Gluing variations

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based on joint work with Wan Cong
arXiv:2302.06928 [gr-qc]

Questions

Throughout this work: vacuum spacetimes with $\Lambda \in \mathbb{R}$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,.$$

Can you

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2. glue together vacuum spacetimes?
3. realise *data on lower dimensional submanifolds* by embedding in a vacuum spacetime?
4. ??? data on lower dimensional submanifolds ???

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Data on lower dimensional manifolds?

The Aretakis-Czimek-Rodnianski question

QUESTION (Aretakis, Czimek and Rodnianski (2021))

Can you find vacuum characteristic initial data interpolating between two **sphere data sets***?*

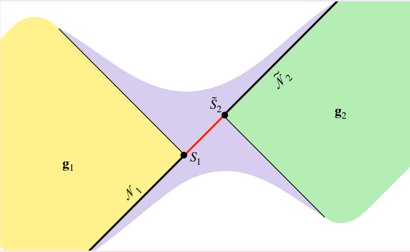

Figure: Gluing construction of Aretakis-Czimek-Rodnianski

Answer: “kind of”, for sphere data near spheres lying on a Minkowskian light cone

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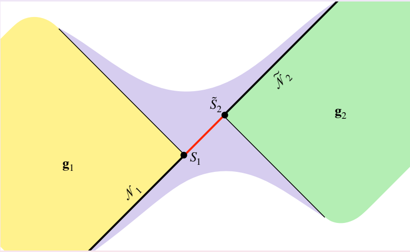

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Let P be a submanifold of M.
Let $k \in \mathbb{N} \cup \{\infty\}$.

Definition

Let g be any smoothly differentiable metric defined in a neighborhood of P. The collection

$$j^k g := \{\partial_{\alpha_1} \cdots \partial_{\alpha_\ell} g_{\mu\nu}|_P \,,\ 0 \le \ell \le k\}$$

will be called *jet of order k of g at P*.

Einstein equations *and their derivatives up to order $k-2$* provide equations, differential and/or algebraic, relating the jets of order k at P.
A jet will be called vacuum if all such equations are satisfied.

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A jet of order k of a metric g will be called $\begin{cases} \textit{spacelike}, \\ \textit{timelike}, \\ \textit{null}, \end{cases}$ if

the metric induced by g on P is.

We similarly define spacelike/timelike/null/characteristic vacuum submanifold data of order k at P.

Question

Given a submanifold $P \subset M$ and a vacuum jet $j^k g$ in $\Psi[P,k]$, is there a vacuum metric on M which realises $j^k g$?

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Let P be a point, $P = \{p\}$

$j^k g =$ the coefficients of the Taylor series of a metric g at p.

vacuum $\equiv$ algebraic conditions on the Taylor coefficients

Now: *in normal coordinates the Taylor coefficients can be expressed in terms of the Riemann tensor and its covariant derivatives.*

For example, using normal coordinates,

$$j^2 g|_p \approx \{g_{\mu\nu}|_p, W^\alpha{}_{\beta\gamma\delta}|_p\}\,,$$

where $W^\alpha{}_{\beta\gamma\delta}$ has the symmetries of the Weyl tensor.

$\Psi[\{p\}, 2] \approx$ all such pairs

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Is there a vacuum metric which realises $\Psi[\{p\},]$?

Data on lower dimensional manifolds

Example: Vacuum jets at a point

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Initial data surface Σ, Riemannian metric g_{ij}, $i, j = 1, \ldots n$, symmetric tensor K_{ij} ("initial time derivative of the metric")

the scalar constraint equation (Λ is the *cosmological constant*):

$$\boxed{R(g_{ij}) = 2\Lambda + |K|^2 - (\mathrm{tr} K)^2 \ ,}$$

and the *vector constraint equation*:

$$\boxed{D_j K^j{}_k - D_k K^j{}_j = 0 \ .}$$

Corollary

Spacelike vacuum hypersurface data

$$\Psi[\Sigma, \infty] \approx \Psi[\Sigma, k] \approx \Psi[\Sigma, 2] \approx \{\textit{all vacuum } (g, K)\}.$$

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Extension theorem

Can one extend vacuum spacelike initial data on a manifold with boundary beyond the boundary?

