Hamiltonian formalism for anisotropic cosmological perturbation theory

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Motivation

- CPT is a working model of gravity at cosmological scales. Consistent quantization may explain (to some extent) the origin of primordial structure. Anisotropy could play a significant role in the primordial universe. The fewer primordial symmetries the better model.
- Hamiltonian form of CPT provides a simple laboratory for quantizations of gravity. Such issues as quantization prescription, diffeomorphism invariance and time problem, semiclassical spacetime reconstruction, etc can be studied within this framework.
- Dirac method has purely classical applications: representations of gravitational waves (the tensor part of the three-metric & three-momentum perturbation is constrained in anisotropic universe).

Dirac's method

ADM formalism for cosmological perturbations

Split the geometric and matter variables in the ADM formalism:

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij},\ \delta\pi^{ij} = \pi^{ij} - \bar{\pi}^{ij},\ N \mapsto N + \delta N,\ N^i \mapsto N^i + \delta N^i.$$

Expand the ADM Hamiltonian:

$$H = N\mathbf{H}_0^{(0)} + \int_\Sigma \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathbf{H}_0 + \delta N^i \delta\mathbf{H}_i \right),$$

where the constraints are first-class (up to first order):

$$\{\delta\mathbf{H}_i, \delta\mathbf{H}_j\} = 0,\ \ \{\delta\mathbf{H}_j, \delta\mathbf{H}_0\} = 0,$$
$$\{\mathbf{H}_0^{(0)} + \int_\Sigma \mathcal{H}^{(2)}, \delta\mathbf{H}_0\} = -ik^j\delta\mathbf{H}_j,\ \ \{\mathbf{H}_0^{(0)} + \int_\Sigma \mathcal{H}^{(2)}, \delta\mathbf{H}_i\} = 0.$$

Dirac method for cosmological perturbations

Reduction of a constrained system = establishing a reduced phase space with a physical Hamiltonian.

At the background level:

$$\mathbf{H}_0^{(0)} = 0, \quad \mathbf{t}^{(0)} = t, \quad \{\mathbf{H}_0^{(0)}, \mathbf{t}^{(0)}\} \neq 0$$

At the perturbation level:

$$\delta\mathbf{H}_\mu = 0, \quad \delta\mathbf{C}_\nu = 0, \quad |\{\delta\mathbf{H}_\mu, \delta\mathbf{C}_\nu\}| \neq 0$$

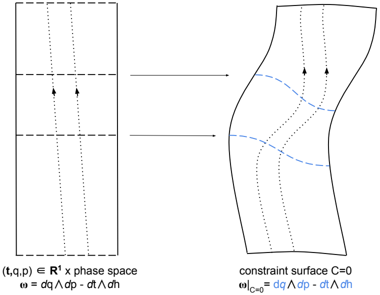
The Dirac bracket:

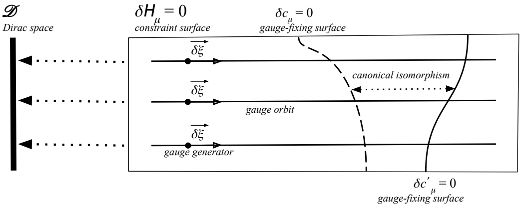
$$\{\cdot,\cdot\}_D = \{\cdot,\cdot\} - \{\cdot, \delta\mathbf{\Phi}_\mu\}\{\delta\mathbf{\Phi}_\mu, \delta\mathbf{\Phi}_\nu\}^{-1}\{\delta\mathbf{\Phi}_\nu, \cdot\}, \;\; \delta\mathbf{\Phi}_\nu \in \{\delta\mathbf{H}_0, \dots, \delta\mathbf{C}_0, \dots\}$$

The choice of physical variables:

