\hbar}[\widehat{Q}, \widehat{D}] = \widehat{Q}$. $U(q,p) := e^{ip\widehat{Q}} e^{-i\ln(q)\widehat{D}}$ is UIR of the **affine group** of real line $U(q',p')U(q,p) = U(qq', \frac{p}{q'} + q')$.
- The quantum Hamiltonian, $p^2 \mapsto (\frac{\widehat{D}}{\widehat{Q}})^2$ is self-adjoint for a wide class of symmetric orderings. In generally,

$$p^2 \mapsto \widehat{P}^2 + \hbar^2 \frac{K}{\widehat{Q}^2}, \quad K > \frac{3}{4},$$

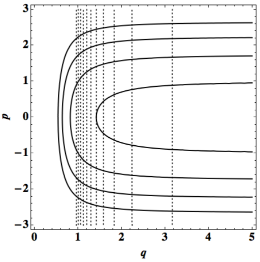
  where the repulsive potential $\propto \widehat{Q}^{-2}$ **removes the singularity**.
- Closed algebra:

$$[\widehat{Q}^2, \widehat{\mathbf{H}}^{(0)}] = 4i\widehat{D}, \;\; [\widehat{D}, \widehat{\mathbf{H}}^{(0)}] = 2i\widehat{\mathbf{H}}^{(0)}, \;\; [\widehat{Q}^2, \widehat{D}] = 2i\widehat{Q}^2.$$

Quantization

The **nonsingular** semiclassical dynamics,

$$q = q_b\sqrt{(k_{max}T)^2+1}, \;\; p = \frac{2(q_b k_{max})(k_{max}T)}{\sqrt{(k_{max}T)^2+1}}.$$



Free parameters: r, w, K.

- Observable volume: $V_b \propto \frac{K^{\frac{1}{1-w}}}{r^{\frac{2}{1-w}}}$. Energy density: $\rho_b \propto \frac{r^{\frac{2(1+w)}{1-w}}}{K^{\frac{1+w}{1-w}}}$.
- The quantum perturbation Hamiltonian reads:

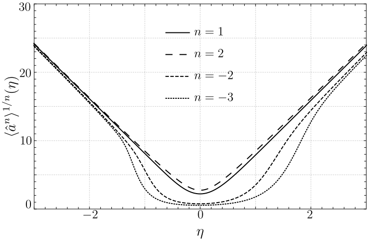
$$\widehat{\mathbf{H}}^{(2)}_{\vec{k}} = \frac{1}{2}\widehat{Q}^{-2}|\widehat{\pi}_{\pm}(\vec{k})|^2 + \frac{k^2}{2}\widehat{Q}^{\frac{6w+2}{3-3w}}|\widehat{h}_{\pm}(\vec{k})|^2.$$

Quantization

We will assume: $|\psi\rangle = |\psi_B\rangle \cdot |\psi_P\rangle$,

$$i\hbar \frac{\partial}{\partial T} |\psi_P\rangle = \langle \psi_B | \widehat{\mathbf{H}}^{(2)} | \psi_B \rangle \cdot |\psi_P\rangle,$$

where $\langle \psi_B | \widehat{\mathbf{H}}^{(2)}_{\vec{k}} | \psi_B \rangle = \frac{1}{2} \langle \widehat{Q}^{-2} \rangle |\widehat{\pi}_\pm(\vec{k})|^2 + \frac{k^2}{2} \langle \widehat{Q}^{\frac{6w+2}{3-3w}} \rangle |\widehat{h}_\pm(\vec{k})|^2.$



No SEMICLASSICAL spacetime!

Quantum dynamics

In analogy with the classical case, we introduce

$$\widehat{\mu}_{\pm,k} = \langle \widehat{\mathbf{Q}}^{-2} \rangle^{\frac{1}{3w-3}} \widehat{h}_{\pm,k}, \quad \eta = \int \langle \widehat{\mathbf{Q}}^{-2} \rangle^{\frac{3w-1}{3w-3}} \mathrm{d}T.$$

The choice of the moments is arbitrary and physically irrelevant.
The Heisenberg form of e.o.m.:

$$\boxed{\widehat{\mu}''_{\pm,\vec{k}} + \left(k^2 c_g^2 - \frac{\left(\langle \widehat{Q}^{-2} \rangle^{\frac{1}{3w-3}} \right)''}{\langle \widehat{Q}^{-2} \rangle^{\frac{1}{3w-3}}} \right) \widehat{\mu}_{\pm,\vec{k}} = 0,}$$

where $c_g^2 = \langle \widehat{Q}^{\frac{6w+2}{3-3w}} \rangle \, \langle \widehat{Q}^{-2} \rangle^{\frac{3w+1}{3-3w}}$. The essential moment is $\langle \widehat{Q}^{-2} \rangle \equiv \langle \hat{a}^{3w-3} \rangle$. The physical scale factor is $a|_{sem} = \langle \hat{a}^{3w-3} \rangle^{\frac{1}{3w-3}}$.