Theorem

Let (Σ, g, K) be spacelike vacuum initial data on a manifold with boundary $\partial\Sigma$.
There exists a manifold without boundary $\check{\Sigma}$ and vacuum initial data $(\check{g}, \check{K})$ on $\check{\Sigma}$ such that $\Sigma \subset \check{\Sigma}$, with

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Characteristic Cauchy problem

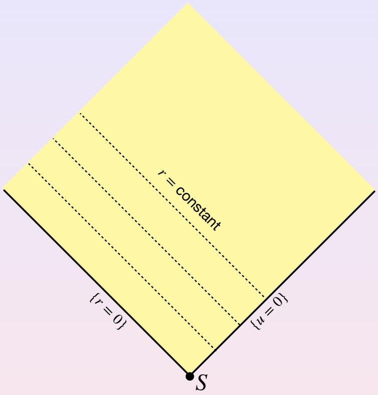
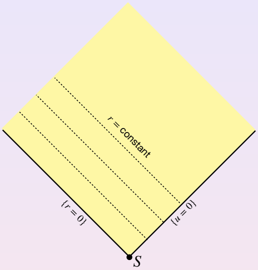


Figure: Characteristic Cauchy problem with intersecting null hypersurfaces

Characteristic Cauchy problem

Isenberg-Moncrief coordinates



The hypersurfaces $\mathscr{N} = \{u = 0\}$ and $\underline{\mathscr{N}} = \{r = 0\}$ are characteristic for the metric

$$\mathbf{g}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = 2\left(-\mathrm{d}u + u\alpha\mathrm{d}r + u\beta_A\mathrm{d}x^A\right)\mathrm{d}r + g_{AB}\mathrm{d}x^A\mathrm{d}x^B\,. \tag{0.1}$$

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For each set of characteristic initial data

$$\beta_A \text{ on } \mathbf{S} := \mathscr{N} \cap \underline{\mathscr{N}} \text{ and } (\alpha, g_{AB}) \text{ on } \mathscr{N} \cup \underline{\mathscr{N}}\,,$$

subject to the Raychaudhuri equation on $\mathscr{N} \cup \underline{\mathscr{N}}$,

$$0 = -\frac{1}{2}g^{AB}\partial_r^2 g_{AB} + \frac{1}{4}g^{CA}g^{BD}(\partial_r g_{AB})\partial_r g_{CD} + \frac{1}{2}\alpha\, g^{AB}\partial_r g_{AB} \tag{0.2}$$

there exists a vacuum metric in a future neighborhood of $\mathscr{N} \cup \underline{\mathscr{N}}$.

Embedding a null hypersurface in a vacuum spacetime?

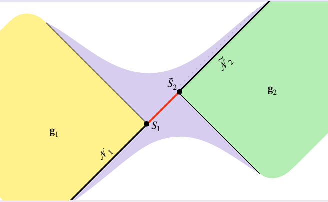

Figure: Gluing construction of Aretakis-Czimek-Rodnianski

QUESTION

What about a vacuum metric in a whole neighborhood of $\mathscr{N} \cup \underline{\mathscr{N}}$? What about a single characteristic hypersurface?

Answer: one needs to understand data of order k on $\mathscr{N}$.

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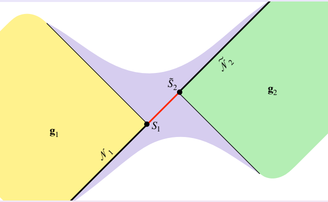

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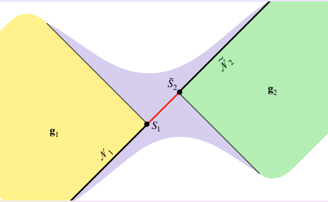

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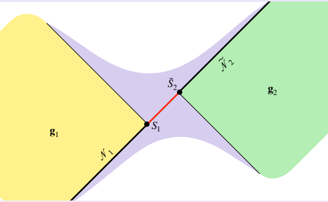

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Proposition

In the Isenberg-Moncrief coordinate system the vacuum characteristic initial data $\Psi(\mathscr{N}, k)$ can be reduced to

$$\Phi_{\mathrm{IM}}[\mathscr{N}, k] := \{(\partial_u{}^j g_{AB}, \beta_A)_{0\le j\le k} \text{ on } \mathbf{S} \text{ and } (g_{AB}, \alpha) \text{ on } \mathscr{N}\}\,, \tag{0.3}$$

where **S** *is a cross-section of $\mathscr{N}$.* □

Proof: transverse derivatives of the metric on a characteristic hypersurface $\mathscr{N}$ are determined uniquely by the above data through ODEs along the null geodesics threading $\mathscr{N}$ or through algebraic equations.

