Quasi-stationary routes to black holes: exterior and interior perspective

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1. Introduction

2. Black hole limit of relativistic figures of equilibrium
   (a) Necessary and sufficient conditions
   (b) Extreme Kerr uniqueness

3. Rigorous results for discs of dust

4. Numerical results for fluid rings with various equations of state

5. A $3n$-parameter family of spacetimes with extreme Kerr near-horizon asymptotics generated by Bäcklund transformations

1. Introduction

• In the case of spherical symmetry, an “unstoppable gravitational collapse” (resulting in a spacetime singularity – within classical general relativity) necessarily leads to a Schwarzschild black hole (cf. Oppenheimer & Snyder 1939) or, if there is some electric charge, to a Reissner-Nordstrøm black hole.

• Without spherical symmetry, for example in the case of a collapsing *rotating* star, the result of a complete collapse is much more difficult to predict. The (weak) cosmic censorship conjecture (Penrose 1969), combined with (i) the assumption that the exterior gravitational field settles down to a stationary state and (ii) the black hole uniqueness (“no-hair theorem”: Israel, Carter, Hawking, Robinson, Mazur, . . .), predicts the formation of a Kerr (or Kerr-Newman) black hole.

• With rotation (and/or electric charge), *quasi-stationary* collapse scenarios, described by sequences of equilibrium configurations becoming more and more compact, are possible. Do they lead to black holes or to naked singularities?

• A continuous sequence of stationary and axisymmetric, uniformly rotating perfect fluid bodies reaches a black hole limit if and only if the relation

$$M = 2\Omega J \qquad\qquad (G = c = 1)$$

is satisfied in the limit (Meinel 2006). The limit leads to an *extreme* Kerr black hole.

• The existence of such a limit was first demonstrated for rotating discs of dust, numerically by Bardeen & Wagoner (1971) and analytically by Neugebauer & M. (1995). This result is in remarkable agreement with cosmic censorship.

• Further numerical examples, for genuine fluid bodies, were provided by the "relativistic Dyson rings" (Ansorg et al. 2003) and their generalizations.

• The black hole limit appears from the "exterior perspective". From the "interior perspective" the limit leads to a spacetime with extreme Kerr near-horizon asymptotics. This motivates the systematic study of solutions to the Einstein equations with this asymptotics.

2. Black hole limit of relativistic figures of equilibrium
(a) Necessary and sufficient conditions

Four-velocity of the fluid:

$$u^i = e^{-V}(\xi^i + \Omega\,\eta^i), \quad \Omega = \text{constant}$$

with Killing vectors: $\xi = \partial/\partial t,\ \eta = \partial/\partial\varphi$

[$\xi^i\xi_i \to -1$ at spatial infinity. We assume asymptotic flatness; the spacetime signature is chosen to be $(+\ +\ +\ -)$. The orbits of the spacelike Killing vector η are closed and η is zero on the axis of symmetry.]

$$\Omega = u^\varphi/u^t, \quad e^{-V} = u^t$$

$$u^i u_i = -1 \quad \Rightarrow \quad (\xi^i + \Omega\,\eta^i)(\xi_i + \Omega\,\eta_i) = -e^{2V}$$

Energy-momentum tensor: $T_{ik} = (\mu + p)\,u_i u_k + p\,g_{ik}$

"Cold" equation of state, $\mu = \mu(p)$, following from

$$p = p(\mu_{\rm b}, T), \quad \mu = \mu(\mu_{\rm b}, T)$$

for $T = 0$, where $\mu_{\rm b}$ is the "baryonic mass-density" [with $(\mu_{\rm b} u^i)_{;i} = 0$] and T the temperature. The specific enthalpy

$$h = \frac{\mu + p}{\mu_{\rm b}}$$

can be calculated from $\mu(p)$ via the thermodynamic relation

$$dh = \frac{1}{\mu_{\rm b}}\, dp \qquad (T = 0)$$

leading to

$$\frac{dh}{h} = \frac{dp}{\mu + p} \quad \Rightarrow \quad h(p) = h(0)\, \exp\left[\int_0^p \frac{dp'}{\mu(p') + p'}\right].$$

[$h(0) = 1$ in most cases.]

$T^{ik}{}_{;k} = 0 \quad \Rightarrow \quad h(p)\, e^{V} = h(0)\, e^{V_0} = \text{constant}$

Relative redshift z of zero angular momentum photons emitted from the surface of the fluid and received at infinity:

$$z = e^{-V_0} - 1$$

Equilibrium models, for a given equation of state, are fixed by two parameters, for example Ω and V_0. (When we discuss a “sequence” of solutions, what is meant is a curve in the two-dimensional parameter space.)

