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Conditional Probabilities with Evolving Observables and the Problem of Time In Quantum Gravity

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INTRODUCTION

There is by now extensive literature addressing the problem of time in classical and quantum gravity (e.g., Kuchař ‘s review, IJMPD 20, 3 (2011)).

The heart of the problem lies in the fact that Einstein’s theory is a totally constrained system whose *Hamiltonian vanishes*, it is a constraint. Since observable quantities are those that commute with the constraints (Dirac Observables) they therefore, *do not evolve*.

We will propose a solution to this in this talk but before that we will discuss two previous proposals to solve this problem.

Both have in common their relational character. In fact, one of the basic ingredients in the different proposals to describe evolution is the use of *relations* between different degrees of freedom in the theory .

- *Evolving Dirac observables.* (Bergmann, DeWitt, Rovelli, Marolf…) also known as *evolving constants of the motion*.

- *Conditional probabilities approach* proposed by Page and Wootters.

We will see that both approaches present problems and do not provide a completely satisfactory solution to the issue of the evolution.

Problems are particularly acute when we try to compute propagators or assign probabilities to histories.

We will show that **a combination of both approaches** addresses the problems they present.

1) Evolving Dirac Observables in totally constrained systems:

$$S = \int [p_a \dot{q}^a - \mu^\alpha \phi_\alpha(q,p)] d\tau$$

In the case of GR, the constraints are first class

$$\phi_\alpha(q,p) = 0$$

$$\{\phi_\alpha(q,p), \phi_\beta(q,p)\} = C^\gamma_{\alpha\beta} \phi_\gamma(q,p)$$

$$H_T = \mu^\alpha \phi_\alpha(q,p)$$

The Hamiltonian vanishes: the generator of the evolution also generates gauge transformations

Dirac observables are gauge invariant quantities

$$\{O(q,p), \phi_\beta(q,p)\} \approx 0 \qquad \{O(q,p), H_T(q,p)\} \approx 0$$

Therefore, they are constants of the motion.

The issue of time: If the physically relevant quantities in totally constrained systems as general relativity are constants of the motion, how can we describe the evolution?

Evolving Dirac observables (evolving constants of the motion): Bergmann, DeWitt, Rovelli, Marolf …

$$\{Q_i(t), \phi_\alpha\} \approx 0 \qquad\qquad Q_i(t, q^a, p_a)\big|_{t=q^1} = q_i$$

For instance, for the relativistic particle. $\phi = p_0^2 - p^2 - m^2$

Two independent observables:

$$p, X \equiv q - \frac{p}{\sqrt{p^2+m^2}} q^0, \qquad\qquad Q(t, q^a, p_a) = X + \frac{p}{\sqrt{p^2+m^2}} t$$

$$Q(t = q^0, q^a, p_a) = q$$

Problem: The issue of the parameter t

Evolving observables depend on a real parameter t. That is, we are assuming that there is an external quantity t , that is not represent by any quantum operator nor belongs to any physical Hilbert space.

One may wonder about the meaning of the condition $q^0=t$. On the one hand, t is supposed to be a parameter whereas q^0 is one of the canonical variables.

Generically, the latter will not be a well-defined quantity at a quantum level

$$q^0 \,|\psi>_{ph} \notin H_{ph}$$

2) Conditional probabilities.

The second alternative we want to consider is a description of the evolution in terms of conditional probabilities.

The idea is that one promotes all variables to quantum operators and computes conditional probabilities among them. This idea appears simple, natural and attractive in a closed system.

Unfortunately, one runs into problems due to the totally constrained nature of gravity. Which variables to promote? Dirac observables? Page and Wootters proposed using kinematical variables, not Dirac observables. That way they had some form of evolution. **Phys.Rev.D27:2885,(1983)**

Kuchař in his review on the problem of time, noted that this procedure faces important difficulties, in particular it does not lead to the correct propagators in model systems. The root of the problem is the distributional nature of physical states and the attempt to compute expectation values of kinematical quantities with them.

