The Ambient Metric and Tractors

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This talk will be mainly an introduction to two mathematical concepts in conformal geometry. Both of them have been forgotten and rediscovered.

- The Ambient Metric
  - Haantjes-Schouten 1936
  - Fefferman-Graham 'Conformal Invariants' (1985) and 'The ambient metric' (2011)
- Tractor Bundle and Connection
  - Tracy Thomas 1926
  - Bailey-Eastwood-Gover 'Thomas's Structure Bundle for Conformal and Related Structures' (1994) generalizing Penrose's twistors

We will consider conformal manifolds $(M, [g])$, where

- M is a smooth n-dimensional manifold ($n > 2$),
- $[g]$ a conformal equivalence class of metrics of signature (p, q) on M, i.e.,

$$g \sim \hat{g} \quad \text{iff} \quad \hat{g} = \Omega^2 g \quad \text{for some pos. function} \quad \Omega \in C^\infty(M).$$

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A motivation for the conformally invariant constructions was in particular the aim to construct and understand conformal invariants and conformally invariant operators (e.g. GJMS operators, that have leading part the power of a Laplacian).

Consider $\mathbb{R}^{n+2}$ with the Minkowski metric $\tilde{g} = \sum_{i=1}^{n+1}(dx^i)^2 - (dx^0)^2$ and the null-cone

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The Minkowski metric $\tilde{g}$ induces

- a degenerate metric on $\mathcal{N}$, and
- a well-defined conformal structure $[g]$ on $\mathbb{S}^n$: any section of (1) determines a metric and different sections lead to conformally related metrics (the usual round metric arises from the section $x^0 = 1$).

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In particular, $\mathcal{N}/\pm 1$ can be identified with the space of all conformally related metrics on $\mathbb{S}^n$

Remarks

- An analogous construction works if one starts with the flat metric $\tilde{g}$ of signature $(p+1, q+1)$ on $\mathbb{R}^{n+2}$; then $\mathbb{P}(\mathcal{N}) \cong \mathbb{S}^p \times \mathbb{S}^q / \mathbb{Z}_2$.

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- The group $O(p+1, q+1)$ acts linearly on $\mathbb{R}^{n+2}$ and by isometries, and hence descends to an action on $\mathbb{P}(\mathcal{N})$ by conformal transformations. This leads to an identification

$$O(p+1, q+1)/P \cong \mathbb{P}(\mathcal{N})$$

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The idea of Fefferman-Graham is to invert the construction in a more general setting and construct an analogue of $\tilde{g}$ for any conformal manifold $(M, [g])$.

Given a conformal manifold $(M, [g])$, one has at each point $x \in M$ a ray of quadratic forms. Together these form the metric bundle

$$\mathcal{Q} = \{(x, g_x) : x \in M, g \in [g]\} \xrightarrow{\pi} M$$

It comes equippes with

- $\mathbb{R}_+$-action (dilations) $\phi_s : \mathcal{Q} \to \mathcal{Q}, \quad \phi_s(g_x, x) := (s^2 g_x, x)$

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- tautological degenerate metric: for $X, Y \in T_{(p,g_p)}\mathcal{Q}$

$$h(X, Y) = g_p(\pi_*(X), \pi_*(Y)).$$

It is homogeneous of degree two, i.e. $\phi_s^* h = s^2 h$.

Definitions

Ambient space $\tilde{M}$: dilation-invariant neighborhood of $\mathcal{Q}$ in $\mathcal{Q} \times \mathbb{R}$

Ambient metric $\tilde{g}$: smooth metric on $\tilde{M}$ of signature $(p+1, q+1)$ s.t.

1. $\tilde{g}$ is homogeneous of degree two, i.e., $\phi_s^* \tilde{g} = s^2 \tilde{g}$,
2. for each $p \in \tilde{M}$ the curve $p \mapsto \delta_s(p)$ is a geodesic,
3. $\tilde{g}$ restricts to tautological tensor h on $\mathcal{Q}$,
4. $\mathrm{Ric}(\tilde{g})$ vanishes to infinite order along $\mathcal{Q}$, i.e.,

$$\partial_\rho^k \mathrm{Ric}(\tilde{g})|_{\rho=0} = 0 \quad \text{for all } k{=}0{,}1{,}\ldots,$$

where ρ denotes a coordinate transversal to the metric bundle.

