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Singular story of non-singular spacetimes with the NUT parameter

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based on:

Jerzy Lewandowski, MO, 2020 Class. Quantum Grav. 37 205007

Jerzy Lewandowski, MO, 2020 Phys. Rev. D 102, 124055

Jerzy Lewandowski, MO, arXiv:arXiv:2101.05802 2021, accepted at Phys. Rev. D

Schwarzschild & Singularities

Let's start with basics:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$f(r) = 1 - \frac{2m}{r} \implies$ singularity at r= 2m ?
$(t, r, \theta, \phi) \in \mathbb{R} \times (2m, \infty) \times S^2$?

Solution: (outgoing) Eddington–Finkelstein coordinates:
$dt = du + f(r)^{-1}\,dr$

$$ds^2 = -f(r)\,du^2 - 2\,du dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$(u, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_{>0} \times S^2$

Works also with charges, rotation, Λ and NUT parameter.

Taub-NUT & Singularities

The Taub-N(ewman)-U(nti)-T(amburino) metric tensor (1963):
l- NUT parameter.

$$ds^2 = -f(r)(dt^{\circ} + 2l\cos\theta\, d\phi)^2 + \frac{dr^2}{f(r)} + (l^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2} \quad , r_{\pm} = m \pm \sqrt{m^2 + l^2}$$

The poles $\theta = 0$ and $\theta = \pi$ are not regular.
But $r = 0$ is! $\implies$ Taub-NUT is a smooth "regularizer" of Schwarzschild.

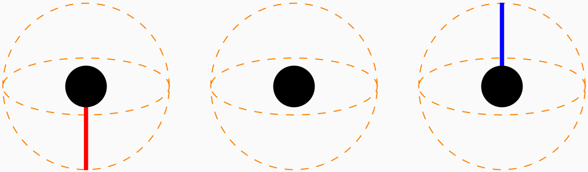
Cosmic string along the z axis (Bonnor 1969).

Taub-NUT & Misner's interpretation

Another solution: Misner glueing (1963)

$$t^\circ = t + 2l\phi$$
$$t^\circ = t' - 2l\phi$$

$$ds^2 = -f(r)(dt + 2l(\cos\theta - 1)\,d\phi)^2 + \frac{dr^2}{f(r)} + (l^2 + r^2)(d\theta^2 + \sin^2\theta\,d\phi^2),$$
$$ds'^2 = -f(r')(dt' + 2l(\cos\theta' + 1)\,d\phi')^2 + \frac{dr'^2}{f(r')} + (l^2 + r'^2)(d\theta'^2 + \sin^2\theta'\,d\phi'^2)$$



Price: $t = t' - 4l\phi \implies (t, r, \theta, \phi) \in [0, 8\pi l) \times \mathbb{R} \times S^2$

Taub-NUT & cosmological interpretation

When CTC are a problem?

$g_{t^\circ t^\circ} = -f(r) = -\frac{r^2-2mr-l^2}{r^2+l^2}, \quad g_{rr} = f(r)^{-1}, \quad r_\pm = m \pm \sqrt{m^2+l^2}$
∂_{t° is timelike for $r > r_+$ or $r < r_-$ ($\text{NUT}_\pm$ regions)
∂_{t° is spacelike otherwise (Taub region)

Cosmological coordinates:

$(t^\circ, r) = (2l(\psi' - \phi), \tau)$, possibly with $d\psi' = d\psi - \frac{1}{2l} f(r)^{-1}\, dr$

$$ds^2 = -\frac{d\tau^2}{-f(\tau)} + (2l)^2(-f(\tau))(d\psi' - \cos\theta)^2 + (\tau^2 + l^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\int_{r_-}^{r_+} \frac{d\tau}{\sqrt{-f(\tau)}} = \int_{r_-}^{r_+} \frac{\sqrt{r^2+l^2}}{\sqrt{r-r_-}\sqrt{r_+-r}}\, dr$$

$\implies$ homogeneous cosmological model of Bianchi type IX, i.e. with the symmetry group SU(2)

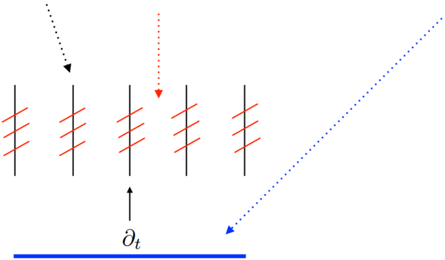
Taub-NUT as a U(1)-principal bundle

singular at $\theta = \pi$

$$-f(r)\left(dt + 2l\left(\cos\theta - 1\right)d\phi\right)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

singular at $\theta = 0$

$$-f(r)\left(dt' + 2l\left(\cos\theta + 1\right)d\phi\right)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2)$$