Baryonic mass M_{b}, gravitational mass M and angular momentum J:

$$M_{\mathrm{b}} = -\int_{\Sigma} \mu_{\mathrm{b}}\, u_i\, n^i d\mathcal{V}, \quad M = 2\int_{\Sigma} (T_{ik} - \frac{1}{2} T^j_j\, g_{ik}) n^i \xi^k d\mathcal{V}, \quad J = -\int_{\Sigma} T_{ik}\, n^i \eta^k d\mathcal{V},$$

where Σ is a spacelike hypersurface ($t = \text{constant}$) with the volume element $d\mathcal{V} = \sqrt{^{(3)}g}\, d^3x$ and the future pointing unit normal n^i.

A combination of the previous relations leads to the formula

$$M = 2\Omega J + h(0)\, e^{V_0} \int \frac{\mu + 3p}{\mu + p}\, dM_{\rm b}\,.$$

We assume μ and p to be non-negative and $0 < M_{\rm b} < \infty,\ 0 < h(0) < \infty$.

$\Rightarrow \quad 1 \le (\mu + 3p)/(\mu + p) \le 3 \quad \Rightarrow$

$$M = 2\Omega J \quad \Leftrightarrow \quad V_0 \to -\infty \quad (z \to \infty)$$

• This condition is necessary and sufficient for approaching a black hole limit.

Surface of the fluid: $\quad (\xi^i + \Omega\, \eta^i)(\xi_i + \Omega\, \eta_i) = -e^{2V_0}$

Black hole horizon: $\quad (\xi^i + \Omega_{\rm h}\, \eta^i)(\xi_i + \Omega_{\rm h}\, \eta_i) = 0$

$\Omega_{\rm h}$: “angular velocity of the horizon” ; $\quad V_0 \to -\infty$: $\ \Omega \to \Omega_{\rm h}$

$M = 2\Omega J \quad \Rightarrow$ Impossibility of black hole limits of non-rotating (uncharged) equilibrium configurations, cf. “Buchdahl’s inequality”.

Together with

$$\Omega = \Omega_{\mathrm{h}} = \frac{J}{2M^2\left[M + \sqrt{M^2 - (J/M)^2}\right]}$$

$\Rightarrow \quad J = M^2$ (*extreme* Kerr black hole).

Note: The last conclusion makes use of the Kerr black hole uniqueness *including the extreme case.*

(b) Extreme Kerr uniqueness

In Weyl's canonical coordinates, the stationary and axisymmetric *vacuum* line element takes the form

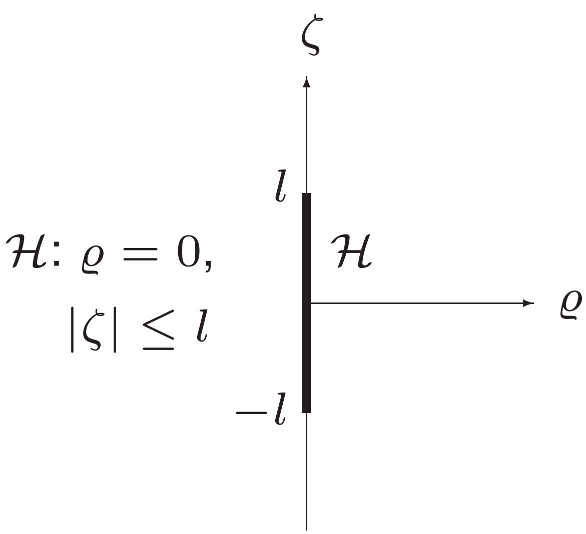
$$ds^2 = e^{2\alpha}(d\varrho^2 + d\zeta^2) + \varrho^2 e^{-2\nu}(d\varphi - \omega\, dt)^2 - e^{2\nu} dt^2,$$