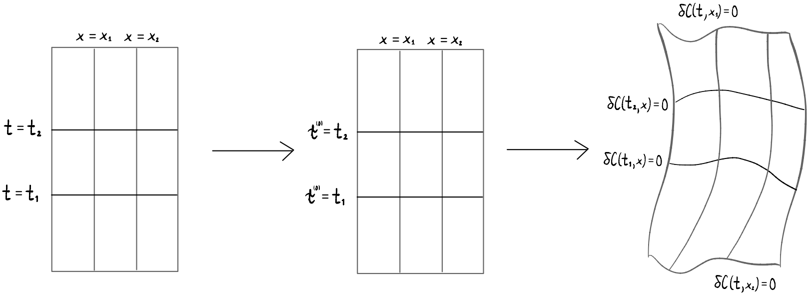
$$H_{phys} = H_{phys}^{(0)}(v_{phys}) + \int_\Sigma \mathcal{H}_{phys}^{(2)}(\delta v_{phys}).$$

Dirac method: phase space picture





Dirac method: spacetime picture



background manifold

homogeneous spacetime

inhomogeneous spacetime

Dirac method: gauge-invariant description

Dirac observables:

$$\{\delta D_I, \delta \mathbf{H}_\mu\} \approx 0 \quad \text{for all } \mu.$$

Express them in terms the physical variables δv_{phys}

$$\delta D_I + \xi_I^\mu \delta \mathbf{C}_\mu + \zeta_I^\mu \delta \mathbf{H}_\mu = \delta v_{phys,I},$$

$$\begin{aligned}
\{\delta D_I, \delta D_J\} &= \{\delta D_I, \delta D_J\}_D \\
&= \{\delta D_I + \xi_I^\mu \delta \mathbf{C}_\mu + \zeta_I^\mu \delta \mathbf{H}_\mu, \delta D_J + \xi_J^\mu \delta \mathbf{C}_\mu + \zeta_J^\mu \delta \mathbf{H}_\mu\}_D \\
&= \{\delta v_{phys,I}, \delta v_{phys,J}\}_D,
\end{aligned}$$

Substitute:

$$H^{(2)}_{phys}(\delta v_{phys,I}) \quad \longrightarrow \quad H^{(2)}_{phys}(\delta D_I)$$

Dirac method: spacetime reconstruction

The stability of gauge-fixing conditions:

$$\{\delta\mathbf{C}_\nu, H\} = 0 \;\Rightarrow\; \frac{\delta N^\mu}{N} = -\{\delta\mathbf{C}_\nu, \delta\mathbf{H}_\mu\}^{-1}\{\delta\mathbf{C}_\nu, \mathbf{H}_0^{(0)} + H^{(2)}\} \;\Rightarrow\; \frac{\delta N^\mu}{N}(\delta D_I)$$

Reconstruction of the three-surfaces:

$$(\delta\mathbf{C}_\nu, \delta\mathbf{H}_\mu, \delta D_I) \leftrightarrow (\delta q_{ij}, \delta\pi^{ij}) \;\Rightarrow\; \delta q_{ij}(\delta D_I)$$

Kuchař parametrization

In kinematical phase space introduce a **canonical** parametrization:

$$\underbrace{(\delta q_{ij}, \delta\pi^{ij})}_{ADM} \mapsto \underbrace{(\delta\mathbf{H}_\mu, \delta\mathbf{C}_\mu, \overbrace{\delta Q_I, \delta P_I}^{\delta D_J})}_{Kuchar}$$

The total Hamiltonian is given by $H_K = H_{ADM} + K$, where $K = \int_\Sigma N\mathcal{K}^{(2)}$.

$$H_K = N\int_\Sigma \underbrace{\mathcal{H}^{(2)}_{phys}(\delta Q_I, \delta P_I)}_{\text{physical part}} + \underbrace{(\lambda_{1I\mu}\delta Q^I + \lambda_{2I\mu}\delta P^I + \lambda_{3\mu\nu}\delta\mathbf{H}^\nu + \lambda_{4\mu\nu}\delta\mathbf{C}^\nu)\delta\mathbf{H}^\mu}_{\text{weakly vanishing part}}$$

1. λ_1 and λ_2 depend on $\delta\mathbf{C}_\mu$,

$$\left.\frac{\delta N_\mu}{N}\right|_{\delta\mathbf{C}_\mu} = \left.\frac{\partial H_K}{\partial\delta\mathbf{H}^\mu}\right|_{\delta\mathbf{C}_\mu} \approx -\lambda_{1I\mu}\delta Q^I - \lambda_{2I\mu}\delta P^I.$$

2. λ_3 is completely irrelevant and can be disregarded.
3. λ_4 is implied by the constraint algebra (gauge-invariant).

