Hamiltonian charges in spacetimes with a positive cosmological constant

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Plan of the talk

We consider Hamiltonian charges (boundary terms included) the following theories on de Sitter background:

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- Weak gravitational field
  - Problems with convergence of charges and renormalisation procedure. Discussed by the example of Energy and its flux.
  - Scalar Field

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- Maxwell theory
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    - Problems with convergence of charges and renormalisation procedure. Discussed by the example of Energy and its flux.
    - Scalar Field

In each case, the asymptotics of the field is modeled on the behavior of solutions of the spacelike Cauchy problem with smooth initial data.

de Sitter spacetime

- We consider four-dimensional de Sitter spacetime $\mathcal{M} = \mathbb{R} \times \mathbb{S}^3$ in Bondi coordinates (u, r, x^A). In these the metric takes the form

$$g \equiv g_{\alpha\beta} dx^\alpha dx^\beta = -(1 - \alpha^2 r^2) du^2 - 2du\, dr + r^2 \mathring{\gamma}_{AB} dx^A dx^B \,, \quad (1)$$

where $\alpha = \sqrt{\Lambda/3}$ with $\Lambda > 0$.

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  where $\alpha = \sqrt{\Lambda/3}$ with $\Lambda > 0$.
- Cosmological horizon $(1 - \alpha^2 r_H^2) = 0$. The vector field ∂_u timelike $\rightarrow$ spacelike.

Killing vectors in de Sitter spacetime

Definition of Killing vector field:

$$\mathcal{L}_X g = 0 \tag{2}$$

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We use the following basis of the space of Killing vectors in de Sitter spacetime

$$\mathcal{T} = \partial_u \,, \tag{3}$$

$$\mathcal{R} = \varepsilon^{BA} \mathring{D}_A \big(R_i n^i \big) \partial_B \,, \tag{4}$$

$$\mathcal{P}_{dS} = e^{\alpha u} \Big[p_i n^i \partial_u - \big(\alpha r + 1 \big) p_i n^i \partial_r - \frac{\alpha r + 1}{r} \mathring{D}^A (p_i n^i) \partial_A \Big] \,, \tag{5}$$

$$\mathcal{L}_{dS} = e^{-\alpha u} \Big[l_i n^i \partial_u + \big(\alpha r - 1 \big) l_i n^i \partial_r + \frac{\alpha r - 1}{r} \mathring{D}^A (l_i n^i) \partial_A \Big] \,, \tag{6}$$

where R_i, p_i and l_i are constants. $n^i = n^i(x^A)$ – dipole functions.

Maxwell field

Maxwell equations:

$$\begin{aligned} \nabla_\nu F^{\mu\nu} &= 4\pi j^\mu \\ \nabla_\mu * F^{\mu\nu} &= 0 \end{aligned}$$

$*$ – denotes Hodge duality

James Clerk Maxwell
(1831-1879)

Lagrangian density is given by

$$\mathcal{L}(A_\mu, \partial A_\mu) = -\frac{1}{16\pi}\sqrt{|-\det g|}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\,. \quad (7)$$

Hamiltonian current

The canonical momentum density reads

$$\pi^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial A_{\beta,\alpha}} = -\frac{1}{4\pi}\sqrt{|-\det g|}F^{\alpha\beta} \,. \tag{8}$$

The standard Hamiltonian current

$$\mathcal{H}^{\mu}_{c}[X] = \pi^{\mu\beta}\mathcal{L}_X A_\beta - X^\mu \mathcal{L} \tag{9}$$

is gauge dependend. This can be fixed by replacing $\mathcal{L}_X A$ by

$$\mathbf{L}_X A_\mu := X^\nu F_{\nu\mu} \tag{10}$$

$$\begin{aligned} \mathcal{H}^\mu[X] &:= \pi^{\mu\beta}\mathbf{L}_X A_\beta - X^\mu \mathcal{L} \\ &= -\frac{1}{4\pi}\sqrt{|-\det g|}\left(F^{\mu\beta}X^\alpha F_{\alpha\beta} - \frac{1}{4}\left(F^{\nu\beta}F_{\nu\beta}\right)X^\mu\right). \end{aligned} \tag{11}$$