U(1)-principal bundle: $\mathbb{R} \times S^3 \to \mathbb{R} \times S^2$

Taub-NUT in 1-forms

Hopf fibration: $S^3 \xrightarrow{\Pi} S^2$ Left (X^μ) and right (Y^μ) invariant vector fields and 1-forms in Euler angles:

$$
\begin{aligned}
X_1 &= \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \frac{\sin\psi}{\tan\theta}\partial_\psi, \\
X_2 &= -\sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \frac{\cos\psi}{\tan\theta}\partial_\psi, \\
X_3 &= \partial_\psi, \\
Y_3 &= \partial_\phi. \\
\omega^1 &= \cos\psi\, d\theta + \sin\theta\sin\psi\, d\phi, \\
\omega^2 &= -\sin\psi\, d\theta + \sin\theta\cos\psi\, d\phi, \\
\omega^3 &= d\psi + \cos\theta\, d\phi,
\end{aligned}
$$

$$
ds^2 = -f(r)4l^2(\omega^3)^2 + \frac{(\omega^0)^2}{f(r)} + (l^2 + r^2)\left((\omega^1)^2 + (\omega^2)^2\right).
$$

Kerr-NUT-(anti-) de Sitter

$$g = -\frac{\mathcal{Q}}{\Sigma}(dt - A\,d\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(a\,dt - \rho\,d\phi)^2,$$

$$\Sigma = r^2 + (l + a\cos\theta)^2,$$

$$A = a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta,$$

$$\rho = r^2 + (l+a)^2 = \Sigma + aA,$$

$$\mathcal{Q} = (a^2 - l^2) - 2mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big),$$

$$P = 1 + \frac{4}{3}\Lambda al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.$$

a - Kerr rotation parameter.

- $\Sigma = 0$: ring-like singularity, but only for $l^2 < a^2$!
- Lorentzian signature $\implies P > 0$
- $A\,d\phi$ is singular at $\theta = \pi$
- $\mathcal{Q}(r_0) = 0$: Killing horizons (up to 4)
- Killing vector fields: ∂_t, ∂_ϕ

General bundle structure

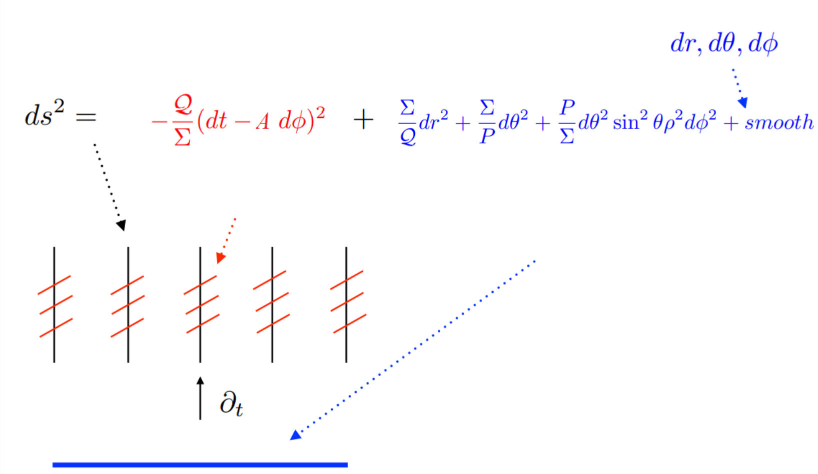
U(1)-principal bundle:

$$\mathbb{R} \times S^3 \xrightarrow{\Pi} \mathbb{R} \times S^2$$

- ▷ ξ - generator of $U(1)$-action $\mathbb{R} \times S^3$
- ▷ ω - connection 1-form on $\mathbb{R} \times S^3$
- ▷ $f : \mathbb{R} \times S^2 \to \mathbb{R}$
- ▷ q - 3D metric tensor on the space of the orbits

$$g = -(\Pi^* f)\omega \otimes \omega + \Pi^* q, \quad \mathcal{L}_\xi g = 0, \quad \xi^\mu \xi_\mu = -(\Pi^* f)$$

Kerr-NUT-(anti-) de Sitter as a



New problem: base space may have conical singularity.
Removable iff $P(0) = P(\pi)$. i.e. $al\Lambda = 0$.