Gauge transformations

Consider a **canonical** transformation:

$$(\delta \mathbf{H}_\mu, \delta \mathbf{C}_\mu, \delta Q_I, \delta P_I) \mapsto (\delta \mathbf{H}_\mu, \delta \tilde{\mathbf{C}}_\mu, \delta \tilde{Q}_I, \delta \tilde{P}_I).$$

Hence,

$$\{\delta \tilde{\mathbf{C}}_\mu - \delta \mathbf{C}_\mu, \delta \mathbf{H}_\nu\} = 0,$$

implying

$$\delta \tilde{\mathbf{C}}_\mu = \delta \mathbf{C}_\mu + \alpha_{\mu I} \delta P^I + \beta_{\mu I} \delta Q^I + \gamma_{\mu\nu} \delta \mathbf{H}^\nu,$$

where α and β are any dynamical parameters.

- $\delta \tilde{\mathbf{C}}_\mu - \delta \mathbf{C}_\mu$ depend essentially on Dirac observables and shift the vanishing of the gauge-fixing functions for each gauge orbit independently
- Gauge-fixing functions $\delta \mathbf{C}_\mu$ are dynamical (chosen for each moment of time)
- At each moment of time the space of each gauge-fixing condition is isomorphic to the (linear) space of Dirac observables, a choice of the origin has to be made (FSG).

Gauge transformations

The new Kuchař parametrization reads:

$$\delta\tilde{\mathbf{C}}_\mu = \delta\mathbf{C}_\mu + \alpha_{\mu I}\delta P^I + \beta_{\mu I}\delta Q^I + \underbrace{\frac{1}{2}\left(\alpha_{\nu I}\beta_\mu{}^I - \alpha_{\mu I}\beta_\nu{}^I\right)}_{\gamma_{\mu\nu}}\delta\mathbf{H}^\nu,$$

$$\delta\tilde{Q}_I = \delta Q_I + \alpha_{\mu I}\delta\mathbf{H}^\mu,$$

$$\delta\tilde{P}_I = \delta P_I - \beta_{\mu I}\delta\mathbf{H}^\mu,$$

$$\Delta\mathcal{K}^{(2)} = \frac{1}{2}\left(\dot{\beta}_{\nu I}\alpha_\mu{}^I - \beta_{\nu I}\dot{\alpha}_\mu{}^I\right)\delta\mathbf{H}^\nu\delta\mathbf{H}^\mu + (\dot{\alpha}_{\mu I}\delta P^I + \dot{\beta}_{\mu I}\delta Q^I)\delta\mathbf{H}^\mu.$$

Compute the stability of a new gauge:

$$\left.\frac{\delta N_\mu}{N}\right|_{\delta\tilde{\mathbf{C}}_\mu} = \left.\frac{\partial H_K}{\partial\delta\mathbf{H}^\mu}\right|_{\delta\tilde{\mathbf{C}}_\mu},$$

$$\frac{\partial}{\tilde{\partial}\delta\mathbf{H}_\mu} = \frac{\partial}{\partial\delta\mathbf{H}_\mu} + \frac{\partial\delta\mathbf{C}_\nu}{\tilde{\partial}\delta\mathbf{H}_\mu}\frac{\partial}{\partial\delta\mathbf{C}_\nu} + \frac{\partial\delta Q_I}{\tilde{\partial}\delta\mathbf{H}_\mu}\frac{\partial}{\partial\delta Q_I} + \frac{\partial\delta P_I}{\tilde{\partial}\delta\mathbf{H}_\mu}\frac{\partial}{\partial\delta P_I}.$$