Initial state

We introduce the mode functions:

$$\widehat{\mu}_{\vec{k}}(t) = \frac{1}{\sqrt{2}}\left(\widehat{a}_{\vec{k}}\mu_k^*(t) + \widehat{a}^{\dagger}_{-\vec{k}}\mu_k(t)\right), \quad [\widehat{a}_{\vec{l}}, \widehat{a}^{\dagger}_{\vec{k}}] = \delta_{\vec{l},\vec{k}}, \quad W(\mu_k, \mu_k^*) = 2i\hbar.$$

The Bunch-Davies vacuum is set as the initial condition:

$$\mu_k = \sqrt{\frac{\hbar}{c_g^{\infty} k}}, \quad \acute{\mu}_k = i\sqrt{\hbar c_g^{\infty} k},$$

$(\lim_{t\to\pm\infty} c_g = c_g^{\infty})$. The mode functions h_k are obtained from the formal solution:

$$h_k = A_1(k) + A_2(k)\int^{\eta} \frac{d\eta'}{\langle \hat{Q}^{-2}\rangle^{\frac{2}{3w-3}}} - k^2 \int^{\eta} \mathrm{d}\eta' c_g^2 \langle \widehat{Q}^{-2}\rangle^{\frac{2}{3w-3}} h_k \int^{\eta'} \frac{\mathrm{d}\eta''}{\langle \widehat{Q}^{-2}\rangle^{\frac{2}{3w-3}}}$$

or numerically.

$\Longrightarrow$ First I will study the semiclassical bounce. Then I will study the WKB bounce. I will use the WKB approximation to get an analytical result depending on σ.

Semiclassical dynamics

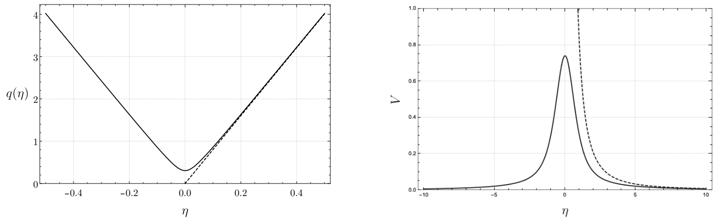
In the semiclassical dynamics,

$$\rho(x, T) = \delta(x - q(T)), \qquad q(T) = \langle \widehat{Q} \rangle (T),$$

$\langle \widehat{Q}^n \rangle = \langle \widehat{Q} \rangle^n$ for any n. It follows that $c_g^2 = 1$ and the e.o.m. reads

$$\boxed{\widehat{\mu}''_{\pm,\vec{k}} + \left(k^2 - \frac{\left(q^{\frac{2}{3-3w}}\right)''}{q^{\frac{2}{3-3w}}} \right) \widehat{\mu}_{\pm,\vec{k}} = 0,}$$

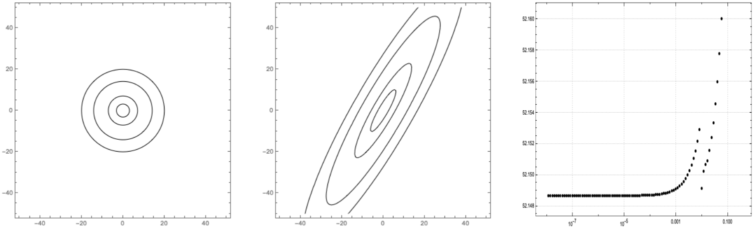
where the scale factor is replaced with $a|_{sem} = q^{\frac{2}{3-3w}}$ depending on a single parameter, K.