- Choice of representative $g = g(x^i) \in [g]$ determines identification

$$\mathcal{Q} \cong \mathbb{R}_+ \times M,\ (t^2 g_x, x) \mapsto (t, x) \quad \text{and} \quad h = t^2 g(x^i)$$

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- Any ambient metric can be put into *normal form* with respect to the chosen representative $g = g(x^i) \in [g]$:

$$\tilde{g} = 2dt\, d(t\rho) + t^2 g(x^i, \rho), \quad \text{with} \quad g(x^i, \rho)|_{\rho=0} = g(x^i),$$

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- Next one makes a power series ansatz for $g(x^i, \rho)$ in ρ and tries to determine the terms iteratively from Ricci flatness of $\tilde{g}$.

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The general result depends on the parity of the dimension $n = \dim(M)$.

Theorem (Fefferman-Graham)

- *n* odd:
  There exists a formal power series solution and it is uniquely determined to infinite order at $\rho = 0$.

  If $(M, [g])$ is real analytic, the power series converges and defines an ambient metric satisfying $\mathrm{Ric}(\tilde{g}) = 0$ in neighborhood of $\mathcal{Q}$.

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- n even:
  There is a conformally invariant tensor (obstruction tensor) that obstructs the existence of smooth solutions to (1)-(4). (When $n = 4$, this is the Bach tensor). However, if (4) is replaced by the condition that $\mathrm{Ric}(\tilde{g})$ vanish to order $\frac{n}{2} - 1$ at $\rho = 0$, then there exists a formal power series solution, which is unique to order $\frac{n}{2}$.

  Even if the obstruction tensor vanishes, and an infinite order ambient metric exists, there is an abiguity at order $\frac{n}{2}$.

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- A geometric construction of explicit ambient metrics for a class of conforal structures determined by projective structures was given in "Fefferman-Graham Ambient Metrics of Patterson-Walker Metrics" (Hammerl, Sagerschnig, Šilhan, Taghavi-Chabert, Žadnik)

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Definition:

A Poincaré-Einstein metric for $(M, [g])$ is a metric g_+ of signature $(p+1, q)$ on the interior M^o satisfying

- g_+ has conformal infinity $[g]$:
  there exists a (local) defining function $r \in C^\infty(M_+)$ for M (zero locus of r is M and $dr \neq 0$ there) such that $r^2 g_+$ extends smoothly to M_+ and $r^2 g_+|_M \in [g]$,
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- g_+ is Einstein, $\mathrm{Ric}(g_+) = -n g_+$ to infinite order along M.

Alternatively, one can consider metrics g_- of signature $(p, q+1)$ and $\mathrm{Ric}(g_-) = n g_-$.

Flat model

Restrict the Minkowski metric $\tilde{g} = \sum_{i=1}^{n+1} (dx^i)^2 - (dx^0)^2$ to the hyperboloid

$$\mathcal{H} = \{x \in \mathbb{R}^{n+2} : \sum_{i=1}^{n+1} (x^i)^2 - (x^0)^2 = -1\}.$$

This gives the hyperbolic metric g_+. Under an appropriate identification of one sheet of $\mathcal{H}$ with the unit ball, it can be realized as the Poincaré metric

$$g_+ = 4(1 - |y|^2)^{-2} \sum_{i=1}^{n+1} (dy^i)^2,$$

which has the conformal structure on $\mathbb{S}^n$ as conformal infinity.

Consider an ambient metric in normal form on $\mathbb{R} \times M \times \mathbb{R}$ w.r.t. g,

$$\tilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_\rho, \quad g_\rho|_{\rho=0} = g.$$

A change of variables

$$\rho = -\tfrac{r^2}{2}, \quad t = \tfrac{s}{r} \quad \text{on } \{\rho \leq 0\}$$

takes it to the form $\tilde{g} = s^2 g_+ - ds^2$ and we restrict it to the hypersurface $\{s = 1\}$. The metric

$$g_+ = r^{-2}\left(dr^2 + g_{-\frac{r^2}{2}}\right) \tag{2}$$

obtained that way has conformal infinity $[g]$.

Poincaré-Einstein from ambient metrics

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If $\mathrm{Ric}(\tilde{g}) = 0$ then $\mathrm{Ric}(g_+) = -n g_+$, and so $\tilde{g}_+$ is a Poincare-Einstein metric for $(M, [g])$.