Gauge transformations

$$\left.\frac{\delta N_\mu}{N}\right|_{\delta\tilde{\mathbf{C}}_\mu} - \left.\frac{\delta N_\mu}{N}\right|_{\delta\mathbf{C}_\mu} \approx$$

$$\left(\frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q^I\partial\delta Q^J}\alpha_{J\mu} - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q^I\partial\delta P^J}\beta_{\mu J} - \dot{\beta}_{\mu I} + \lambda_{4\mu\nu}\beta_{\nu I}\right)\delta Q^I$$

$$+\left(\frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta Q^J}\alpha_{\mu J} - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta P^J}\beta_{J\mu} - \dot{\alpha}_{\mu I} + \lambda_{4\mu\nu}\alpha_{\nu I}\right)\delta P^I.$$

Plug in the gauge transformation coefficients α and β, make use of the physical Hamiltonian $\mathcal{H}^{(2)}_{phys}$ and the constraint algebra λ_4, and obtain the new lapse and shifts.

Partial gauge-fixing

Replace all or some of the gauge-fixing conditions with conditions on the lapse and shift functions (example: the synchronous gauge).

It is "partial" because $\frac{\delta N_\mu}{N}\Big|_{\delta\tilde{\mathbf{C}}_\mu} - \frac{\delta N_\mu}{N}\Big|_{\delta\mathbf{C}_\mu} \approx 0$ implies

$$\dot{\alpha}_{\mu I} = \alpha_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial \delta Q^J \partial \delta P^I} - \beta_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial \delta P^J \partial \delta P^I} + \lambda_{4\mu\nu}\alpha^\nu{}_I,$$

$$\dot{\beta}_{\mu I} = \alpha_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial \delta Q^J \partial \delta Q^I} - \beta_\mu{}^J \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial \delta P^J \partial \delta Q^I} + \lambda_{4\mu\nu}\beta^\nu{}_I.$$

- For any $(\alpha_{\mu I}(t_0), \beta_{\mu I}(t_0))$ a unique solution $t \mapsto (\alpha_{\mu I}(t), \beta_{\mu I}(t))$ exists. Hence, there is complete freedom in fixing $\delta\tilde{\mathbf{C}}_\mu(t_0)$.
- Phase space picture: once $\delta\tilde{\mathbf{C}}_\mu(t_0)$ is fixed at one time, it is determined at all times.
- Spacetime picture: $\delta\tilde{\mathbf{C}}_\mu(t_0)$ are needed in order to unambiguously move from the Kuchař to the ADM parametrization of the (intrinsic and extrinsic) geometry. Hence, once an arbitrary initial three-surface with space coordinates is chosen, it is propagated uniquely through the four-dimensional spacetime if the lapse and shifts are fixed everywhere.

Application:

Perturbed Bianchi I universe

Perturbed Bianchi I universe

The background metric of the Bianchi Type I model reads:

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \sum a_i^2(\mathrm{d}x^i)^2, \quad a = (a_1 a_2 a_3)^{\frac{1}{3}},$$

where the coordinates $(x^1, x^2, x^3) \in [0,1)^3$ are assumed.

The canonical perturbation variables read

$$\delta q_{ij} = q_{ij} - a_i^2\delta_{ij}, \quad \delta\pi^{ij} = \pi^{ij} - p^i\delta^{ij}, \quad \delta\phi = \phi - \bar{\phi}, \quad \delta\pi_\phi = \pi_\phi - \bar{\pi}_\phi,$$

The Fourier transform of a perturbation variable δX,

$$\delta\check{X}(\underline{k}) = \int_\Sigma \delta X(\overline{x}) e^{-ik_i x^i}\mathrm{d}^3x,$$

yields

$$\{\delta\check{\phi}(\underline{k}), \delta\check{\pi}^\phi(\underline{k}')\} = \delta_{\underline{k},-\underline{k}'}, \quad \{\delta\check{q}_{ij}(\underline{k}), \delta\check{\pi}^{lm}(\underline{k}')\} = \delta_{(i}^{\ l}\delta_{j)}^{\ m}\delta_{\underline{k},-\underline{k}'},$$

$(\Sigma \simeq \mathbb{T}^3,\ k_i = 2\pi n_i,\ n_i \in \mathbb{Z})$.

