The Physical Relevance of the Fiducial Cell in Loop Quantum Cosmology

Johannes Münch[1]

Relativity Seminar University of Warsaw

Warsaw, Poland

based on

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In collaboration with Fabio M. Mele[2]

[1]Aix-Marseille Université, Université de Toulon, CNRS, CPT, France
[2]Okinawa Institute of Science and Technology, Japan

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Motivation

Spacially flat cosmology

$$\mathrm{d}s^2 = -N(t)\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \qquad (1)$$

Full theory perspective:

- Field theory:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- Insert Eq. (1)

No fiducial structures needed!

Mini-superspace:

- Eq. (1) into $S_{EH} = \int_M \mathrm{d}^4x\sqrt{-g}R$
- Effective point particle
- Divergences: $\int_\Sigma \mathrm{d}^3x \to \infty$

Need a fiducial regulator $V_o \subset \Sigma$!

How do both pictures fit together? How "fiducial" is the cell V_o?

Plan of the Talk

- **Part I: Classical Theory**
  - Problem and Setup
  - Dependences on Fiducial Structures
  - Classical Dynamics
- **Part II: Quantum Cosmology**
  - Representations
  - Quantum Dynamics
  - Fluctuations

Part I: Classical Theory

The Problem

Spacially flat cosmology

Spacetime: $M = \mathbb{R} \times \Sigma$

$$\mathrm{d}s^2 = -N(t)\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{x}^2$$

Action: $(\kappa = 8\pi G,\ c = 1)$ GR + massless scalar field ϕ

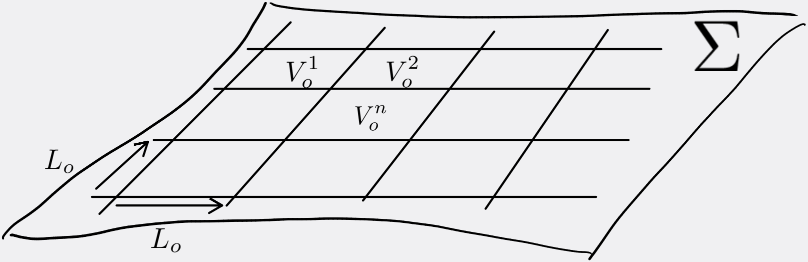
$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_M = \int \mathrm{d}t\, \mathcal{L} + \text{boundary terms}$$

$$\mathcal{L} = \int_\Sigma \mathrm{d}^3x\, \mathscr{L} = \underbrace{\int_\Sigma \mathrm{d}^3x}_{\to\infty} \left(-\frac{3}{\kappa}\frac{a\,\dot{a}^2}{N} + \frac{a^3\dot{\phi}^2}{2N}\right)$$

Full Homogeneity is too strong!

Fiducial Structures

Impose homogeneity only on finite regions:

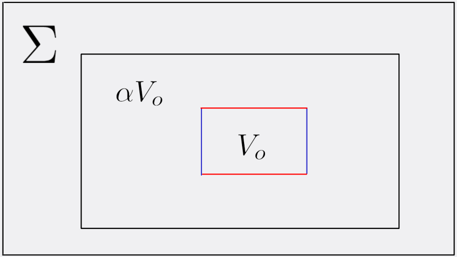


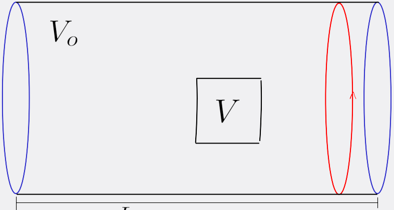
Topological decomposition in boxes of *coordinate* edge length L_o/volume V_o

Fields q_{ab}, P^{ab}, ϕ, p_ϕ are constant only in V_o^n
Decompose e.g.

$$\phi(x) = \sum_n \phi^n \chi_{V_o^n}(x) \quad , \quad \chi_{V_o^n}(x) = \begin{cases} 1 & x \in V_o^n \\ 0 & \text{else} \end{cases}$$

Boundary Conditions and Freedom





L_o in x , $\frac{L_o}{\beta}$ in $\beta \cdot x$, $L_{phys} = L_o a$

Remaining freedom

1. Coordinate transformations: $x \mapsto \beta x \Rightarrow L_o \mapsto \frac{L_o}{\beta}$ not considered
2. Rescaling of scale of homogeneity/periodicity

$$V_o \mapsto \alpha V_o \quad , \quad L_o \mapsto \alpha^{1/3} L_o$$

Does the physics change under 2.?