Current of Hamiltonian flux

Let Y be an arbitrary vector field. Lie derivative of Hamiltonian current reads

$$
\begin{aligned}
-\frac{4\pi}{\sqrt{|-\det g|}}\mathcal{L}_Y\mathcal{H}^\mu[X] &= 2\nabla_\sigma\left[Y^{[\sigma}F^{\mu]\alpha}X^\kappa F_{\kappa\alpha} - \frac{1}{4}Y^{[\sigma}X^{\mu]}F^{\alpha\beta}F_{\alpha\beta}\right] \\
&\quad +4\pi Y^\mu j^\alpha X^\sigma F_{\alpha\sigma} - \frac{1}{4}Y^\mu F^{\alpha\sigma}F_{\alpha\sigma}\nabla_\kappa X^\kappa\,, \\
&\quad +Y^\mu F^{\sigma\alpha}\Delta_{\alpha\sigma}(X,A)\,. \qquad (12)
\end{aligned}
$$

where $\Delta_{\beta\delta}(V,A) := [\nabla_\delta, \mathcal{L}_V]A_\beta$. If $j^\alpha = 0$ and $\mathcal{L}_X g = 0$ then $\mathcal{L}_Y\mathcal{H}^\mu[X]$ leads to boundary term.

Asymptotics

The fields F_{Ar} which are associated with a conformally smooth Maxwell field have expansions of the form

$$F_{Ar} \quad = \quad -\overset{(0)}{F}_{Ax} r^{-2} + \ldots = \overset{(2)}{F}_{Ar} r^{-2} + \ldots \,, \tag{13}$$

where $\overset{(i)}{F}_{Ar} = \overset{(i)}{F}_{Ar} (u, x^A)$.

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where $\overset{(i)}{F}_{Ar} = \overset{(i)}{F}_{Ar}(u, x^A)$. Sourceless, vacuum Maxwell equations restrict asymptotics of the other components:

$$F_{AB} = \overset{(0)}{F}_{AB} + \frac{\partial_A \overset{(2)}{F}_{Br} - \partial_B \overset{(2)}{F}_{Ar}}{r} + \frac{\partial_A \overset{(3)}{F}_{Br} - \partial_B \overset{(3)}{F}_{Ar}}{2r^2} + \ldots , \qquad (14)$$

$$F_{uA} = \overset{(0)}{F}_{uA} + \frac{\alpha^2 \overset{(3)}{F}_{Ar} - \mathring{D}_A \overset{(2)}{F}_{ur} - \mathring{D}^B \overset{(0)}{F}_{BA}}{2r} + \ldots , \qquad (15)$$

$$F_{ur} = \frac{\overset{(2)}{F}_{ur}}{r^2} - \frac{\mathring{D}^A \overset{(2)}{F}_{Ar}}{r^3} - \frac{\mathring{D}^A \overset{(3)}{F}_{Ar}}{2r^4} + \ldots , \qquad (16)$$

Asymptotic results for Maxwell theory

- All Hamiltonian charges $\int_{\mathcal{C}_u} \mathcal{H}^\mu[X,F]dS_\mu$ and their fluxes are well defined.
- Examples