Perturbed Bianchi I universe

(Fix conformal metric: $\gamma_{ij} = a^{-2}\bar{q}_{ij}$):

$$A^1_{ij} = \gamma_{ij}, \quad A^2_{ij} = \hat{k}_i\hat{k}_j - \frac{1}{3}\gamma_{ij},$$
$$A^3_{ij} = \frac{1}{\sqrt{2}}\left(\hat{k}_i\hat{v}_j + \hat{v}_i\hat{k}_j\right), \quad A^4_{ij} = \frac{1}{\sqrt{2}}\left(\hat{k}_i\hat{w}_j + \hat{w}_i\hat{k}_j\right),$$
$$A^5_{ij} = \frac{1}{\sqrt{2}}\left(\hat{v}_i\hat{w}_j + \hat{w}_i\hat{v}_j\right), \quad A^6_{ij} = \frac{1}{\sqrt{2}}\left(\hat{v}_i\hat{v}_j - \hat{w}_i\hat{w}_j\right).$$

$$\delta q_n = \delta\check{q}_{ij}A^{ij}_n, \quad \delta\pi^n = \delta\check{\pi}^{ij}A^n_{ij}.$$

The Poisson bracket now reads

$$\{\delta\check{\phi}(\underline{k}), \delta\check{\pi}^{\phi}(\underline{k}')\} = \delta_{\underline{k},-\underline{k}'}, \quad \{\delta q_n(\underline{k}), \delta\pi^m(\underline{k}')\} = \delta_n{}^m\delta_{\underline{k},-\underline{k}'}.$$

A^n_{ij}'s and A^{ij}_n's are in general time-dependent as γ_{ij} and $(\hat{k}_i, \hat{v}_i, \hat{w}_i)$ evolve.

The Fermi-Walker basis

The Fourier transform fixes a slicing of the spatial coordinate space (x^1, x^2, x^3) with the wavefronts of plane waves. In the physical space, the wavefronts are not fixed but being continuously tilted and anisotropically contracted or expanded. The tangent basis $(\hat{v}, \hat{w})$ can be Fermi-Walker transported along the (future-oriented) null vector field $\vec{p}$ whose spatial component is dual to the wavefront $\underline{k}$ of a gravitational wave,

$$\vec{p} = \overline{k} + |\overline{k}|\partial_\eta \ ,$$

where $\nabla_{\vec{p}}\vec{p} = 0$. Field $\vec{p}$ may be identified with tangents to null geodesics associated with rays of gravitational waves in the eikonal approximation (i.e., for large wavenumbers):

$$\frac{\mathrm{d}\hat{v}^j}{\mathrm{d}\eta} = -\sigma_{vv}\hat{v}^j - \sigma_{vw}\hat{w}^j, \quad \frac{\mathrm{d}\hat{w}^j}{\mathrm{d}\eta} = -\sigma_{ww}\hat{w}^j - \sigma_{wv}\hat{v}^j,$$

(unfortunately, not suitable for quantization).

Reduction of the ADM formalism

Set gauge-fixing functions (flat slicing gauge):

$$\delta \mathbf{C}_1 := \delta q_1, \quad \delta \mathbf{C}_2 := \delta q_2, \quad \delta \mathbf{C}_3 := \delta q_3, \quad \delta \mathbf{C}_4 := \delta q_4.$$

A complete set of second-class constraints:

$$\delta \mathbf{\Phi}_\rho = \{\delta \mathbf{C}_1, \delta \mathbf{C}_2, \delta \mathbf{C}_3, \delta \mathbf{C}_4, \delta \mathbf{H}_0, \delta \mathbf{H}_k, \delta \mathbf{H}_v, \delta \mathbf{H}_w\}, \quad \det\{\delta \mathbf{\Phi}_\rho, \delta \mathbf{\Phi}_\sigma\} \neq 0.$$