V_o-dependence: 1) Variables and Observables

Variables [Bodendorfer '16]

From full theory:

$$v(x) = \sqrt{q}(x) \quad , \quad b(x) = -\frac{2q_{ab}P^{ab}(x)}{3\sqrt{q}}$$

Partial homogeneity:

$$b(x) = \sum_n \chi_{V_o^n}(x) b^n \quad , \quad v = \sqrt{\mathring{q}} \sum_n \chi_{V_o^n}(x) v^n$$

Poisson bracket: (see e.g. [Mele, JM to be published])

$$\{b^n, v^m\}_D = \frac{\delta_{nm}}{V_o} \quad , \quad \{\phi^n, p_\phi^m\}_D = \frac{\delta_{nm}}{V_o}$$

Observables

Extensive:

$$\text{vol}(V) = \int_V \mathrm{d}^3x \sqrt{q} \overset{V\subset V_o^n}{\approx} V \cdot v^n \quad , \quad p_\phi(V) = \int_V \mathrm{d}^3x \, p_\phi \overset{V\subset V_o^n}{\approx} V \cdot p_\phi^n$$

Intensive:

$$b(V) = \frac{1}{\text{vol}(V)} \int_V \mathrm{d}^3x \sqrt{q}\, b(x) \overset{V\subset V_o^n}{\approx} b^n$$

$$\phi(V) = \frac{1}{\text{vol}(V)} \int_V \mathrm{d}^3x \sqrt{q}\, \phi(x) \overset{V\subset V_o^n}{\approx} \phi^n$$

V_o-dependence: 2) Poisson Structure

Choose $V, V' \subset V_o^n$ and $V, V' \subset \alpha V_o^n$ $(\alpha > 1)$

- vol(V) is the same observable from full theory perspective for V_o^n and αV_o^n
- similar $b(V) = b^n$ for both V_o^n and αV_o^n

$$\{b(V'), \text{vol}(V)\}_{(D,V_o^n)} = \frac{V}{V_o^n}$$

$$\{b(V'), \text{vol}(V)\}_{(D,\alpha V_o^n)} = \frac{V}{\alpha V_o^n}$$

Poisson structure itself is V_o-dependent:

$$\{\,\cdot\,,\,\cdot\,\}_D \mapsto \frac{1}{\alpha}\{\,\cdot\,,\,\cdot\,\}_D$$

V_o-dependence: 3) Hamiltonian

Inserting piecewise homogeneity into the full theory Hamiltonian:

$\mathcal{C}_a \sim$ boundary terms at ∂V_o^n

$$\mathcal{H} = \sum_n \left(-\frac{3\kappa}{4} v^n \left(b^n\right)^2 + \frac{\left(p_\phi^n\right)^2}{2v^n} \right) + \text{ boundary terms at } \partial V_o^n$$

Total Hamiltonian: $(\kappa = 8\pi G,\ c = 1)$

$$H = \int_\Sigma \mathrm{d}^3x \left(N\mathcal{H} + N^a \mathcal{C}_a\right) \approx \sum_n V_o^n N \mathcal{H}^n \ \rightarrow \text{ divergent}$$

$\Rightarrow$ restrict to one cell only!

Truncation

- Neglect of boundary contributions (cross cell interactions)
- Restrict to only one cell (neglect of modes larger than one cell)

Truncation

Full theory:

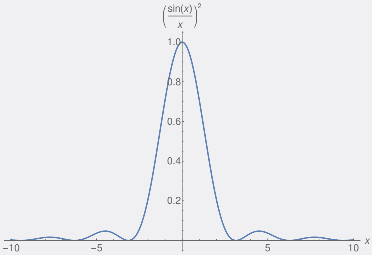
$$H = \sum_n \int_{V_o^n} \mathrm{d}^3x \, N\mathcal{H}^n$$
$$+ \text{ boundary terms at } \partial V_o^n \quad (2)$$