$$\frac{dE[\mathcal{C}_u]}{du} = \frac{1}{2\pi}\int_{\partial\mathcal{S}_\tau}\left[\mathcal{T}^{[\sigma}F^{\mu]\alpha}\mathcal{T}^\kappa F_{\kappa\alpha} - \frac{1}{4}\mathcal{T}^{[\sigma}\mathcal{T}^{\mu]}F^{\alpha\beta}F_{\alpha\beta}\right]dS_{\sigma\mu}$$
$$= -\lim_{R\to\infty}\frac{1}{4\pi}\int_{S_R}\left[r^2F_{ur}^2 + \mathring{\gamma}^{AB}F_{uA}F_{rB} + \epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\right]_{r=R}d\mu_{\mathring{\gamma}}$$
$$= -\frac{1}{4\pi}\int_{S_\infty}\left[\mathring{\gamma}^{AB}\left(\alpha^2\overset{(2)}{F}_{Ar}\overset{(0)}{F}_{Bu} + \overset{(0)}{F}_{Au}\overset{(0)}{F}_{Bu}\right)\right]d\mu_{\mathring{\gamma}}\,. \tag{17}$$

where:

$\mathcal{C}_u$	light cone $u = \mathrm{const}$ emanating from $r = 0$,
$S(R)$	sphere of radius R.

Linearized gravity

- New description of canonical charges of gravitational waves emitted by an isolated system.

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Linearized gravity

- New description of canonical charges of gravitational waves emitted by an isolated system.
- Asymptotic conditions on linearised fields have been modeled on the asymptotic behavior of the full solutions of the Einstein equations with $\Lambda > 0$.
- All boundary terms are taken into consideration.
- The gauge conditions for linearised fields

$$h_{rr} = 0 = h_{rA}\,, \quad \mathring{\gamma}^{AB} h_{AB} = 0\,. \tag{18}$$

Lagrangian and canonical energy

- From $L = \frac{\sqrt{|\det g|}}{16\pi}\left(R - \frac{\Lambda}{2}\right)$, we obtain the Lagrangian for weak fields

$$\mathcal{L}[h] = \frac{1}{32\pi}\sqrt{|\det g|}\left(P^{\alpha\beta\gamma\delta\epsilon\sigma}\nabla_\alpha h_{\beta\gamma}\nabla_\delta h_{\epsilon\sigma} + Q(h)\right), \quad (19)$$

where Q is a quadratic polynomial in h, and

$$\begin{aligned} P^{\alpha\beta\gamma\delta\epsilon\sigma} &= \frac{1}{2}\big(g^{\alpha\epsilon}g^{\delta\beta}g^{\gamma\sigma} + g^{\alpha\epsilon}g^{\sigma\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\epsilon}g^{\sigma\gamma} - g^{\alpha\beta}g^{\gamma\delta}g^{\epsilon\sigma} \\ &\quad - g^{\beta\gamma}g^{\alpha\epsilon}g^{\sigma\delta} + g^{\beta\gamma}g^{\alpha\delta}g^{\epsilon\sigma}\big). \end{aligned} \quad (20)$$

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- For a given a Lagrangian field theory of fields ϕ^A, the canonical energy is defined as

$$\mathcal{H}[\mathcal{S}, X, \phi] \;:=\; \int_{\mathcal{S}} \underbrace{\Big(\frac{\partial \mathcal{L}}{\partial {\phi^A}_\mu}\pounds_X \phi^A - X^\mu \mathcal{L}\Big)}_{=:\mathcal{H}^\mu} dS_\mu\,, \tag{21}$$

where ${\phi^A}_\mu := \partial_\mu \phi^A$.

Example: Canonical energy in Bondi gauge

The canonical energy $E_c[h, \mathcal{C}_{u,R}]$ is given by

$$\begin{aligned} E_c[h, \mathcal{C}_{u,R}] &:= \mathcal{H}[\mathcal{C}_{u,R}, \partial_u, h] \\ &= \frac{1}{64\pi} \int_{\mathcal{C}_{u,R}} g^{BE} g^{FC} \big(\partial_u h_{BC} \partial_r h_{EF} - h_{BC} \partial_r \partial_u h_{EF} \big) \, r^2 \sin\theta \, dr \, d\theta \, d\varphi \\ &\quad - \frac{1}{32\pi} \int_{S(R)} P^{r(\beta\gamma)\delta(\epsilon\sigma)} \nabla_\delta h_{\epsilon\sigma} \, h_{\beta\gamma} \, r^2 \sin\theta \, d\theta \, d\varphi \,, \end{aligned} \tag{22}$$

where:

$h_{\mu\nu}$	solution of the linearised vacuum Einstein equations,
$\mathcal{C}_u$	light cone $u = \mathrm{const}$ emanating from $r = 0$,
$\mathcal{C}_{u,R}$	light cone truncated at radius $r = R$,
$S(R)$	sphere of radius R.