Poincaré-Einstein from ambient metrics

- In this way one obtains *even* Poincaré metrics $g_+ = r^{-2}(dr^2 + g_r)$, where
$$g_r = g - \mathrm{P} r^2 + \ldots$$

- Fefferman-Graham also have a more general result describing all formal expansions of Poincaré metrics.
- For $n \geq 3$ odd: Let γ be a symmetric 2-tensor on M such that ${\gamma^i}_i = 0$ and $\nabla^i \gamma_{ij} = 0$, then there exists a Poincaré metric in normal form such that $\mathrm{tf}(\partial_r^n g_r|_{r=0}) = \gamma$. These conditions uniquely determine g_r to infinite order at $r = 0$.

Remark on tractors

Fefferman-Graham ambient construction: conformal manifold $(M, [g])$ of signature (p, q), $p + q = n \rightsquigarrow$ manifold $\tilde{M}$ of dimension $n + 2$ with hypersurface $\mathcal{Q} \subset \tilde{M}$, dilations δ_s, and metric $\tilde{g}$ of sig. $(p + 1, q + 1)$.

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For a conformal manifold $(M,[g])$ there is a natural vector bundle over M of rank $n+2$ with a linear connection determined only by the conformal structure (tractor bundle $\mathcal{T}$ and tractor connection $\nabla^{\mathcal{T}}$).

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These can also be obtained from the ambient picture:

- Restict $T\tilde{M}$, $\tilde{g}$, Levi-Civita $\nabla^{\tilde{g}}$ to $\mathcal{Q}$
- One can show that they pass down to a bundle $\mathcal{T} \to M$, a bundle metric H on $\mathcal{T}$ and a linear connection $\nabla^{\mathcal{T}}$. (In particular, sections X of $\mathcal{T}$ correspond to sections of $T\tilde{M}|_{\mathcal{Q}}$ s.t. $(\delta_s)^* X = s^{-1} X$.)
- Čap-Gover: These coincide with the usual tractor bundle and connection to be introduced next.

Let $\mathcal{E}[w]$ be the bundle of densities of conformal weight w. Its sections can be identified with functions f on $\mathcal{C}$ homogeneous of degree w, i.e. $(\delta_s)^* f = s^w f$. A metric $g \in [g]$ is a section of $\mathcal{C}$, so $f \circ g$ is a function on M. Changing the metric $\hat{g} = \Omega^2 g$ leads to $f \circ \hat{g} = \Omega^w f \circ g$.

Conformal-to-Einstein operator

Now consider the operator $D : \mathcal{E}[1] \to S^2 T^* M \otimes \mathcal{E}[1]$,

$$D(\sigma) = \text{trace-free part}(\nabla_a \nabla_b \sigma + P_{ab}\sigma),$$

where $\mathrm{P}_{ij} = \frac{1}{(n-2)}(Ric_{ij} - \frac{1}{2(n-1)} S\, g_{ij})$ denotes the Schouten tensor. Then

- D is conformally invariant.

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- D is conformally invariant.
- Nowhere vanishing solutions $\sigma \in \mathcal{E}_+[1]$ to $D(\sigma) = 0$ correspond to Einstein metrics in the conformal class via $\sigma \mapsto \sigma^{-2} g$.

The standard tractor bundle via prolongation

Fix a metric $g \in c$ is fixed. Rewrite the second order PDE

$$D(\sigma) = \nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma + g_{ab}\rho = 0$$

as a first order closed system.

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Proposition

Solutions to $D(\sigma) = 0$ are in bijective correspondence with parallel sections of the linear connection

$$\nabla^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \mathrm{P}_{ab}\sigma + g_{ab}\rho \\ \nabla_a \rho - \mathrm{P}_a{}^b \mu_b \end{pmatrix} \quad \text{on} \quad [\mathcal{T}]_g = \begin{pmatrix} \mathbb{R} \\ T^*M \\ \mathbb{R} \end{pmatrix}$$

via

$$\sigma \mapsto (\sigma, \nabla_a \sigma, -\tfrac{1}{n}(\Delta\sigma + \mathrm{P}\sigma)).$$

Tractor bundle and tractor connection

Define

$$\mathcal{T} := \bigsqcup_{g \in \mathbf{c}} [\mathcal{T}]_g / \sim \qquad \leftarrow \text{suitable equivalence relation}$$

$\nabla^{\mathcal{T}}$ determines a conformally invariant connection on $\mathcal{T}$.