Symmetry reduced: (n fixed)

$$H = V_o^n N\mathcal{H}^n \quad (3)$$

(2) contains non-homogeneous modes:

$$\tilde{b}(k) = \int_\Sigma \mathrm{d}^3x \sqrt{\mathring{q}}\, b(x)\, e^{-i\vec{k}\cdot\vec{x}}$$
$$\leq C \prod_{\xi=x,y,z} \frac{\sin\left(V_o^{\frac{1}{3}} k_\xi/2\right)}{V_o^{\frac{1}{3}} k_\xi/2}$$



(3) has only $k = 0$ mode

- $V_o^{\frac{1}{3}} k_\xi \gg 1$ suppressed by homogeneity
- $V_o^{\frac{1}{3}} k_\xi \ll 1$ ignored / irrelevant for large volumes

What is the dynamical relevance of small modes?

V_o-dependence: Summary

The choice of V_o

1. affects the available subset of full theory observables
2. transforms the Poisson structure inversely $\propto 1/V_o$
3. enters linearly the Hamiltonian

- V_o labels a family of symmetry reduced theories (no canonical transformation)
- there is a well defined map between different V_o's

Implications on Classical Dynamics

(drop index n from now on)

Choose $V_o^{(1)}$ and $V_o^{(2)}$ + observable $\mathcal{O}$ defined in both

$$\dot{\mathcal{O}} = \left\{\mathcal{O}\,,\, H_T^{(1)}\right\}_{(1)} = \frac{V_o^{(2)}}{V_o^{(1)}}\left\{\mathcal{O}\,,\, \frac{V_o^{(1)}}{V_o^{(2)}} \cdot H_T^{(2)}\right\}_{(2)} = \left\{\mathcal{O}\,,\, H_T^{(2)}\right\}_{(2)}$$

$\Rightarrow$ Dynamics is independent of V_o

On-shell the regulator can be removed $V_o \to \infty$

Alternative

- Start with full theory: $G_{\mu\nu} = \kappa T_{\mu\nu}$
- Restrict to FLRW-metrics + solve the system
- construct dynamics of $\mathcal{O}$

$\to$ No reference to any V_o needed!

Classical physics is local (up to boundary conditions)

Part II: Quantum Cosmology

Quantisation

Quantisation of the V_o-family of theories

Setup

- ϕ-clock de-parametrisation:

$$p_\phi(V_o) = \sqrt{\frac{3\kappa}{2}} V_o v b =: H_{true}$$

- polymer quantisation in b
- Hilbert spaces (different for each V_o)

$$\mathscr{H}_{LQC} = L^2\left(\mathbb{R}_{Bohr}, \mathrm{d}\mu_{\mathsf{Bohr}}\right) \quad , \quad |\psi\rangle = \sum_{\nu\in\mathbb{R}} \psi(\nu) \, |\nu\rangle$$

$$\langle \nu \mid \nu' \rangle = \delta_{\nu,\nu'} \quad , \quad \langle b \mid \nu \rangle = e^{i\lambda b\nu} \quad , \quad \psi(\nu) = \langle \nu \mid \psi \rangle \; ,$$

$|\nu\rangle$: Eigenstate of $\hat{v}$

$|b\rangle$: Eigenstate of $\widehat{e^{i\nu\lambda b}}$

Operator Representations

Weyl Canoical Commutation Relations

$$\widehat{e^{-i\xi v}}\,\widehat{e^{-i\mu b}} = \widehat{e^{-i\mu b}}\,\widehat{e^{-i\xi v}}\,e^{-\frac{i\mu\xi}{V_o}}$$

$\Rightarrow$ operator representations have to contain V_o!

$$\hat{v}\,|\nu\rangle = \frac{\eta^\gamma}{V_o^\gamma}\nu\,|\nu\rangle \qquad , \qquad \widehat{e^{-i\lambda\mu b}}\,|\nu\rangle = |\nu - \frac{\lambda\mu}{\eta^\gamma V_o^\delta}\rangle$$

$\gamma + \delta = 1$, $\eta = \kappa^{3/2}$, $\mu \in \mathbb{R}$, polymerisation scale λ
Units $\hbar = 1$, $[\kappa] = L^2$, $[\nu] = [\hat{v}] = [\mu] = 1$, $[b] = [\lambda^{-1}] = L^{-3}$.