Asymptotic behavior for large r

- There exists a dynamically consistent class of fields[1] $h_{AB} = r^2 \check{h}_{AB}$ which have an asymptotic expansion of the form

$$\check{h}_{AB} = \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \dots \tag{23}$$

[1] H. Friedrich, *On the existence of n–geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure,* Commun. Math. Phys. **107** (1986), 587-609.

Boundary term in canonical energy

The remaining $h_{\mu\nu}$'s are determined by the linearised version of constraint equations in Bondi coordinates:

$$h_{ru} \equiv 0\,, \tag{24}$$

$$\check{h}_{uA} = \overset{(0)}{\check{h}}_{uA} + \overset{(2)}{\check{h}}_{uA}/r^2 + \ldots\,, \tag{25}$$

$$\check{h}_{AB} = \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \ldots\,. \tag{26}$$

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One finds the following form of the boundary term in $E_c[h, \mathcal{C}_{u,R}]$:

$$\begin{aligned}
&-\frac{\Lambda R}{192\pi}\int_{S^2}\mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\overset{(1)}{\check{h}}_{BD}\sin(\theta)\,d\theta\,d\varphi\\
&-\frac{1}{64\pi}\int_{S^2}\mathring{\gamma}^{AB}\Big(\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD} - 6\overset{(0)}{\check{h}}_{uA}\overset{(3)}{\check{h}}_{uB}\Big)\sin(\theta)\,d\theta\,d\varphi\\
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1. the divergence of the boundary integral is compensated by that of the volume integral and, if not,
2. whether the boundary integral is needed at all in the definition of energy and, if so,
3. can one obtain consistent solutions by restricting oneself to a set of fields with $\overset{(1)}{\check{h}}_{AC} \equiv 0$.

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Linearised evolution equation for g_{AB}

Denoting by $TS[\cdot]$ the traceless symmetric part of a tensor, we have in vacuum

$$r\partial_r[r(\partial_u \check{h}_{AB})] - \frac{1}{2}\partial_r[N^2(\partial_r \check{h}_{AB})] + TS\big[\mathring{D}_A\big(\partial_r(r^2\check{h}_B)\big)\big] = 0\,, \qquad (25)$$

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$$\begin{aligned}\partial_u\check{h}_{AB}(r,\cdot) &= \frac{1}{r}\int_0^r \frac{1}{s}\Big(\frac{1}{2}\partial_r[N^2(\partial_r\check{h}_{AB})] - TS\big[\mathring{D}_A\big(\partial_r(r^2\check{h}_B)\big)\big]\Big)(s,\cdot)ds \\ &= \frac{\partial_u\overset{(1)}{\check{h}}_{AB}(\cdot)}{r} + \frac{\partial_u\overset{(3)}{\check{h}}_{AB}(\cdot)}{r^3} + o(r^{-3})\,, \end{aligned} \qquad (26)$$

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where

$$\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot) = \int_0^\infty \frac{1}{s}\Big(\frac{1}{2}\partial_r[N^2(\partial_r\check{h}_{AB})] - \mathring{\gamma}_{CA}\mathring{D}_B[\partial_r(r^2\check{h}^C)]\Big)(s,\cdot)ds\,, \qquad (27)$$