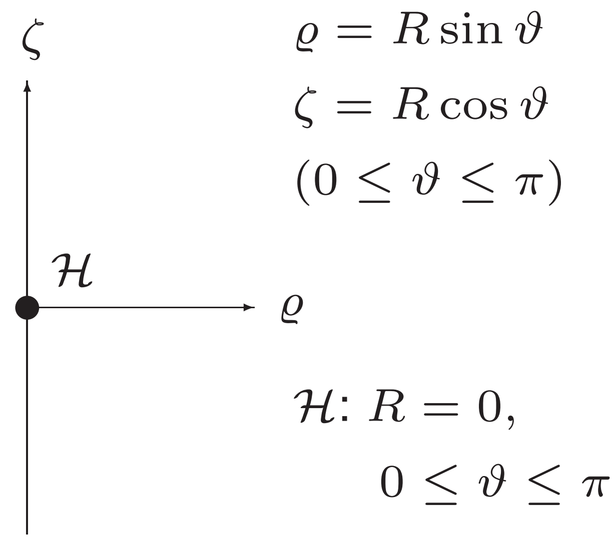
where

$$\varrho^2 = (\xi^i \eta_i)^2 - \xi^i \xi_i \eta^k \eta_k = (\chi^i \eta_i)^2 - \chi^i \chi_i \eta^k \eta_k.$$

$\Rightarrow \quad \varrho = 0$ on the horizon ($\mathcal{H}$: $\chi^i \chi_i = 0$, $\chi^i \eta_i = 0$ with $\chi^i \equiv \xi^i + \Omega_{\mathrm{h}} \eta^i$)

Therefore, the $t = \mathrm{constant}$, $\varphi = \mathrm{constant}$ slice of the horizon of a single stationary and axisymmetric black hole surrounded by a vacuum can only be a finite intervall or a single point on the ζ-axis. In *both* cases, the corresponding boundary value problem can uniquely be solved by means of the “inverse scattering method” .





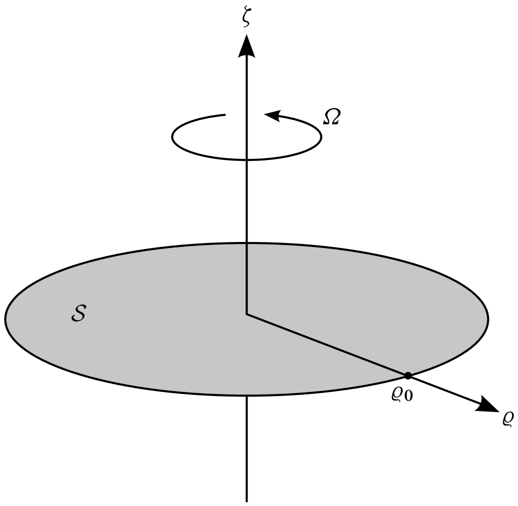
Result: Kerr with $J < M^2$ — Kerr with $J = M^2$

$[\, l = \sqrt{M^2 - (J/M)^2}\,]$

• The Kerr (-Newman) black holes – including the extreme case – are the only stationary and axisymmetric black holes (with a single connected horizon) surrounded by an asymptotically flat (electro-) vacuum (Meinel et al. 2008, Meinel 2012).

Other proofs of the extreme Kerr (-Newman) uniqueness have been published by Amsel et al. (2010), Figueras & Lucietti (2010) and Chruściel & Nguyen (2010).

3. Rigorous results for discs of dust

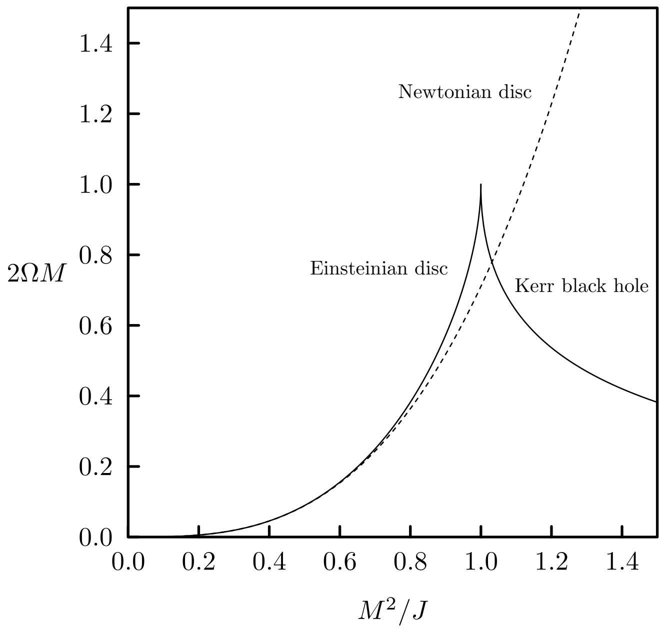


Two parameters: ϱ_0, Ω

The exact solution to this problem has been found in terms of hyperelliptic theta functions by solving the corresponding boundary value problem via the "inverse scattering method" (Neugebauer & M. 1995). It depends on the normalized coordinates ϱ/ϱ_0, ζ/ϱ_0 or ϱ/M, ζ/M and the previously introduced parameter V_0, which is given here by

$$e^{2V_0} = -(\xi^i + \Omega\,\eta^i)(\xi_i + \Omega\,\eta_i)\Big|_{\mathcal{S}} = \text{constant}.$$