Transformation behaviour

$$\hat{v}\big|_{V_o^{(1)}} = \left(\frac{V_o^{(2)}}{V_o^{(1)}}\right)^\gamma \hat{v}\big|_{V_o^{(2)}} \;,\; \widehat{e^{-i\lambda\mu^{(1)}b}}\Big|_{V_o^{(1)}} = \widehat{e^{-i\lambda\mu^{(2)}b}}\Big|_{V_o^{(2)}} \;,$$

$$\mu^{(1)} = \left(\frac{V_o^{(2)}}{V_o^{(1)}}\right)^\delta \mu^{(2)}$$

Quantum transformation behaviour is different from classical!

Quantum Dynamics

$$i\frac{\partial}{\partial\phi}\left|\psi\right\rangle = \hat{H}_{true}\left|\psi\right\rangle \quad , \quad \left|\psi;\phi\right\rangle = e^{i\phi\hat{H}_{true}}\left|\psi\right\rangle$$

Hamiltonian

Regularisation [Martín-Benito, Maruguán, Olmedo '09]:

$$\hat{H}_{true} = \sqrt{\frac{3\kappa}{2}}V_o\sqrt{|\hat{v}|}\left(\widehat{\frac{\sin(\lambda b)}{2\lambda}}\text{sign}(\hat{v}) + \text{sign}(\hat{v})\widehat{\frac{\sin(\lambda b)}{2\lambda}}\right)\sqrt{|\hat{v}|}$$

Action:

$$\hat{H}_{true}\psi(\nu) = \frac{i}{4}\sqrt{\frac{3\kappa}{2}}\cdot\Big(s_+(n)\sqrt{|n||n+1|}\psi(\theta\cdot(n+1)) - s_-(n)\sqrt{|n||n-1|}\psi(\theta\cdot(n-1))\Big)$$

$$\nu = \frac{\lambda}{\eta^\gamma V_o^\delta}n \quad , \quad n\in\mathbb{R} \quad , \quad \theta = \frac{\lambda}{\eta^\gamma V_o^\delta} \quad , \quad s_\pm(n) = \text{sign}(n\pm 1)+\text{sign}(n)$$

Eigenstates depend only on $n = \nu/\theta$: Look for $\Psi_E : \mathbb{R}\to\mathbb{C}$ with

$$-\frac{i}{2}\sqrt{\frac{3\kappa}{2}}\cdot\Big(s_+(n)\sqrt{|n||n+1|}\Psi_E(n+1) - s_-(n)\sqrt{|n||n-1|}\Psi_E(n-1)\Big) = E\Psi_E(n)\ .$$

It follows: $\psi_E(\nu) = \Psi_E\left(\frac{\nu}{\theta}\right)$

Quantum Dynamics preserving Isomorphism

Consider two quantisations with $V_o^{(1)}$ and $V_o(2)$ Eigenstates:

$$\psi_E^{(1)}(\nu) = \Psi_E\left(\frac{\nu}{\theta^{(1)}}\right) = \Psi_E\left(\frac{\theta^{(2)}}{\theta^{(1)}}\frac{\nu}{\theta^{(2)}}\right) = \psi_E^{(2)}\left(\left(\frac{V_o^{(1)}}{V_o^{(2)}}\right)^{\delta}\nu\right)$$

Identification of E-eigenstates in different Hilbert spaces

Isomorphism

$$\mathscr{I}\,:\,\mathscr{H}_{LQC}^{(1)}\longrightarrow\mathscr{H}_{LQC}^{(2)}\quad\text{by}\quad\psi^{(1)}\longmapsto\psi^{(2)}=\mathscr{I}\left(\psi^{(1)}\right)$$

$$\psi^{(2)}(\nu)=\psi^{(1)}\left(\left(\frac{V_o^{(2)}}{V_o^{(1)}}\right)^{\delta}\nu\right)\;.$$

Dynamics of $\psi^{(1)}$ and $\psi^{(2)}$ is the same as

$$\left\langle\psi_E^{(1)}\,\middle|\,\psi^{(1)}\right\rangle_{(1)}=\left\langle\psi_E^{(2)}\,\middle|\,\psi^{(2)}\right\rangle_{(2)}\quad\forall E$$

Quantum dynamics can be made V_o independent!