Final state

The final state is a squeezed vacuum for two modes of a standing wave. The perturbations emerge coherently. In the standard coherent state representation, the probability distribution reads:

$$\rho(z) = \pi^{-1}|\langle z|0_{BD}\rangle|^2 = \frac{e^{-|z|^2} e^{Re[\bar{z}^2 \frac{\beta_{|k|}}{\alpha_{|k|}}]}}{\pi|\alpha_{|k|}|^2}$$



The observable we study is the primordial amplitude spectrum δ_h.

$$\delta_h(k) = \frac{|h_k|}{2\pi} k^{\frac{3}{2}} \ .$$

Semiclassical dynamics

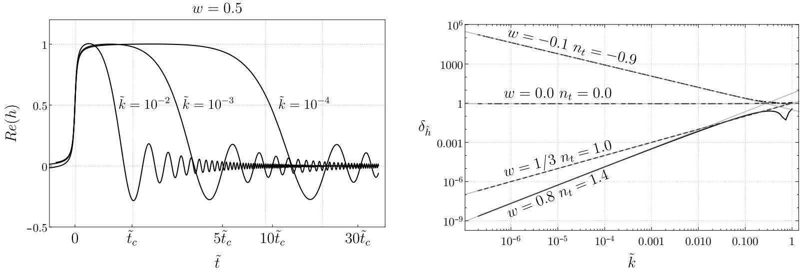


Figure: The evolution of the amplitude of a few selected modes (LEFT) and the primordial amplitude spectrum δ_h (RIGHT) in the semiclassical universe.

One finds the primordial amplitude spectrum:

$$\delta_{h,Semi} \propto \frac{r}{\sqrt{K}} \left(\frac{k}{k_*} \right)^{\frac{6w}{3w+1}}$$

Semiclassical dynamics

The primordial amplitude at the pivot scale is constrained:

$$\delta_{h,Semi} \propto \frac{r}{\sqrt{K}} \lesssim 10^{-5}.$$

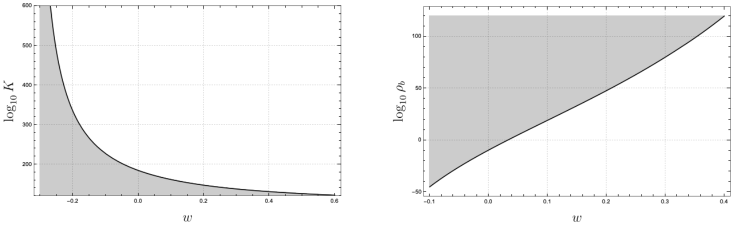
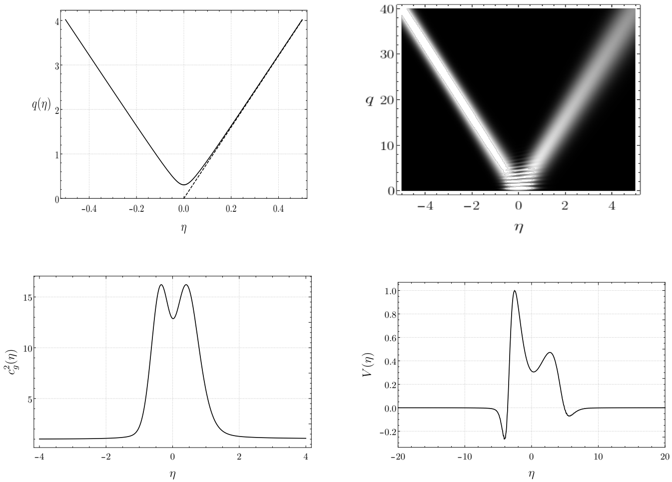


Figure: The white regions represent the admissible values of the parameter K and the energy density ρ_b at the bounce as functions of w given the upper bound on the primordial GW amplitude from CMB data ($r = 2$).

Quantum spread

Semiclassical vs. quantum description of the background geometry:



Quantum states produce nontrivial moments $\langle \widehat{Q}^n \rangle$ to which the gravity-wave speed c_g^2 and the gravitational potential V are sensitive.

WKB dynamics

We assume the solution to the background geometry

$$\langle x|\psi_B\rangle(t) = A(x,t)\exp\left[iS(x,t)/\hbar\right], \quad A, S \in \mathbb{R}.$$

The Schrödinger eq. expanded in $\hbar$ yields at lowest order,

$$\partial_t S = -\frac{1}{2}\left(S_{,x}^2 + \frac{\hbar^2 K}{x^2}\right), \quad \partial_t A^2 = -\partial_x(A^2 S_{,x}),$$

where S is Hamilton's principal function

$$S(t,x) = \frac{1}{2}\int^{t,x}\left(\frac{1}{4}x_{,t'}^2 - \frac{K}{x^2}\right)\mathrm{d}t',$$

where the integral is taken over the **semiclassical trajectories** with fixed initial conditions, and A^2 behaves like the density of particles following the semiclassical trajectories.