Volume integral from $E_c[h, \mathcal{C}_{u,R}]$

Using the obtained field behavior

$$\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot) = \frac{\partial_u \overset{(1)}{\check{h}}_{AB}(\cdot)}{r} + \frac{\partial_u \overset{(3)}{\check{h}}_{AB}(\cdot)}{r^3} + o(r^{-3}), \quad (28)$$

$$\check{h}_{AB} = \frac{\overset{(1)}{\check{h}}_{AB}}{r} + \frac{\overset{(2)}{\check{h}}_{AB}}{r^2} + \dots, \quad (29)$$

one finds a finite volume contribution to the canonical energy

$$\int_{\mathcal{C}_{u,R}} g^{BE} g^{FC} \left(\partial_u h_{BC} \partial_r h_{EF} - h_{BC} \partial_r \partial_u h_{EF}\right) r^2 \sin\theta \, dr \, d\theta \, d\varphi \quad (30)$$

Energy flux formula

The flux formula is also divergent

$$\frac{dE_c[h,\mathcal{C}_{u,R}]}{du} = -\frac{\Lambda R}{96\pi}\int_{S^2} \mathring{\gamma}^{AB}\mathring{\gamma}^{CD}\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD}\,\sin(\theta)\,d\theta\,d\varphi$$
$$-\frac{1}{32\pi}\int_{S^2}\mathring{\gamma}^{AB}\left(\mathring{\gamma}^{CD}\partial_u\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD} - 6\overset{(3)}{\check{h}}_{uA}\partial_u\overset{(0)}{\check{h}}_{uB}\right)\,\sin(\theta)\,d\theta\,d\varphi$$
$$+o(1)\,. \tag{31}$$

Renormalised energy and flux

We propose to introduce a *renormalised canonical energy*, obtained by removing the divergent terms in the canonical energy

$$
\begin{aligned}
\hat{E}_c[h,\mathcal{C}_u] :=& \\
&\frac{1}{64\pi}\int_{\mathcal{C}_u} g^{BE}g^{FC}\left(\partial_u h_{BC}\partial_r h_{EF} - h_{BC}\partial_r\partial_u h_{EF}\right) r^2\, dr\,\sin(\theta)\,d\theta\,d\varphi \\
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which has its own finite flux formula

$$\frac{d\hat{E}_c[h,\mathcal{C}_{u,R}]}{du} = -\frac{1}{32\pi}\int_{S^2}\mathring{\gamma}^{AB}\Big(\mathring{\gamma}^{CD}\partial_u\overset{(1)}{\check{h}}_{AC}\partial_u\overset{(1)}{\check{h}}_{BD} - 6\overset{(3)}{\check{h}}_{uA}\partial_u\overset{(0)}{\check{h}}_{uB}\Big)\sin(\theta)\, d\theta\, d\varphi\,. \quad (33)$$

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For $\Lambda = 0$, we recover the weak-field version of the usual Trautman-Bondi mass loss formula.

Scalar field

- The Lagrangian for scalar field reads

$$\mathcal{L} = -\frac{1}{2}\sqrt{|-\det g|}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2\right), \tag{34}$$

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- The canonical energy-momentum current $\mathcal{H}^\mu$ equals

$$\mathcal{H}^\mu[X] = -\sqrt{|-\det g|}\Big(\nabla^\mu\phi\,\mathcal{L}_X\phi - \frac{1}{2}\big(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\big)X^\mu\Big)\,. \tag{36}$$

Scalar field – Energy

- The scalar fields evolving out of smooth initial data on a Cauchy surface have an asymptotic in the form

$$\phi(u, r, x^A) = \frac{\overset{(1)}{\phi}(u, x^A)}{r} + \frac{\overset{(2)}{\phi}(u, x^A)}{r^2} + \frac{\overset{(3)}{\phi}(u, x^A)}{r^3} + \ldots . \qquad (37)$$