Quantum Fluctuations

A quantum theory is more than dynamics!

Expectation values under $\mathscr{I}$

For $\psi^{(2)} = \mathscr{I}\left(\psi^{(1)}\right)$:

$$\left\langle \hat{H}_{true}\Big|_{V_o^{(1)}} \right\rangle_{\psi^{(1)}} = \left\langle \hat{H}_{true}\Big|_{V_o^{(2)}} \right\rangle_{\psi^{(2)}}$$

$$\left\langle \hat{v}|^n_{V_o^{(1)}} \right\rangle_{\psi^{(1)}} = \left(\frac{V_o^{(2)}}{V_o^{(1)}}\right)^n \left\langle \hat{v}|^n_{V_o^{(2)}} \right\rangle_{\psi^{(2)}}$$

$$\left\langle \widehat{e^{-i\lambda\mu b}}\Big|^n_{V_o^{(1)}} \right\rangle_{\psi^{(1)}} = \left\langle \widehat{e^{-i\lambda\mu b}}\Big|^n_{V_o^{(2)}} \right\rangle_{\psi^{(2)}}$$

Fiducial Cell and Uncertainty Relation

[Ashtekar, Bojowald, Lewandowski '03]

Recall $\widehat{\mathsf{vol}(V_o)} = V_o \hat{v}$

$$\left\langle V_o^{(1)} \hat{v}\Big|_{V_o^{(1)}}^{n} \right\rangle_{\psi^{(1)}} = \left\langle V_o^{(2)} \hat{v}\Big|_{V_o^{(2)}}^{n} \right\rangle_{\psi^{(2)}}$$

Invariant, **but** observable changes!

Uncertainty relation: [Rovelli, Wilson-Ewing '14]

$$\Delta_\psi \left. \widehat{\mathsf{vol}\,(V_o)} \right|_{V_o} \Delta_\psi \left. \frac{\widehat{\sin(\lambda b)}}{\lambda} \right|_{V_o} \geq \frac{1}{2} \left| \left\langle \left. \widehat{\cos(\lambda b)} \right|_{V_o} \right\rangle_\psi \right|$$

When ψ saturates the inequality, $\mathscr{I}(\psi)$ does to, too!

Removing the Regulator

Volume of the fiducial cell is independent of V_o: $(\psi = \delta_{\nu,\nu_o})$

$$\left\langle \widehat{\text{vol}\,(V_o)} \right\rangle_{\delta_{\nu,\nu_o}} = V_o \frac{\eta^\gamma}{V_o^\gamma} \nu_o \overset{\gamma+\delta=1}{=} V_o^\delta \eta^\gamma \nu_o = \lambda n_o$$

Homogeneity on full Σ is obtained for ψ with

$$\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \to \infty$$

Sub-volumes

A reasonable observable is $V \subset V_o$

$$\left\langle \widehat{\text{vol}(V)} \right\rangle_\psi = \frac{V}{V_o} \left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \xrightarrow{\frac{V}{V_o} \to 0\,,\ \left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \to \infty} \text{finite}$$

No fluctuations left:

$$\Delta_\psi \widehat{\text{vol}(V)} \Delta_\psi \frac{\widehat{\sin(\lambda b)}}{\lambda} \geq \frac{V}{2\,V_o} \left| \widehat{\cos(\lambda b)} \right| \to 0$$

Conclusions

Classical theory

- Understanding of the truncation
- Is local and dynamics is independent of V_o
- Fully homogeneous spatial slices can be considered

Quantum theory

- Isomorphism $\mathscr{I}$ makes dynamics V_o independent
- Quantum fluctuations depend on the physical size $\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi$
- Full homogeneity is obtained by choosing ψ s.t. $\left\langle \widehat{\text{vol}(V_o)} \right\rangle_\psi \to \infty$
- Quantum fluctuation of finite volumes become arbitrarily small

Future Directions

- **What is the physical scale of homogeneity?**
- What is the role of inhomogeneities? [Bojowald '20; ...]
- Take renormalisation into account [Bodendorfer, Han, Haneder 21; Bodendorfer, Wuhrer 20; Bodendorfer, Haneder 19;]

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Thank you for your attention!