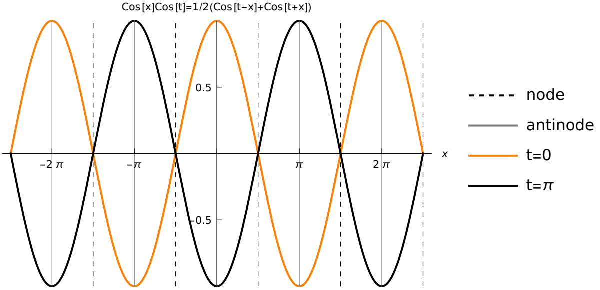
Toy models of standing gravitational waves

Sebastian J. Szybka
(a joint work with Syed Naqvi, Krzysztof Głód and Szymon Sikora)

Obserwatorium Astronomiczne UJ

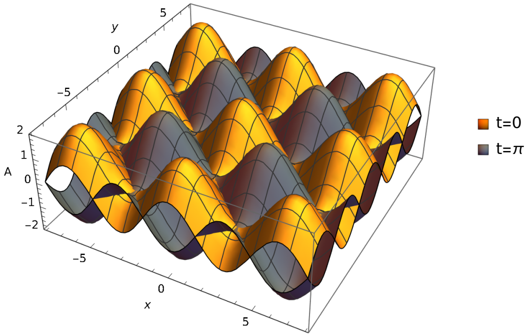
Standing waves are a quite common phenomenon in physics . . .
What is a standing gravitational wave?
Hans Stephani, 2003

standing wave in 1D



standing wave in 2D

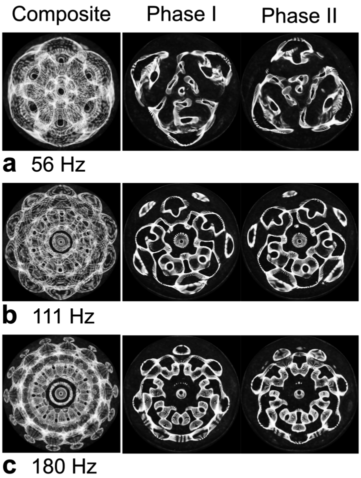
A=Cos[t](Cos[x]+Cos[y])=1/2(Cos[t−x]+Cos[t+x]+Cos[t−y]+Cos[t+y])



examples of standing waves

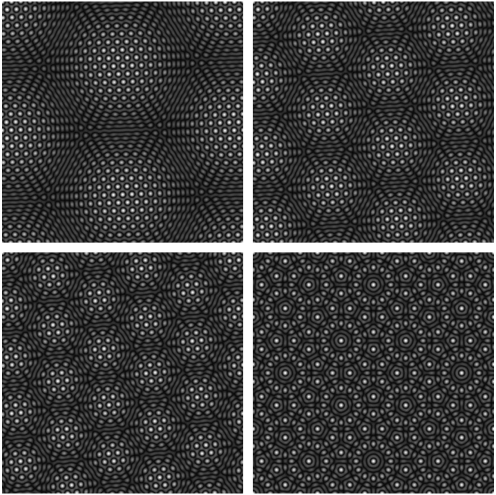
- a wave function of an electron in a box
- a violin string
- seismic waves
- Faraday waves 1831 (Navier-Stokes equations)
- . . .

Faraday waves—nonlinearity



credits: Merlin Sheldrake and Rupert Sheldrake
Determinants of Faraday Wave-Patterns in Water Samples Oscillated Vertically at a Range of Frequencies from 50-200 Hz, WATER 9, 1-27, OCTOBER 25, 2017