Introduce Dirac's bracket:

$$\{\cdot, \cdot\}_D = \{\cdot, \cdot\} - \{\cdot, \delta \mathbf{\Phi}_\rho\}\{\delta \mathbf{\Phi}_\rho, \delta \mathbf{\Phi}_\sigma\}^{-1}\{\delta \mathbf{\Phi}_\sigma, \cdot\}.$$

Reduce the Hamiltonian:

$$H_{phys} = \int_\Sigma \left(N\mathcal{H}_0^{(2)} + \delta N^\mu \delta \mathbf{H}_\mu\right)\Big|_{\delta \mathbf{C}=0} = \int_\Sigma N\mathcal{H}_0^{(2)}\Big|_{\delta \mathbf{C}=0}.$$

By removing $(\delta q_i, \delta \pi_i)$, $i = 1, 2, 3, 4$, we obtain

$$\delta \dot{q} = N \frac{\partial \mathcal{H}_0^{(2)}\big|_{\delta \mathbf{C}}}{\partial \delta \pi}, \quad \delta \dot{\pi} = -N \frac{\partial \mathcal{H}_0^{(2)}\big|_{\delta \mathbf{C}}}{\partial \delta q},$$

where $(\delta q, \delta \pi) \in \{(\delta q_5, \delta \pi_5), (\delta q_6, \delta \pi_6), (\delta \phi, \delta \pi_\phi)\}$.

Physical Hamiltonian

After rescaling $(\delta q_5, \delta\pi_5),\ (\delta q_6, \delta\pi_6),\ (\delta\phi, \delta\pi_\phi)$:

$$
\begin{aligned}
H_{BI} =& \frac{N}{2a}\Big[\delta\tilde{\pi}_5^2 + \delta\tilde{\pi}_6^2 + \delta\tilde{\pi}_\phi^2 + (k^2 + U_5)\delta\tilde{q}_5^2 + (k^2 + U_6)\delta\tilde{q}_6^2 + (k^2 + U_\phi)\delta\tilde{\phi}^2 \\
&+ C_1\delta\tilde{q}_5\delta\tilde{q}_6 + C_2\delta\tilde{q}_5\delta\tilde{\phi} + C_3\delta\tilde{q}_6\delta\tilde{\phi}\Big].
\end{aligned}
$$

Rename the dynamical variables:

$$
\begin{aligned}
H_{BI} =& \frac{N}{2a}\Big[\delta P_1^2 + \delta P_2^2 + \delta P_3^2 + (k^2 + U_5)\delta Q_1^2 + (k^2 + U_6)\delta Q_2^2 + (k^2 + U_\phi)\delta Q_3^2 \\
&+ C_1\delta Q_1\delta Q_2 + C_2\delta Q_1\delta Q_3 + C_3\delta Q_2\delta Q_3\Big],
\end{aligned}
$$

where δQ_I and δP_I are Dirac observables s.t.:

$$
\begin{aligned}
\delta Q_1\big|_{\delta\mathbf{C}} = \delta\tilde{q}_5,\ \ \delta Q_2\big|_{\delta\mathbf{C}} = \delta\tilde{q}_6,\ \ \delta Q_3\big|_{\delta\mathbf{C}} = \delta\tilde{\phi}, \\
\delta P_1\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_5,\ \ \delta P_2\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_6,\ \ \delta P_3\big|_{\delta\mathbf{C}} = \delta\tilde{\pi}_\phi.
\end{aligned}
$$