3) Conditional probabilities in terms of evolving Dirac observables.

As we have seen, both approaches require the use of variables which are not defined in the physical space.

Here we will elaborate upon a different approach where all reference to external parameters is abolished, and evolving constants are used to define correlations between Dirac observables in the theory.

R. Gambini, R. Porto, JP, S. Torterolo PRD 79, 041501 (2009).

We propose to revisit the Page-Wootters construction by **computing relational probabilities among evolving Dirac observables.** The latter are well defined on the physical space of states of the theory and are quantities that one can expect to observe and to be represented by well defined self-adjoint quantum operators.

First you choose an evolving observable as your clock, let us call it *T(t)* .
Then one identifies the set of observables $O_1(t)\ldots O_N(t)$ that commute with T and describes the physical system whose evolution one wants to study and computes

$$\mathcal{P}\left(O \in [O_0 - \Delta O, O_0 + \Delta O] | T \in [T_0 - \Delta T, T_0 + \Delta T]\right) = \lim_{\tau\to\infty} \frac{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}\left(P_{O_0}(t) P_{T_0}(t) \rho P_{T_0}(t)\right)}{\int_{-\tau}^{\tau} dt \, \mathrm{Tr}\left(P_{T_0}(t)\rho\right)}$$

In other words, *t* is the parameter associated to the variable used to define the evolving observables. This variable is treated as an ideal quantity that we do not need to observe.

We have shown that for the example considered by Kuchař (parameterized two particles), this indeed yields the correct propagators.

How does one make contact with ordinary quantum mechanics?

Given a system, first of all you choose some physical variable as your "clock", let us call it T. Such variable will be represented by a quantum operator. Then you choose the variables that will describe the physical system under study. Generically we call them X. One then computes:

$$P(< X >= x_0 |< T >= t_0) = \frac{\int dt \,\mathrm{Tr}(P_x(t) P_T(t) \rho P_T(t))}{\int dt \,\mathrm{Tr}(P_T(t)\rho)}$$

That is, the conditional probability that X takes a value x_0 when T takes a value t_0. The quantity t in the right-hand side will become the "ideal" t of Schrödinger's theory. In the Schrödinger picture the density matrices (quantum states) evolve with the traditional Schrödinger equation we only ask different questions about them than usual. Let us see how that emerges.

How does quantum evolution look like when one casts it in terms of T rather than t? Here one needs to make some assumptions. We assume that the density matrix can be written as a direct product of that of the clock and that of the system under study and that one has a unitary independent evolution for the clock and the system,

$$\rho = \rho_{\rm cl} \otimes \rho_{\rm sys} \quad U = U_{\rm cl} \otimes U_{\rm sys}$$

We also define the probability density that the clock variable takes the value T when the "ideal time" takes the value t,

$$\mathcal{P}_t(T) \equiv \frac{{\rm Tr}\left(P_T(0) U_{\rm cl}(t) \rho_{\rm cl} U_{\rm cl}(t)^\dagger\right)}{\int_{-\infty}^{\infty} dt\, {\rm Tr}\left(P_T(t)\rho_{\rm cl}\right)},$$

And define an evolution in terms of the variable T,

$$\rho(T) \equiv \int_{-\infty}^{\infty} dt U_{\rm sys}(t) \rho_{\rm sys} U_{\rm sys}(t)^\dagger \mathcal{P}_t(T)$$

With these identifications we can rewrite the conditional probability as an ordinary probability in quantum mechanics for the density matrix ρ(T)

$$P(O|T) = \frac{\mathrm{Tr}(P_O \rho(T))}{\mathrm{Tr}(\rho(T))}$$

To get something closer to the usual Schrödinger equation, we assume that the probability for the clock is quite peaked,

$$\mathcal{P}_t(T) = \delta(T-t) + a(T)\delta'(T-t) + b(T)\delta''(T-t) + \dots$$