Eddington-Finkelstein-like coordinates

$$dv := dt + \frac{\rho}{\mathcal{Q}}dr, \quad d\tilde{\phi} := d\phi + \frac{a}{\mathcal{Q}}dr.$$

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}(dv - A d\tilde{\phi})^2 + 2dr(dv - A d\tilde{\phi}) + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(a dv - \rho d\tilde{\phi})^2.$$

Misner gluing along suitable Killing vector field.

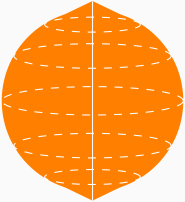
$$\xi = \partial_v + b\partial_{\tilde{\phi}}, \qquad b = \text{const.}$$

$$\xi(\tau) = 1, \quad \xi(x^i) = 0 \quad , (x^\mu) = (\tau, x^i) = (v,\ r,\ \theta,\ \hat{\phi} := -bv + \tilde{\phi})$$

$$\xi = \partial_\tau$$

$(v, r, \theta, \hat{\phi}) \in I_\tau \times \mathbb{R} \times [0, \pi) \times I_{\hat{\phi}}$

Consier q pullbacked to $r =$const.



$$\lim_{\to \text{poles}} \frac{\text{circumference}}{\text{radius}} = 2\pi$$

Continuity condition

$$P(0) = \frac{P(\pi)}{|1-4lb|}, \quad P = 1 + \frac{4}{3}\Lambda a l \cos\theta + \frac{\Lambda}{3} a^2 \cos^2\theta.$$

The condition is r independent! $\varphi = P(0)\,\hat{\phi}$

… solving the continuity condition …

Continuity condition

$$P(0) = \frac{P(\pi)}{|1 - 4lb|}, \quad P = 1 + \frac{4}{3}\Lambda al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.$$

- $l = 0$ - all Killing vectors are fine.
- $l \neq 0, a\Lambda = 0 \implies b = 0\ \xi = \partial_t$.
- $l \neq 0$ - two "principal" Killing vector fields.

$$b_+ = \frac{2a\Lambda}{3 + a^2\Lambda + 4al\Lambda} \quad , b_- = \frac{3 + a^2\Lambda}{2l(3 + a^2\Lambda + 4al\Lambda)}$$
$$\xi_+ = \partial_v + b_+\partial_{\tilde{\phi}}, \qquad \xi_- = \partial_v + b_-\partial_{\tilde{\phi}}$$

… gluing metrics.

$$\omega = d\tau - \frac{(1-Ab)\Sigma dr + \big(A\mathcal{Q}(1-Ab) - P\sin^2\theta\rho(a-b\rho)\big)\, d\varphi/P(0)}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2}.$$

where $A d\phi = (a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta)d\phi$ is well defined at $\theta = 0$, fails at $\theta = \pi$.

$$\omega_\varphi(r, \theta = 0) = 0, \quad \omega_\varphi(r, \theta = \pi) = -\frac{4l}{1-4lf}\frac{1}{P(0)} = \text{sgn}(1-4lb)\frac{-4l}{P(\pi)}.$$

$\omega_\phi(r, \theta = \pi)$ is r-independent!

$$\tau = \tau' + \frac{4l}{P(\pi)}\varphi'$$

We have continuity. What about smoothness? Check by inspection.

(Projectively) non-singular Killing horizons

Projective non-singularity

$\iff$ space of null generators in non-singular.

We want $S^3 \xrightarrow{\Pi} S^2$, where S^2 is a space of null generators.
Necessarily $b = \frac{a}{r_0^2+(a+l)^2}$, i.e. the Killing vector field developing a horizon

- $\Lambda = \frac{3}{a^2+2l^2+2r_0^2} \Rightarrow$ only de Sitter spacetimes
- At most 1 horizon is non-singular - the one with $r = r_0$
- Always outer-/innermost - but possibly with $m < 0$ due to symmetry $(r, m) \leftrightarrow (-r, -m)$
- Always non-extremal.