Newtonian limit: $|V_0| \ll 1$,   Black hole limit: $V_0 \to -\infty$



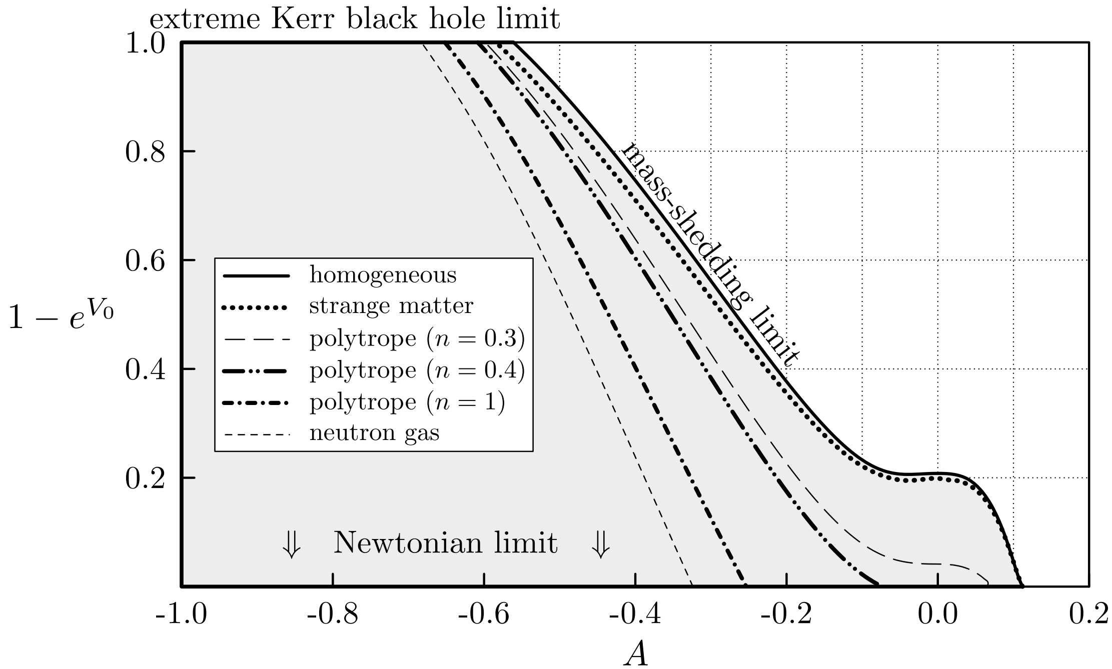
In the black hole limit, the disc shrinks to the origin of the ϱ/M, ζ/M coordinate system, since $\varrho_0/M \to 0$; and the solution becomes precisely the extreme Kerr solution (outside the horizon).

Note that the limit in the ϱ/ϱ_0, ζ/ϱ_0 coordinates is different: It gives a non-asymptotically flat solution with the extreme Kerr "throat geometry" at spatial infinity!

4. Numerical results for fluid rings with various equations of state

$$ds^2 = e^{2\alpha}(d\varrho^2 + d\zeta^2) + W^2 e^{-2\nu}(d\varphi - \omega\, dt)^2 - e^{2\nu} dt^2$$