Dirac observables

$$
\begin{aligned}
\delta Q_1 &= \underbrace{\frac{1}{\sqrt{2}a}\delta q_5} + \frac{2P_{vw}}{aP_{kk}}(\delta q_1 - \frac{1}{3}\delta q_2), \\
\delta Q_2 &= \underbrace{\frac{1}{\sqrt{2}a}\delta q_6} + \frac{P_{vv} - P_{ww}}{aP_{kk}}(\delta q_1 - \frac{1}{3}\delta q_2), \\
\delta Q_3 &= \underbrace{a\delta\phi + \frac{p_\phi}{aP_{kk}}(\delta q_1 - \frac{1}{3}\delta q_2),} \\
\delta P_1 &= \underbrace{\sqrt{2}a\delta\pi_5 + \frac{\frac{5}{6}(TrP) - P_{kk}}{\sqrt{2}a^3}\delta q_5} - \frac{2P_{vw}}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right), \\
&+ \mathcal{F}(P_{vw}, P_{kv}P_{kw})(\delta q_1 - \frac{1}{3}\delta q_2) - \frac{P_{vw}}{a^3 P_{kk}}\left(3P_{kk}\delta q_1 + a^2 p_\phi \delta\phi\right) + \frac{\sqrt{2}}{a^3}\left(P_{kw}\delta q_3 + P_{kv}\delta q_4\right) \\
\delta P_2 &= \underbrace{\sqrt{2}a\delta\pi_6 + \frac{\frac{5}{6}(TrP) - P_{kk}}{\sqrt{2}a^3}\delta q_6} - \frac{P_{vv} - P_{ww}}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right) \\
&+ \mathcal{F}\left(\frac{P_{vv} - P_{ww}}{2}, \frac{P_{kv}^2 - P_{kw}^2}{2}\right)(\delta q_1 - \frac{1}{3}\delta q_2) - \frac{P_{vv} - P_{ww}}{2a^3 P_{kk}}\left(3P_{kk}\delta q_1 + a^2 p_\phi \delta\phi\right) + \frac{\sqrt{2}}{a^3}\left(P_{kv}\delta q_3 - P_{kw}\delta q_4\right) \\
\delta P_3 &= \underbrace{\frac{1}{a}\delta\pi_\phi - \frac{(TrP)P_{kk} + 3p_\phi^2}{6aP_{kk}}\delta\phi - \frac{3p_\phi}{2a^3}\delta q_1 + \frac{2(TrP)P_{kk}p_\phi - 6a^6 P_{kk}V_{,\phi} - 3p_\phi^3}{6a^3 P_{kk}^2}(\delta q_1 - \frac{1}{3}\delta q_2)} \\
&- \frac{p_\phi}{\sqrt{2}a^3 P_{kk}}\left(\frac{P_{vv} - P_{ww}}{2}\delta q_6 + P_{vw}\delta q_5\right) - \frac{p_\phi\left((P_{vv} - P_{ww})^2 + 4P_{vw}^2\right)}{2a^3 P_{kk}^2}(\delta q_1 - \frac{1}{3}\delta q_2).
\end{aligned}
$$

Dirac observables

The rotation of the Dirac observables in the $(\hat{v}, \hat{w})$-plane:

$$R_{\hat{k}}(\theta)\delta Q_1 = \cos(2\theta)\delta Q_1 - \sin(2\theta)\delta Q_2$$

$$R_{\hat{k}}(\theta)\delta P_1 = \cos(2\theta)\delta P_1 - \sin(2\theta)\delta P_2$$

$$R_{\hat{k}}(\theta)\delta Q_2 = \cos(2\theta)\delta Q_2 + \sin(2\theta)\delta Q_1$$

$$R_{\hat{k}}(\theta)\delta P_2 = \cos(2\theta)\delta P_2 + \sin(2\theta)\delta P_1$$

$$R_{\hat{k}}(\theta)\delta Q_3 = \delta Q_3$$

$$R_{\hat{k}}(\theta)\delta P_3 = \delta P_3$$

where $R_{\hat{k}}(\theta)$ is the rotation around $\hat{k} = \hat{v} \times \hat{w}$ by the angle θ.