Cosmological interpretation

Can we get

$$ds^2 = -\frac{\Sigma}{-\mathcal{Q}}dr^2 + \frac{-\mathcal{Q}}{\Sigma}(dt - A\,d\phi)^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(a\,dt - \rho\,d\phi)^2$$

such that $\mathcal{Q} < 0$ and $l^2 > a^2$ (no curvature singularity)?
Yes: for $\Lambda > 0$, sufficiently small m and some inequalities on $a,\ l,\ \Lambda$.
Inhomogeneous, non-extendable cosmological models without CTC.

Scri and principal killing vector fields

Geometry of the scri:

$$\frac{\Lambda}{3}\left((1-bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} + P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2$$

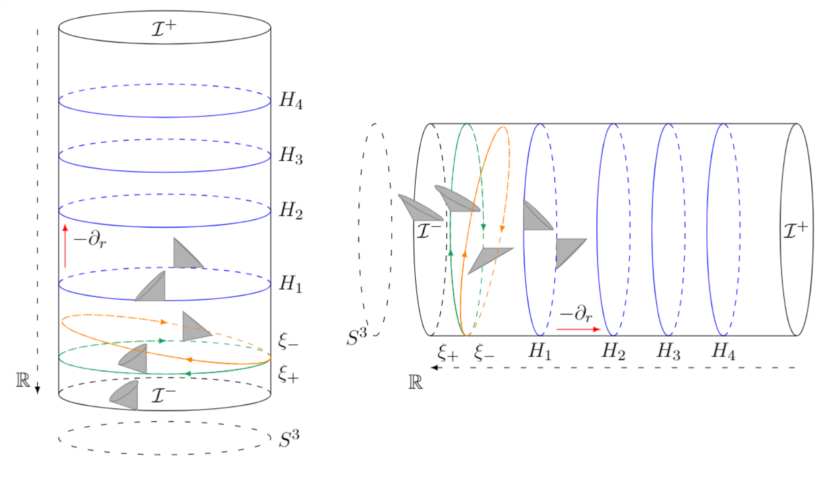
What about ξ_- ?

For Taub-NUT $\xi_+ = \tau,\ \xi_- = \partial_v + \frac{1}{2l}\partial_{\tilde{\phi}}$.

Left and right invariant fields.

In general: ξ_+ and ξ_- define the same orbits ans thus equivalent gluings.

Global structure $\Lambda > 0$ || $\Lambda < 0$



Accelerated KN(a)dS

Physical parameters $(a, m, l, \Lambda, \alpha)$. α - acceleration. ω residual gauge.

$$ds^2 = \frac{1}{F^2}\left\{ -\frac{\mathcal{Q}}{\Sigma}(dt - A d\phi)^2 + \frac{\Sigma}{\mathcal{Q}} dr^2 + \frac{\Sigma}{P} d\theta^2 + \frac{P}{\Sigma}\sin^2\theta (a dt - \rho d\phi)^2 \right\}$$

$$F = 1 - \frac{\alpha}{\omega}(l + a\cos\theta) r, \quad P = 1 - a_3\cos\theta - a_4\cos^2\theta,$$

$$\mathcal{Q} = \omega^2 k - 2mr + \epsilon r^2 - 2\frac{\alpha n}{\omega} r^3 - (\alpha k + \frac{1}{3}\Lambda) r^4,$$

$$a_3 = 2\frac{\alpha a m}{\omega} - \alpha^2 a l k - \frac{4}{3}\Lambda a l, \quad a_4 = -\alpha^2 a^2 k - \frac{1}{3}\Lambda a^2,$$

$$\epsilon = \frac{\omega^2 k}{(a^2 - l^2)} + 4\frac{\alpha l m}{\omega} - (a^2 + 3l^2)(\alpha^2 k + \Lambda/3),$$

$$n = \frac{\omega^2 k l}{(a^2 - l^2)} - \frac{\alpha m (a^2 - l^2)}{\omega} + (a^2 - l^2)(\alpha^2 k + \Lambda/3),$$

$$k = \frac{1 + 2\alpha l m/\omega - l^2\Lambda}{3\alpha^2 l^2 + \omega^2/(a^2 - l^2)}.$$

Accelerated KN(a)dS

Useful gauge: $\omega = \frac{a^2+l^2}{a}$.
Many results still hold (bundle structure, gluing, equivalence of principal vector fields …).
Continuity condition for $l = 0$ reduces to $m\alpha = 0$, but with $l \neq 0$ accelerated Kerr-NUT-(anti-) de Sitter spacetime can be without conical singularity.

The end

Thank you for your attention!