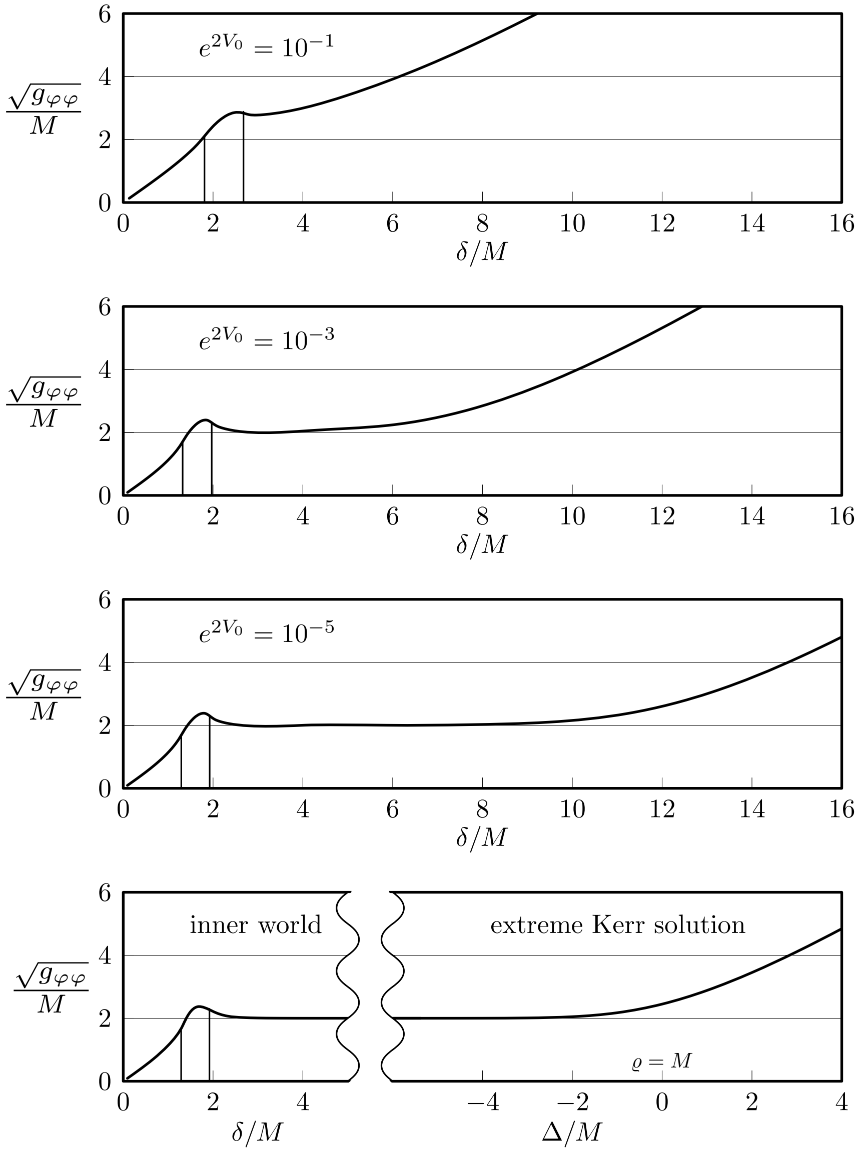
$$\alpha = \alpha(\varrho, \zeta)$$
$$\nu = \nu(\varrho, \zeta)$$
$$\omega = \omega(\varrho, \zeta)$$
$$W = W(\varrho, \zeta)$$



toroidal bodies: $A := -\varrho_{\rm i}/\varrho_{\rm o} < 0$

sheroidal bodies: $A := \zeta_{\rm p}/\varrho_{\rm o} > 0$

Ansorg et al. (2003), Fischer et al. (2005), Labranche et al. (2007)



$$2\pi\sqrt{g_{\varphi\varphi}(\varrho,0)} = 2\pi W(\varrho,0)e^{-\nu(\varrho,0)}:$$

proper circumference of a circle $\varrho = \mathrm{constant}$ in the "equatorial plane" ($\zeta = 0$, $t = \mathrm{constant}$)

$$\delta = \int_0^{\varrho} e^{\alpha(\varrho',0)}d\varrho'$$: proper radius of that circle

($\mu = \mathrm{constant}$, $A = -0.7$)

$$\Delta = \int_M^{\varrho} e^{\alpha(\varrho',0)}d\varrho'$$

• The extreme Kerr throat geometry or "extreme Kerr near-horizon geometry" can be obtained from the extreme Kerr metric in Boyer-Lindquist coordinates $\tilde{r}$, $\tilde{\theta}$, $\tilde{\varphi}$, $\tilde{t}$ by means of the coordinate transformation

$$r = \frac{\tilde{r} - M}{\gamma}, \quad \theta = \tilde{\theta}, \quad \varphi = \tilde{\varphi} - \Omega \tilde{t}, \quad t = \gamma \tilde{t} \quad (\text{with} \quad \Omega = \Omega_{\rm h} = \frac{1}{2M})$$

in the limit $\gamma \to 0$ (Bardeen & Horowitz 1999).