Which gives for the evolution,

$$\rho(T) = \rho_{\mathrm{sys}}(T) + a(T)[H, \rho_{\mathrm{sys}}(T)] - b(T)[H, [H, \rho_{\mathrm{sys}}(T)]],$$

And the differential equation that gives the time evolution of the density matrix is given by,

$$-i\hbar\frac{\partial\rho}{\partial T}=[\hat{H},\rho]+\sigma(T)[\hat{H},[\hat{H},\rho]]+...$$

Where $\sigma(T)$ is the rate of spread of the wavefunction of the clock. We have assumed one started with a clock in a quantum state such that the variable T has a distribution that is very peaked around t. In fact, the above expression is approximate, as the spreads increase one gets higher order terms with more commutators.

Class.Quant.Grav.21:L51-L57,2004

New J.Phys.6:45,2004

What are the consequences of the extra term? If we assume σ is constant, the equation can be solved exactly, and one gets that the density matrix in an energy eigen-basis evolves as,

$$\rho_{nm}(t) = \rho_{nm}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t}$$

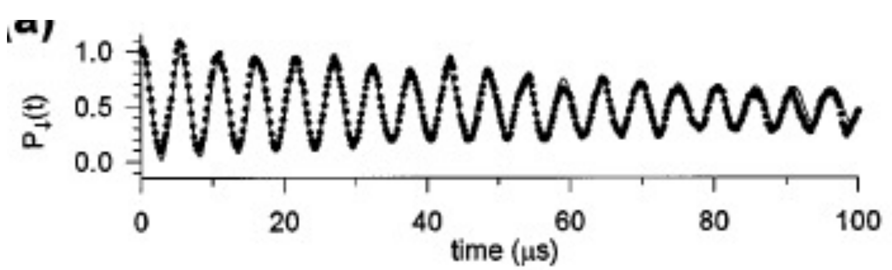
Where the omegas are the Bohr frequencies associated with the eigenvalues of H. $\omega_{mn} = E_m - E_n$

Therefore, the off-diagonal elements of the density matrix decay to zero exponentially, and pure states generically evolve into mixed states.

Quantum mechanics with real clocks therefore does not have a unitary evolution.

The effect can be made arbitrarily large simply choosing “lousy clocks” to do physics. This is not usually done, but an interpretation of experiments with Rabi oscillations indicates the effect is there,

R . Bonifacio, S . Olivares, P. Tombesi et. al., J. Mod. Optics, 47 2199 (2000)
PRA61, 053802 (2000).
D. Meekhof, C. Monroe, B. King, W. Itano, D. Wineland PRL76, 1796 (1996).



Can the effect be eliminated just by choosing better and better clocks? And if not, how much does reality depart from traditional quantum theory? To estimate this, we have to ask ourselves the question “what is the best clock we can build”?

There are many phenomenological arguments based on quantum and gravitational considerations that lead to estimates of such a limitation,
(Salecker-Wigner and Ng, Karolyhazy, Lloyd, Hogan, Amelino Camelia) $\delta T = T^{1/3} t_P^{2/3}$

We will not enter into the analysis of these phenomenological estimations, (which have been questioned in the literature). But it is important to remark that the evolution with real clocks will not be unitary if the spread in the error of the clock grows with time with some power of *T*.

That is, if $\delta T = T_{planck}$ the evolution is unitary, but if $\delta T = T^a T_{Planck}^{1-a}$ with a>0 there will exist a fundamental loss of unitarity.

R. Gambini, R. Porto, J. Pullin, GRG 39, 1143 (2007)

We have recently reformulated our approach using POVM’s with the framework introduced by Höhn, Smith and Lock, arXiv:2007.00580, 1912.00033

R. Gambini, JP, Universe 6, 236 (2020).

Conclusions:

- Using evolving constants of the motion in the conditional probability interpretation of Page and Wootters allows to correctly compute the propagator and assign probabilities to histories.
- The resulting description is entirely in terms of Dirac observables.
- There are corrections to the propagator due to the use of “real clocks and rods” to measure space and time that include a fundamental loss of coherence.