Faraday waves—nonlinearity



credits: Stéphan Fauve and Gérard Ioos, Quasipatterns versus superlattices resulting from the superposition of two hexagonal patterns, Comptes Rendus Mécanique Volume 347, Issue 4, April 2019, Pages 294-304

standing waves and gravity

- Gowdy standing waves
- standing wave boundary conditions
- primordial gravitational waves from inflation (Tensor perturbations)
- Hans Stephani, *Some remarks on standing gravitational waves*, General Relativity and Gravitation 35, 467 (2003)

definitions

- a superposition of waves moving in opposite directions
  problem: Einstein equations are nonlinear—no direct superpositions, e.g. Khan–Penrose impulsive waves and singularities
- a definition in terms of gravitational energy density
  problem: energy density of gravitational waves does not exist because of the equivalence principle
- a definition in terms of "no spatial energy flux"
  problem: no obvious gravitational analogue of the Poynting vector

high frequency limit and null dust

- Trautman 1958, *Radiation and Boundary Conditions in the Theory of Gravitation*, Bull. Acad. Polon. Sci., 6, 407-412 (1958)
- Isaacson 1968 → Burnett 1989 → Green and Wald 2011
- standing waves in a high frequency limit correspond to a superposition of null dusts (SJS, A. Cieślik, Phys. Rev. D 100, 064025, 2019)

$$t^{(0)} = \frac{1}{2}\rho(x^{\alpha})(k^{\flat}{}_{+} \otimes k^{\flat}{}_{+} + k^{\flat}{}_{-} \otimes k^{\flat}{}_{-}) \, ,$$

problem: in the limit information is lost, so one cannot exclude possibility that some non-standing waves solutions also have limit of this form

searching for exact standing gravitational waves

- let $(M, \boldsymbol{g})$ be a $1+3$ dimensional spacetime which admits the time function t and belongs to the G_2 group — there exist two linearly independent Killing fields $\boldsymbol{\xi}$, $\boldsymbol{\eta}$
- we are interested in the Abelian G_2 group denoted as G_2I
- the orthogonally transitive G_2I metric on non-null orbits can be written as

$$g = e^f(\epsilon dt^2 + dz^2) + W\left[-\epsilon e^p(dx + \omega dy)^2 + e^{-p}dy^2\right]$$

where $W > 0$ and f, W, p are functions of t , z only ($\boldsymbol{\xi} = \partial_x$, $\boldsymbol{\eta} = \partial_y$)

toy-models

- for spacelike orbits $\epsilon = -1$ and under an additional assumption $W = t$

$$g = e^{f}(-dt^2 + dz^2) + t\left[e^{p}(dx + \omega dy)^2 + e^{-p}dy^2\right]$$

a remark: a complex substitution leads to ‘an equivalence’

cosmological spacetimes

non-stationary cylindrical spacetimes

stationary axially symmetric spacetimes

definition

let $\mathcal{Q} : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ be a space of functions such that for any $s \in \mathcal{Q}$ we have: $s(t, z)$ is a bounded function which is asymptotically almost periodic in t as $t \to \infty$ (with a non-vanishing amplitude) and strictly periodic in z

exact solutions: standing waves

$$g = e^f(-dt^2 + dz^2) + t\left[e^p(dx + \omega dy)^2 + e^{-p}dy^2\right]$$

$s_i, r_j, q_k \in \mathcal{Q}$ and α, β are real constants

- vacuum (T^3 Gowdy)[1]
  $f = -\beta q_1/\sqrt{t} + \beta^2(s_1 + tr_1)$, $p = -\ln t + \beta q_1/\sqrt{t}$, $\omega = 0$
  Phys. Rev. D 103, 024011 (2021), SJS, S. Naqvi
- vacuum (T^3 "unpolarized" Gowdy)[1]
  $f = \ln\left[\cosh\alpha\cosh(\beta q_1/\sqrt{t}) - \sinh(\beta q_1/\sqrt{t})\right] + \beta^2(s_1 + tr_1)$,
  $p = -\ln t - \ln\left[\cosh\alpha\cosh(\beta q_1/\sqrt{t}) - \sinh(\beta q_1/\sqrt{t})\right]$,
  $\omega = \beta\sqrt{t}q_2\sinh\alpha$
  `arxiv.org/abs/2106.05829`, K. Głód, S. Sikora, SJS
- electro-vac (T^3 electromagnetic Gowdy)[1]
  $f = -\frac{1}{2}\ln t + 2\ln\left(1 + te^{\beta q_1/\sqrt{t}}\right) - \beta q_1/\sqrt{t} + \beta^2(s_1 + tr_1)$,
  $p = -2\ln\left(1 + te^{\beta q_1/\sqrt{t}}\right) + \beta q_1/\sqrt{t}$, $\omega = 0$,
  $A^\flat = \frac{1}{2}\tanh[\ln(\sqrt{t}) + \frac{1}{2}\beta q_1/\sqrt{t}]\,dx$
  work in progress, SJS, S. Naqvi