WKB dynamics

Let us assume the density distribution at the bounce to read

$$\rho(x) = \frac{x}{2q_b^2\sigma}\chi_{[q_b(1-\sigma),q_b(1+\sigma)]}(x),$$

where $\chi_{[q_b(1-\sigma),q_b(1+\sigma)]}(x)$ is the characteristic function, q_b is a fixed bouncing point and $0 < \sigma < 1$ is a free dimensionless parameter. For $\sigma \to 0$,

$$\begin{aligned}
\rho(x) &\to \delta(x - q_b), \\
\langle \widehat{Q}^2 \rangle = q_b^2\left(1 + \sigma^2 + \frac{\ln\left|\frac{1+\sigma}{1-\sigma}\right|}{2\sigma}(k_{max}t)^2\right) &\to q_b^2\left(1 + (k_{max}t)^2\right) = q^2, \\
\langle \widehat{Q}^{-2} \rangle = \frac{1}{8q_b^2\sigma}\ln\left|\frac{(1+\sigma)^4 + (k_{max}t)^2}{(1-\sigma)^4 + (k_{max}t)^2}\right| &\to \frac{1}{q_b^2(1 + (k_{max}t)^2)} = q^{-2}, \\
(\Delta\widehat{Q})^2\Big|_{t=0} = q_b^2\sigma^2 &\to 0.
\end{aligned}$$

The parameter σ has the interpretation of the relative 'volume' dispersion, $\sigma^2 = \left.\frac{(\Delta\widehat{Q})^2}{\langle\widehat{Q}\rangle^2}\right|_{t=0}$.

WKB dynamics

E.o.m.:

$$\boxed{\mu''_{\pm,k} + \left(k^2 c_{g,\sigma}^2 - V_\sigma\right)\mu_{\pm,k} = 0}$$

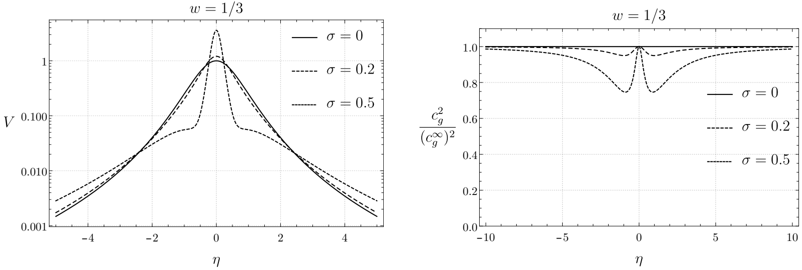


Figure: The gravitational potential V in the semiclassical and WKB universes (LEFT). The gravity-wave speed in the semiclassical and WKB universes (RIGHT).

Analytical result:

$$\delta_{h,WKB} = (1+\sigma^2)^{-\frac{1}{3w+1}}\,\delta_{h,Semi}$$

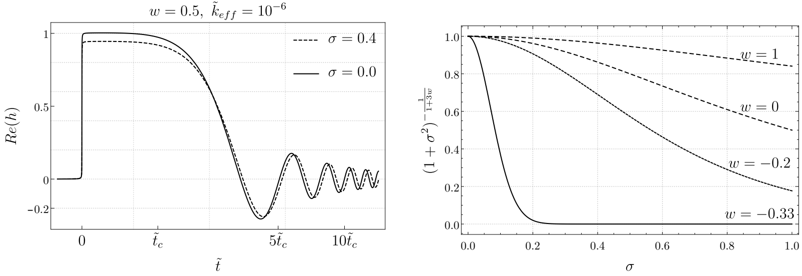


Figure: The evolution of a selected mode in the semiclassical and the WKB universe (LEFT). The suppression of primordial gravitational wave amplitude δ_h, as function of the dispersion σ (RIGHT).

Amplitude suppression with spectral index unaltered.

Discussion

- The full dynamical equation for the evolution of linear perturbations on a quantum background leads to the multiverse scenario.
- The quantum spread of the background geometry influences the propagation of linear perturbations in the universe. A rough estimate is given by the WKB calculation that yields an analytical result as function of the value of spread.
- The result: the spectral index is insensitive to the spread whereas the overall amplitude may be significantly suppressed e.g. for negative pressure fluids. The suppression could alleviate the large K problem.
- Further questions: Other WKB states? What is beyond the WKB approximation? Would the interference produce more effects? How would the matter perturbations be affected by background uncertainties?
- I showed you a relatively small effect. In my opinion, it suggests the possibility that significant physical effects in the primordial universe might be neglected by semiclassical approaches, and more studies are needed. In particular, the full dynamics needs to be studied.