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- The energy on the truncated cone is defined as $E[\mathcal{T},\mathcal{C}_{u,r}] := \int_{\mathcal{C}_{u,R}} \mathcal{H}^\mu[\partial_u] dS_\mu = \int_{\mathcal{C}_{u,R}} \mathcal{H}^u[\partial_u] dS_u$, and reads

$$\begin{aligned} E[\mathcal{T},\mathcal{C}_{u,R}] &= \frac{1}{2}\int_{\mathcal{C}_{u,R}} \left(\mathring{\gamma}^{AB}\mathring{D}_A\phi\mathring{D}_B\phi + m^2 r^2\phi^2 + \left(r^2 - \alpha^2 r^4\right)\left(\partial_r\phi\right)^2 \\ &= \frac{\alpha^2 R}{2}\int_{S_R} (\overset{(1)}{\phi})^2 \, d\mu_{\mathring{\gamma}} + \int_{\mathcal{C}_{u,R}} O(r^{-2})\, dr\, d\mu_{\mathring{\gamma}} \end{aligned}$$

where $\mathcal{C}_{u,R} = \mathcal{C}_u \cap \{r \leq R\}$.

Scalar field – Angular momentum

The total angular-momentum is obtained from the integral $J[\mathcal{R},\mathcal{C}_{u,R}] := \int_{\mathcal{C}_{u,R}} \mathcal{H}^\mu[\mathcal{R}] dS_\mu \equiv R_i J^i[\mathcal{C}_{u,R}]$, where the J^i's are given by

$$J^i[\mathcal{C}_{u,R}] \quad := \quad \int_{\mathcal{C}_{u,R}} r^2 \varepsilon^{AB} \mathring{D}_B n^i \mathring{D}_A \phi \partial_r \phi \, dr \, d\mu_{\mathring{\gamma}} = \int_{\mathcal{C}_{u,R}} O(r^{-2}) \, dr \, d\mu_{\mathring{\gamma}} \,, \quad (39)$$

so that the limit $R \to \infty$ is finite, even though the total energy diverges.

Scalar Field – Energy flux

- If X is a Killing vector and $\Box_g \phi - \underbrace{m^2}_{=2\alpha^2} \phi = 0$ then

$$\frac{\mathcal{L}_Y \mathcal{H}^\mu[X]}{\sqrt{|-\det g|}} = -2\nabla_\sigma \Big(Y^{[\sigma}\nabla^{\mu]}\phi X^\alpha \nabla_\alpha \phi - \frac{1}{2} Y^{[\sigma} X^{\mu]} \left(\nabla^\alpha \phi \nabla_\alpha \phi + m^2 \phi^2\right)\Big) . \tag{40}$$

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- The energy flux:

$$\begin{aligned} \frac{dE[\mathcal{T}, \mathcal{C}_{u,R}]}{du} &= -\int_{S_R} \left[r^2 \left(\partial_u \phi + \left(\alpha^2 r^2 - 1 \right) \partial_r \phi \right) \partial_u \phi \right]_{r=R} d\mu_{\mathring{\gamma}} \\ &= \int_{S_R} \Big[\alpha^2 \overset{(1)}{\phi} \partial_u \overset{(1)}{\phi} R + \alpha^2 \overset{(1)}{\phi} \partial_u \overset{(2)}{\phi} \\ &\quad + \Big(2\alpha^2 \overset{(2)}{\phi} - \partial_u \overset{(1)}{\phi} \Big) \partial_u \overset{(1)}{\phi} + O\Big(\frac{1}{R} \Big) \Big] d\mu_{\mathring{\gamma}} . \end{aligned} \tag{41}$$

Summary

- The asymptotic conditions satisfied by the linearized metric have been modeled on the asymptotic behavior of the full solutions of the Einstein equations with positive cosmological constant.
- Near de Sitter spacetime, the canonical charges of weak fields in Bondi gauge are divergent in general.
- Proposed renormalised energy and flux in the limit $\Lambda = 0$ become classical Trautman-Bondi quantities.

Summary

- The asymptotic conditions satisfied by the linearized metric have been modeled on the asymptotic behavior of the full solutions of the Einstein equations with positive cosmological constant.
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We plan to extend our investigations by:

- Analysis of algebraic structures of renormalised charges.

Thank for your attention!

Sources:

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