[1] assuming that t-constant hypersurfaces are compact without boundary and orientable

exact solutions

$$q_1(t,z) = 2\sqrt{\lambda} J_0(t/\lambda)\sin(z/\lambda)$$
$$s_1(t,z) = q_1(t,z)\sqrt{\lambda} t J_1(t/\lambda)\sin(z/\lambda)$$
$$r_1(t,z) = t(J_0(t/\lambda)^2 + J_1(t/\lambda)^2)$$
$$q_2(t,z) = 2\sqrt{\lambda t} J_1(t/\lambda)\cos(z/\lambda)$$

J_i are Bessel functions of the first kind and λ is a constant

tools

- a geodesic deviation equation in an orthonormal frame $e_{\hat{\alpha}}$ which is a freely falling frame of stationary observers at antinodes (only for the Gowdy type solutions)
- Weyl scalars (using a Newman-Penrose tetrad)
- optical scalars for rays aligned along the propagation direction of gravitational waves
- 'a laser ranging'
- the Bel-Robinson tensor (u^α is a four-velocity of an observer)

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\mu\gamma}{}^{\nu} C_{\delta\nu\beta}{}^{\mu} + \star C_{\alpha\mu\gamma}{}^{\nu} \star C_{\delta\nu\beta}{}^{\mu}$$

  - super-energy density

$$W = T_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta \geq 0$$

  - super-Poynting vector

$$S^\alpha = -T^{\mu}{}_{\beta\gamma\delta}(\delta^\alpha_\mu + u^\alpha u_\mu) u^\beta u^\gamma u^\delta$$

Newman–Penrose tetrad

a complex null tetrad $w_{\breve{\mu}} = \{k, l, m, \bar{m}\}$
real vectors k, l and complex conjugate vectors m, $\bar{m}$ are null
their inner products vanish except

$$k \cdot l = -1 \,, \quad m \cdot \bar{m} = 1$$

the orthonormal tetrad $\{e_{\hat{\alpha}}\}$

$$\begin{aligned} k &= \frac{1}{\sqrt{2}}(e_{\hat{0}} + e_{\hat{1}}) \quad & l &= \frac{1}{\sqrt{2}}(e_{\hat{0}} - e_{\hat{1}}) \\ m &= \frac{1}{\sqrt{2}}(e_{\hat{2}} - ie_{\hat{3}}) \quad & \bar{m} &= \frac{1}{\sqrt{2}}(e_{\hat{2}} + ie_{\hat{3}}) \end{aligned} \tag{1}$$

$$g_{\breve{\alpha}\breve{\beta}} = -2k_{(\breve{\alpha}}l_{\breve{\beta})} + 2m_{(\breve{\alpha}}\bar{m}_{\breve{\beta})}$$

the Weyl tensor has ten independent components which are
determined by the five complex coefficients Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4

geodesic deviation in a freely falling frame

$$\frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}{}_{\hat{0}\hat{\beta}\hat{0}}\xi^{\hat{\beta}}$$

ξ — a deviation vector, τ — a proper time of the observer
by an appropriate choice of a null tetrad we have $\Psi_1 = \Psi_3 = 0$

$$\begin{aligned}
R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{0}} &= \frac{1}{2}\left[R_{\hat{0}\hat{0}} - R_{\hat{1}\hat{1}}\right] + 2\Re\left[\Psi_2\right] \\
R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} &= \frac{1}{2}\left[R_{\hat{0}\hat{0}} - R_{\hat{1}\hat{1}}\right] + \frac{1}{2}\Re\left[-2\Psi_2 + \Psi_0 + \Psi_4\right] \\
R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} &= \frac{1}{2}\left[R_{\hat{0}\hat{0}} - R_{\hat{1}\hat{1}}\right] + \frac{1}{2}\Re\left[-2\Psi_2 - \Psi_0 - \Psi_4\right] \\
R^{\hat{2}}{}_{\hat{0}\hat{3}\hat{0}} &= -\frac{1}{2}R_{\hat{2}\hat{3}} + \frac{1}{2}\Im\left[\Psi_0 - \Psi_4\right]
\end{aligned}$$