• In addition to $\partial / \partial t$ and $\partial / \partial \varphi$ it has two more Killing fields (Wolf 1998):

$$\left(\frac{t^2}{2} + \frac{1}{8 r^2 \Omega^4} \right) \frac{\partial}{\partial t} - r t \frac{\partial}{\partial r} - \frac{1}{2 r \Omega^2} \frac{\partial}{\partial \varphi}, \quad t \frac{\partial}{\partial t} - r \frac{\partial}{\partial r}$$

5. Solutions with extreme Kerr near-horizon asymptotics (M. & Kleinwächter 2020)

• It is well known that the stationary and axially symmetric vacuum Einstein equations are equivalent to the Ernst equation

$$(\Re\,\mathcal{E})\nabla^2\mathcal{E} = (\nabla\mathcal{E})^2$$

[∇ as in Euclidean 3-space with spherical coordinates r, θ and φ and $\mathcal{E} = \mathcal{E}(r,\theta)$]

$$ds^2 = f^{-1}[\,h(dr^2 + r^2 d\theta^2) + r^2\sin^2\theta\,d\varphi^2] - f(dt + a\,d\varphi)^2$$

with $f = \Re\,\mathcal{E}$; the other metric functions h and a can also be obtained from $\mathcal{E}$.

• Ernst potential of the extreme Kerr near-horizon geometry:

$$\mathcal{E}_{\rm NHG} = -\Omega^2 r^2 H(\theta), \quad H(\theta) = \frac{2(1+\mathrm{i}\cos\theta)^2}{1-\mathrm{i}\cos\theta} + \sin^2\theta$$

- Bäcklund transformation formula (Neugebauer 1980):

$$\mathcal{E} = \mathcal{E}_0 \frac{\det \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ \alpha_0 & \alpha_1\lambda_1 & \alpha_2\lambda_2 & \cdot & \cdot & \cdot & \alpha_{2n}\lambda_{2n} \\ 1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdot & \cdot & \cdot & (\lambda_{2n})^2 \\ \alpha_0 & \alpha_1(\lambda_1)^3 & \alpha_2(\lambda_2)^3 & \cdot & \cdot & \cdot & \alpha_{2n}(\lambda_{2n})^3 \\ 1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdot & \cdot & \cdot & (\lambda_{2n})^4 \\ \alpha_0 & \alpha_1(\lambda_1)^5 & \alpha_2(\lambda_2)^5 & \cdot & \cdot & \cdot & \alpha_{2n}(\lambda_{2n})^5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdot & \cdot & \cdot & (\lambda_{2n})^{2n} \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \alpha_1\lambda_1 & \alpha_2\lambda_2 & \cdot & \cdot & \cdot & \alpha_{2n}\lambda_{2n} \\ 1 & (\lambda_1)^2 & (\lambda_2)^2 & \cdot & \cdot & \cdot & (\lambda_{2n})^2 \\ 1 & \alpha_1(\lambda_1)^3 & \alpha_2(\lambda_2)^3 & \cdot & \cdot & \cdot & \alpha_{2n}(\lambda_{2n})^3 \\ 1 & (\lambda_1)^4 & (\lambda_2)^4 & \cdot & \cdot & \cdot & (\lambda_{2n})^4 \\ 1 & \alpha_1(\lambda_1)^5 & \alpha_2(\lambda_2)^5 & \cdot & \cdot & \cdot & \alpha_{2n}(\lambda_{2n})^5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & (\lambda_1)^{2n} & (\lambda_2)^{2n} & \cdot & \cdot & \cdot & (\lambda_{2n})^{2n} \end{pmatrix}}$$

The entries in the two $(2n+1)\times(2n+1)$ matrices are given by

$$\alpha_0 = -\frac{\mathcal{E}_0^*}{\mathcal{E}_0}, \quad \lambda_i = \sqrt{\frac{K_i - re^{\mathrm{i}\theta}}{K_i - re^{-\mathrm{i}\theta}}} \quad (\lambda_i \to e^{\mathrm{i}\theta} \text{ as } r \to \infty)$$

and solutions α_i to the total Riccati equations

$$\begin{aligned}(\mathcal{E}_0 + \mathcal{E}_0^*)\, d\alpha_i \quad &= \quad \left[\tfrac{\partial \mathcal{E}_0^*}{\partial z}(\alpha_i - \lambda_i) + \tfrac{\partial \mathcal{E}_0}{\partial z}\alpha_i(\alpha_i\lambda_i - 1)\right] dz \\ &\quad + \ \left[\tfrac{\partial \mathcal{E}_0^*}{\partial z^*}(\alpha_i - \lambda_i^{-1}) + \tfrac{\partial \mathcal{E}_0}{\partial z^*}\alpha_i(\alpha_i\lambda_i^{-1} - 1)\right] dz^*\end{aligned}$$

with the complex coordinates

$$z = \mathrm{i}re^{-\mathrm{i}\theta}, \quad z^* = -\mathrm{i}re^{\mathrm{i}\theta}$$