Physical metric

$$\left.\frac{\delta N}{N}\right|_{FS} = -\frac{P_{vw}}{aP_{kk}}\delta Q_1 - \frac{P_{vv} - P_{ww}}{2aP_{kk}}\delta Q_2 - \frac{p_\phi}{2aP_{kk}}\delta Q_3$$

$$\left.\frac{\delta N^k}{N}\right|_{FS} = \delta Q_1\left(\frac{P_{vw}}{2a^2} - \frac{2P_{kv}P_{kw}}{a^2 P_{kk}}\right) + \delta Q_2\left(\frac{P_{kw}^2}{a^2 P_{kk}} - \frac{P_{kv}^2}{a^2 P_{kk}} + \frac{P_{vv} - P_{ww}}{4a^2}\right)$$
$$+ \delta Q_3\left(\frac{a^4 V_{,\phi}}{2P_{kk}} - \frac{p_\phi (TrP)}{2a^2 P_{kk}} + \frac{3p_\phi}{4a^2}\right) + \frac{P_{vw}}{P_{kk}}\delta P_1 + \frac{P_{vv} - P_{ww}}{2P_{kk}}\delta P_2 + \frac{p_\phi}{2P_{kk}}\delta P_3$$

$$\left.\frac{\delta N^v}{N}\right|_{FS} = \delta Q_1\left(\frac{2P_{kv}P_{vw}}{a^2 kP_{kk}} + \frac{2P_{kw}}{a^2 k}\right) + \delta Q_2\left(\frac{P_{kv}(P_{vv} - P_{ww})}{a^2 kP_{kk}} + \frac{2P_{kv}}{a^2 k}\right) + \frac{P_{kv}p_\phi}{a^2 kP_{kk}}\delta Q_3$$

$$\left.\frac{\delta N^w}{N}\right|_{FS} = \delta Q_1\left(\frac{2P_{kw}P_{vw}}{a^2 kP_{kk}} + \frac{2P_{kv}}{a^2 k}\right) + \delta Q_2\left(\frac{P_{kw}(P_{vv} - P_{ww})}{a^2 kP_{kk}} - \frac{2P_{kw}}{a^2 k}\right) + \frac{P_{kw}p_\phi}{a^2 kP_{kk}}\delta Q_3$$

$$\delta q_5 = \sqrt{2}a\delta Q_1$$
$$\delta q_6 = \sqrt{2}a\delta Q_2$$

Gravitational waves in anisotropic universe

Scalar gravity gauge:

$$\delta \tilde{\mathbf{C}}_1 = \delta q_2, \quad \delta \tilde{\mathbf{C}}_2 := \delta q_3, \quad \delta \tilde{\mathbf{C}}_3 := \delta q_4, \quad \delta \tilde{\mathbf{C}}_4 := \delta q_5.$$

$$\mathrm{Det}\, \{\delta \mathbf{H}, \delta \tilde{\mathbf{C}}\} = -\frac{8i\sqrt{2}k^2 P_{vw}}{a},$$

$$\delta Q_1 = \frac{2P_{vw}}{aP_{kk}} \delta q_1,$$
$$\delta Q_2 = \frac{1}{\sqrt{2}a} \delta q_6 + \frac{P_{vv} - P_{ww}}{aP_{kk}} \delta q_1,$$

$$\left. \frac{\delta \tilde{N}_\mu}{N} \right|_{SG} - \left. \frac{\delta N_\mu}{N} \right|_{FS} = A_\mu \delta Q_1 + B_\mu \delta P^1.$$

Conclusions

- Hamiltonian form of CPT can be a useful playground for quantizations of gravity. Such issues as quantization prescription, diffeomorphism invariance and time problem, semiclassical spacetime reconstruction, . . . can be studied within this framework.
- The reduced phase space for anisotropic CPT can be easily derived with the Dirac method. Anisotropic CPT brings in some interesting issues: the dynamical triad $(\hat{k}, \hat{v}, \hat{w})$, new gauges including the representation of a gravitational wave by a scalar metric perturbation, richer dynamics of perturbations, . . .
- The structure of the theory is conveniently displayed in the Kuchař parametrization. The gauge transformations can be conveniently formulated and gauge-fixing can be achieved in various way.