Ψ_A — complex Weyl scalars

geodesic deviation in a freely falling frame

at antinodes
$R_{\hat{0}\hat{0}} = 8\pi T_{\hat{0}\hat{0}} = \rho = E^2 + B^2$
$R_{\hat{1}\hat{1}} = 8\pi T_{\hat{1}\hat{1}} = 8\pi\sigma_{xx} = 2E_x^2 + 2B_x^2 - \rho$
$R_{\hat{2}\hat{3}} = 8\pi\sigma_{xy} = 2E_x E_y + 2B_x B_y - \rho\delta_{xy}$
we have $B = 0$ and $E = E_x e_{\hat{2}}$ which implies
$R_{\hat{0}\hat{0}} = R_{\hat{1}\hat{1}}$ and $R_{\hat{2}\hat{3}} = 0$
for the solution studied: $\psi_0 = \psi_4$ (ψ_A are real)

$$R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{0}} = 2\Psi_2$$

$$R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} = -\Psi_2 + \Psi_0$$

$$R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} = -\Psi_2 - \Psi_0$$

$$R^{\hat{2}}{}_{\hat{0}\hat{3}\hat{0}} = 0$$

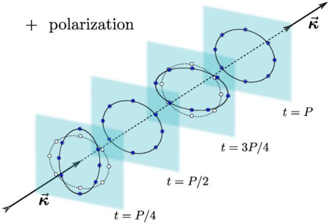
geodesic deviation in a freely falling frame

A remark: The Szekeres theorem implies that there do not exist vacuum solutions to Einstein equations with $\Psi_0 \neq 0$, $\Psi_4 \neq 0$ and $\Psi_1 = \Psi_2 = \Psi_3 = 0$. Since for our electro-vac spacetime $\Psi_0 = \Psi_4 \neq 0$ and $\Psi_1 = \Psi_3 = 0$ (by 'a gauge transformation'), then necessary the Coulomb component Ψ_2 is non-zero. Two transverse gravitational waves cannot be trivially superposed.

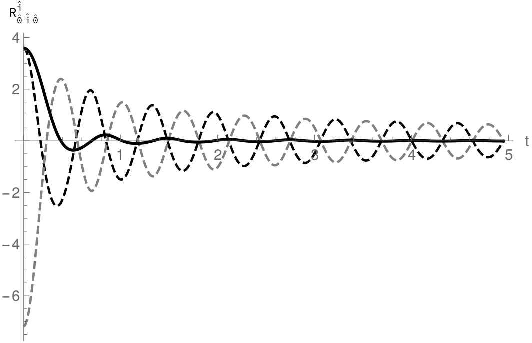
Tissot diagrams

- invented by Nicolas Auguste Tissot in 1859 and 1871 to characterize local distorsions to map projection in cartography
- applied to gravitational waves by Garry Gibbons
- Tissot diagrams for linear waves: N. Bishop, L. Rezzolla,*Extraction of gravitational waves in numerical relativity*, Living reviews in relativity, 2016
- Tissot diagrams for plane waves: P.-M. Zhang, C. Duval, G. W. Gibbons, P. A. Horvathy, *Soft Gravitons & the Memory Effect for Plane Gravitational Waves*, Phys. Rev. D 96, 064013 (2017)
- Tissot diagrams for standing waves: SJS, S. Naqvi, *Freely falling particles in standing wave spacetime*, Phys. Rev. D 103, 024011 (2021)

linearized gravitational waves (N. Bishop, L. Rezzolla)