The (finite) constants K_i must either be real ($K_i = K_i^*$), resulting in $\lambda_i = 1/\lambda_i^*$, or complex conjugate pairs ($K_j = K_i^*$), to ensure $\lambda_j = 1/\lambda_i^*$. The integration constants of the Riccati equations have to be chosen such that $\alpha_i = 1/\alpha_i^*$ or $\alpha_j = 1/\alpha_i^*$, respectively.

For $\mathcal{E}_0 = \mathcal{E}_{\mathrm{NHG}}$ we obtain

$$\alpha_0 = -\frac{H^*}{H}$$

$$\alpha_i = -\frac{\psi(\lambda_i, \theta) - c_i \psi(-\lambda_i, \theta)}{\chi(\lambda_i, \theta) + c_i \chi(-\lambda_i, \theta)} \qquad (i = 1, 2, \ldots, 2n)$$

with

$$\psi(\lambda, \theta) = [\chi(1/\lambda^*, \theta)]^* = \frac{A(1+\lambda^2) + B(1-\lambda^2) + C\lambda}{e^{-\mathrm{i}\theta}(\lambda - e^{\mathrm{i}\theta})^2}$$

$$A = \frac{\cos\theta + \mathrm{i}}{1+\mathrm{i}}, \quad B = \frac{(1-\mathrm{i})\sin\theta}{\cos\theta - \mathrm{i}}, \quad C = -(1+\mathrm{i})(\cos\theta - \mathrm{i})$$

The constants c_i have to satisfy

$c_i = -c_i^*$ (for real K_i) or $c_j = -c_i^*$ (for pairs $K_j = K_i^*$)

They can also be chosen infinite, meaning $\alpha_i = \psi(-\lambda_i, \theta)/\chi(-\lambda_i, \theta)$

Solutions with extreme Kerr near-horizon asymptotics

For the discussion of the asymptotic behaviour as $r \to \infty$ a reformulation in terms of $n \times n$ determinants (cf. Yamazaki 1983) is useful. With

$$r_i \equiv -\lambda_i(K_i - re^{-\mathrm{i}\theta}) = r\sqrt{\left(1 - \frac{K_i e^{\mathrm{i}\theta}}{r}\right)\left(1 - \frac{K_i e^{-\mathrm{i}\theta}}{r}\right)}$$

one obtains

$$\mathcal{E} = \mathcal{E}_0 \, \frac{\det\left(\frac{\alpha_p r_p - \alpha_q r_q}{K_p - K_q} + \alpha_0\right)}{\det\left(\frac{\alpha_p r_p - \alpha_q r_q}{K_p - K_q} + 1\right)}$$

with

$$p = 1, 3, 5, \ldots, 2n-1; \quad q = 2, 4, 6, \ldots, 2n$$

(This means: first row $p = 1$, second row $p = 3$, $\ldots$, n-th row $p = 2n-1$ and first column $q = 2$, second column $q = 4$, $\ldots$, n-th column $q = 2n$)

For $n = 1$:

$$\mathcal{E} = \mathcal{E}_0 \, \frac{\alpha_1 r_1 - \alpha_2 r_2 + \alpha_0 (K_1 - K_2)}{\alpha_1 r_1 - \alpha_2 r_2 + K_1 - K_2}$$

Expansion in powers of r^{-1}:

$$\Rightarrow \quad \alpha_i = -\frac{\psi(\lambda_i, \theta)}{\chi(\lambda_i, \theta)} + \mathcal{O}(r^{-2}) = F(\theta) + \mathcal{O}(r^{-1}) \quad \text{for } c_i \neq \infty$$

and

$$\alpha_i = \frac{\psi(-\lambda_i, \theta)}{\chi(-\lambda_i, \theta)} = G(\theta) + \mathcal{O}(r^{-1}) \quad \text{for } c_i = \infty$$

with

$$F(\theta) = \frac{\mathrm{i}(\cos\theta + \mathrm{i})^2}{(\cos\theta - \mathrm{i})^2}, \quad G(\theta) = \frac{\mathrm{i}(\cos\theta + \mathrm{i})(6\mathrm{i} - 15\cos\theta - 6\mathrm{i}\cos 2\theta - \cos 3\theta)}{(\cos\theta - \mathrm{i})(6\mathrm{i} + 15\cos\theta - 6\mathrm{i}\cos 2\theta + \cos 3\theta)}$$