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Remark: Relationship with the normal Cartan geometry

(M, c) of signature (p, q) determines a canonical normal conformal Cartan geometry:

- P-principal bundle $\mathcal{G} \to M$, where $P = \mathrm{Stab}(\ell) \subset \mathrm{SO}(p+1, q+1)$ of a null line $\ell \subset \mathbb{R}^{p+1,q+1}$,
- Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{so}(p+1, q+1))$.

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- Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{so}(p+1, q+1))$.

The tractor bundle is the associated bundle

$$\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{p+1,q+1}, \quad \text{and } \nabla^{\mathcal{T}} \text{ induced from } \omega.$$

.

The standard tractor bundle via prolongation

Tractor curvature

$$(\nabla_a^{\mathcal{T}}\nabla_b^{\mathcal{T}} - \nabla_b^{\mathcal{T}}\nabla_a^{\mathcal{T}}) \begin{pmatrix} \sigma \\ \mu^c \\ \rho \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ Y_{ab}{}^c & C_{ab}{}^c{}_d & 0 \\ 0 & -Y_{abd} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^d \\ \rho \end{pmatrix} \qquad (3)$$

$C_{ab}{}^c{}_d$ Weyl curvature, $Y_{abc} = 2\nabla_{[a}\mathrm{P}_{b]c}$ Cotton tensor.
Hence the tractor curvature vanishes if and only if the manifold is locally equivalent to the flat conformal model (conformally flat).

Tractor metric

For tractors $U^I = (\sigma, \mu^i, \rho)$ and $V^J = (\alpha, \beta^j, \gamma)$,

$$H_{IJ}U^IV^J = \sigma\gamma + g_{ij}\mu^i\beta^j + \rho\alpha. \qquad (4)$$

If g has signature (p, q), the metric H has signature $(p+1, q+1)$.
It is conformally invariant and preserved by the tractor connection,

$$\nabla^{\mathcal{T}} H = 0.$$

Recall Proposition 1:

$$\left\{ \begin{array}{c} \text{solutions to} \\ \text{tracefree}\,(\nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma) = 0 \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{covariantly constant tractors} \\ S \in \Gamma(\mathcal{T}) \quad \text{s.t.} \quad \nabla^{\mathcal{T}} S = 0 \end{array} \right\}$$

via $\quad \sigma \mapsto S = (\sigma, \nabla_a \sigma, -\frac{1}{n}(\Delta\sigma + \mathrm{P}\sigma))$. Then

$$H(S,S) = -\tfrac{2}{n}\sigma(\Delta\sigma + \mathrm{P}\sigma) + g^{ab}\nabla_a\sigma\nabla_b\sigma,$$

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Suppose $(M,[g])$ be a conformal manifold of sig. (p,q) equipped with a cov. const. tractor $S \in \Gamma(\mathcal{T})$, $\nabla^{\mathcal{T}} S = 0$, $H(S,S) \neq 0$. Then (Gover)

- $\Sigma = \{x | \sigma(x) = 0\}$ is empty or a hypersurface with an induced conformal structure of sig. $(p-1,q)$ resp. $(p,q-1)$.
- $M \setminus \Sigma$ carries an induced Einstein metric

$$g_\sigma = \sigma^{-2} g, \ \mathrm{Ric}(g_\sigma) = \Lambda g_\sigma$$

  with Λ of the opposite sign as $H(S,S)$.
  If $H(S,S) > 0$, this defines Poincare-Einstein metric around Σ.

Remarks

- There are more general results in similar situations (Gover,...)
- Since we have a connection $\nabla^{\mathcal{T}}$, we have a notion of parallel transport, and we can consider its (restricted) holonomy group

$$\mathrm{Hol}_x(\nabla^{\mathcal{T}}) := \left\{ \begin{array}{c} \text{linear transformations } \mathcal{T}_x \to \mathcal{T}_x \text{ obtained by parallel} \\ \text{transport around contractible loops based at } x \end{array} \right\}$$

  This is called the conformal holonomy group.
  In particular, there are general results about holonomy reductions for conformal structures and more general parabolic geometries (Armstrong, Leistner, Čap-Gover-Hammerl,...)
- One can also study the geometric implications of existence of parallel tractors and induced geometries on strata for *special* conformal structures. In joint work with Travis Willse we studied the geometry of conformal structures associated with $(2, 3, 5)$ distributions admitting parallel standard tractors.