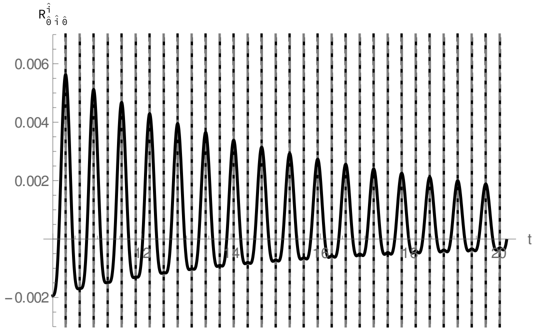


"spring constants" in polarized T^3 Gowdy



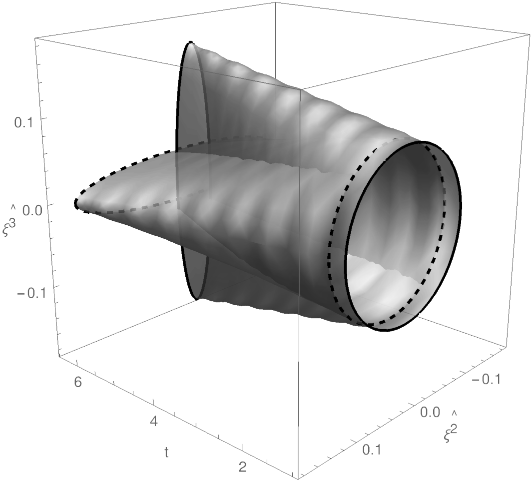
"Spring constants" at an antinode: longitudinal (solid), transverse (dashed).

"spring constants" in polarized T^3 Gowdy



"Spring constants" at an antinode: longitudinal (solid), transverse (dashed).

polarized T^3 Gowdy

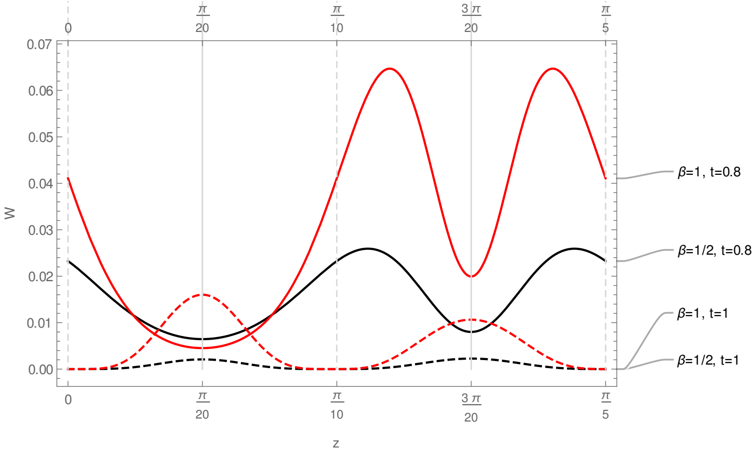


summary

- exact toy models of standing gravitational waves
- the precise definition of standing gravitational waves is still needed
- anitodes attract freely falling particles (in particular models)
- Weyl scalars are useful in the analysis
- a convenient setting to study nonlinear waves

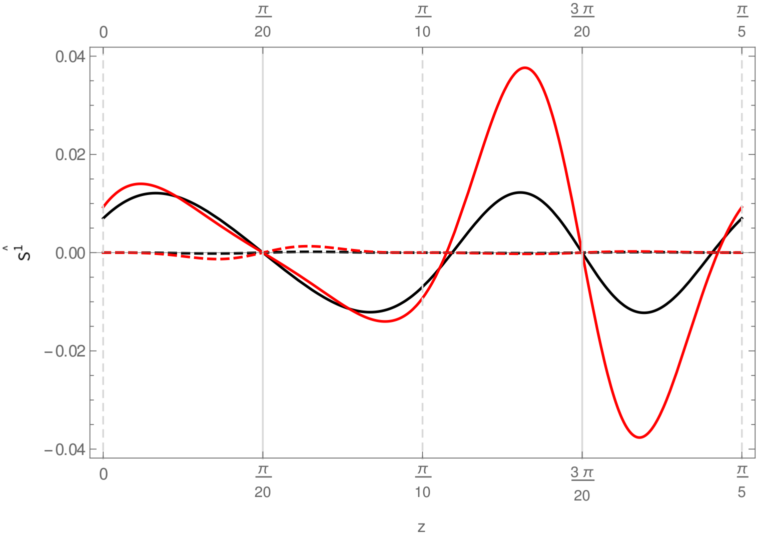
super-energy density

(electro-vac model, $\lambda = 1/10$)



$\hat{1}$ component of super-Poynting vector

(electro-vac model, $\lambda = 1/10$)



$\hat{1}$ component of super-Poynting vector

(electro-vac model, $\lambda = 1/10$)