Note: This is true both to the future or to past along the generators.

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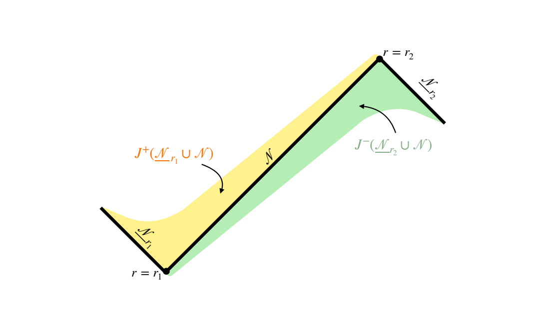
The “hand-crank construction”



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Proof: Use the crank:

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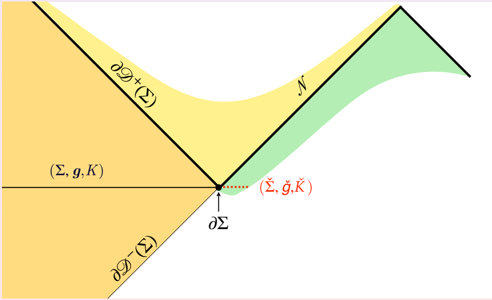

Corollary: Embedding a truncated cone

Use the spacelike data extension

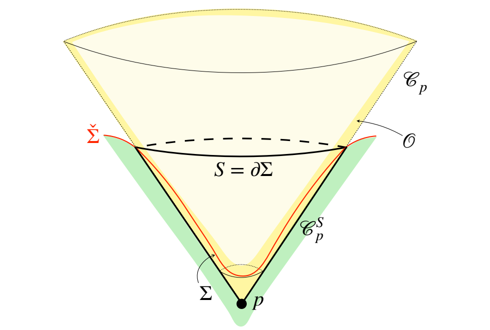

Figure: Extending a vacuum metric on a truncated future cone $J^+(p) \cap J^-(\mathscr{S})$ to a neighborhood thereof.

Corollary: Embedding data at a point

Use the embedding of a cone

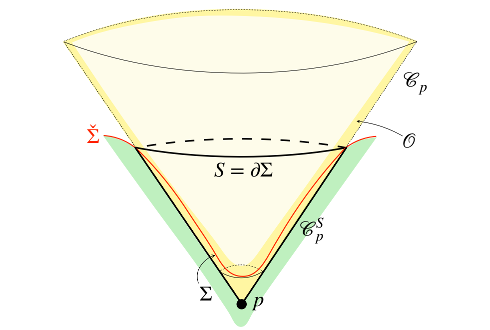

Figure: Extending data at a point

Corollary: Extending a characteristic future development

The Fledermaus = two cranks (and a spacelike extension if need be)

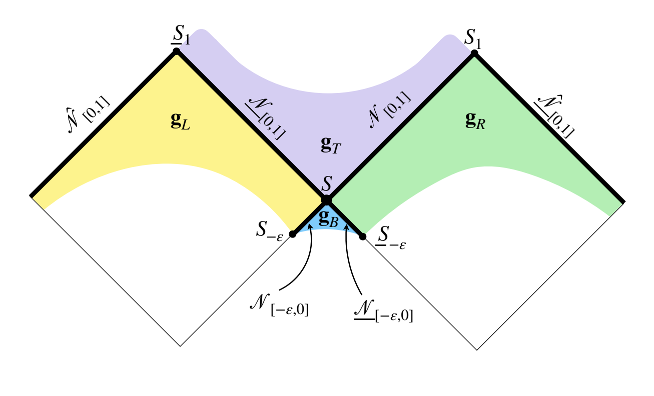

Figure: The Fledermaus.