Because of

$$\lim_{r\to\infty} \frac{r_i}{r} = 1$$

we find (with $\mathcal{E}_0 = \mathcal{E}_{\rm NHG}$)

$$\lim_{r\to\infty} \frac{\mathcal{E}}{\mathcal{E}_{\rm NHG}} = 1$$

for all θ with $F(\theta) \neq G(\theta)$ if n of the $2n$ constants c_i, say c_p (with $p = 1, 3, 5, \ldots, 2n-1$), are chosen finite and the other ones, say c_q (with $q = 2, 4, 6, \ldots, 2n$), are chosen infinite. This reduces the number of free real constants contained in the K_i's and c_i's from $4n$ to $3n$. For $n = 1$, $c_1 \neq \infty$ and $c_2 = \infty$ means that K_1 and K_2 must be real. For $n > 1$, pairs of complex conjugate K_i's are possible as well. It turns out that the so far excluded special values of θ defined by $F(\theta) = G(\theta)$ are the same values for which $\alpha_0 \equiv -H^*/H = 1$ holds ($\cos^2\theta = 2\sqrt{3}-3$), leading obviously to $\mathcal{E} = \mathcal{E}_{\rm NHG}$ for all r. Hence our solutions have the extreme Kerr near-horizon asymptotics whenever precisely n of the $2n$ constants c_i are chosen infinite.

• Explicit expressions for all metric functions can be calculated using the general Bäcklund formalism.

• For the case $n = 1$, the three-parameter family of solutions with extreme Kerr near-horizon asymptotics leads to the following expressions for the metric functions in

$$ds^2 = f^{-1}[\, h(dr^2 + r^2 d\theta^2) + r^2 \sin^2\theta \, d\varphi^2] - f(dt + a\, d\varphi)^2 :$$

$$f = \Re\left(\mathcal{E}_{\rm NHG} \, \frac{\alpha_1 r_1 - \alpha_2 r_2 + \alpha_0 (K_1 - K_2)}{\alpha_1 r_1 - \alpha_2 r_2 + K_1 - K_2} \right)$$
$$h = h_0 \Omega^4 K_1^4 \psi_1 \psi_2 \psi_1^* \psi_2^* (\lambda_1 \lambda_2^* + \lambda_1^* \lambda_2 - \alpha_1 \alpha_2^* - \alpha_1^* \alpha_2) f_0^{-2}$$
$$a = (a_0 f_0 - 2\, \Im\, Q) f^{-1}$$

with

$$h_0 = h_{\rm NHG} = \frac{1}{4}(\cos^4\theta + 6\cos^2\theta - 3)$$

$$f_0 = f_{\rm NHG} = \frac{4\Omega^2 r^2 h_0}{\cos^2\theta + 1}, \quad a_0 = a_{\rm NHG} = -\frac{2r\sin^2\theta}{f_0(\cos^2\theta + 1)}$$

and

$$\psi_1 = \psi(\lambda_1, \theta) - c_1 \psi(-\lambda_1, \theta), \quad \psi_2 = -\mathrm{i}\psi(-\lambda_2, \theta)$$

$$Q = \frac{\alpha_1 r_1 (K_2 - r\cos\theta) - \alpha_2 r_2 (K_1 - r\cos\theta) + \mathrm{i} a_0 f_0 (K_1 - K_2)}{\alpha_1 r_1 - \alpha_2 r_2 + K_1 - K_2}$$

Note that

$$\lim_{r\to\infty} \frac{f}{f_0} = \lim_{r\to\infty} \frac{h}{h_0} = \lim_{r\to\infty} \frac{a}{a_0} = 1$$

References

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R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer & D. Petroff, *Relativistic Figures of Equilibrium*, Cambridge University Press 2008

R. Meinel, *Constructive proof of the Kerr-Newman black hole uniqueness including the extreme case*, Classical and Quantum Gravity 29 (2012) 035004

R. Meinel & A. Kleinwächter, *Bäcklund transforms of the extreme Kerr near-horizon geometry*, Physics Letters A 384 (2020) 126572

• A much simpler scenario of quasi-stationary routes to black holes including the exterior and interior perspective is discussed in

R. Meinel & M. Hütten, *On the black hole limit of electrically counterpoised dust configurations*, Classical and Quantum Gravity 28 (2011) 225010