Stationary vacuum metrics with smooth null scri

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Origins of the null scri

- Trautman 1958: outgoing radiation conditions in null directions, $E_{,u} \leq 0$
- Bondi 1960, Bondi, van der Burg and Metzner 1962: foliation by null surfaces, asymptotic expansions, Bondi mass and news (axially symmetric case)
- Sachs 1962: general case
- Penrose 1963: conformal compactification, diagrams with null boundaries (scri) $\mathscr{I}^+$, $\mathscr{I}^-$ etc.

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About work of Trautman see Chruściel 2002 Editor's Note: Lectures on General Relativity by Andrzej Trautman, Gen. Relat. Grav. 34

About the Bondi-Sachs approach see Mädler and Winicour 2016, Bondi-Sachs formalism, Scholarpedia, 11(12):33528

About Penrose's compactification see any book

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$$u = t - r - 2M \ln(r - 2M) \ , \quad u^{,\alpha} u_{,\alpha} = 0 \ .$$

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Extra gauge: the luminosity condition

$$\sqrt{\det \tilde{g}_{AB}} = r^2 \sqrt{\det s_{AB}}$$

Assumption of asymptotical flatness, $E(u) = \frac{1}{4\pi} \int M d\sigma$.

The conformal approach

Let metric g be smooth in a neighbourhood of $\mathscr{I}^+ = R \times S_2$. Define coordinates u, x^A on $\mathscr{I}^+$ such that $u = const$ on leaves S_2 and ∂_u is orthogonal to them. Extend foliation to the neighbourhood emitting past null geodesics orthogonally to u=const. Propagate coordinates u, x^A along these geodesics and choose Ω to be the affine parameter of geodesics such that

$$g = du(g_{00}du - 2d\Omega + 2g_{0A}dx^A) + g_{AB}dx^A dx^B$$

$$\hat{g}_{0A} = 0 \ on \ \mathscr{I}^+ \ .$$

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The affine gauge is more geometrical and $g^{\mu\nu}$ is simpler

$$g^{01} = -1 \ , \ \ g^{11} = -g_{00} + g_{0A}g_0{}^A \ , \ \ g^{1A} = g_0{}^A \ , \ \ g^{AB} \ (raises \ A, B) \ .$$

The Einstein equations

Equations for the physical metric $\tilde{g} = \Omega^{-2} g$:

$$\tilde{R}_{\mu\nu} = 0$$

or

$$R_{\mu\nu} - 2Y_{\mu\nu} - Y g_{\mu\nu} = 0 \ ,$$

where

$$Y_{\mu\nu} = -\frac{1}{\Omega}\Omega_{|\mu\nu} + \frac{1}{2\Omega^2}\Omega_{|\alpha}\Omega^{|\alpha} g_{\mu\nu} \ , \quad Y = Y_{\alpha}^{\ \alpha} \ .$$

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No second order pole $\Rightarrow \Omega_{|\alpha}\Omega^{|\alpha} = g^{11} = O(\Omega) \Rightarrow g_{00} = O(\Omega)$.

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Regularity of $Y_{AB} \Rightarrow \hat{g}_{AB,0} = a\hat{g}_{AB} \Rightarrow \hat{g}_{AB} = -e^{\alpha} s_{AB} \ , \ a = \alpha_{,0}$.

We eliminate α by a change of coordinates Ω and u.

Summary of regularity:

$$g_{00} = b\Omega^2 - 2M\Omega^3 + O(\Omega^4)$$

$$g_{0A} = q_A\Omega^2 + 2L_A\Omega^3 + O(\Omega^4)$$

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The Bianchi identity $\Rightarrow$ the independent Einstein equations

$$\tilde{R}^{(k)}_{11} = \tilde{R}^{(k)}_{1A} = \tilde{R}^{(k)}_{AB} = 0$$

$$\tilde{R}^{(2)}_{00} = \tilde{R}^{(2)}_{0A} = 0\ .$$

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$\tilde{R}_{11} = 0$

$$-\frac{1}{2}(\ln|g|)_{,11} + \frac{1}{4}g_{AB,1}g^{AB}{}_{,1} = 0\ ,$$

hence

$$\hat{g}^{AB}g^{(k+2)}_{AB} = \langle g^{(l)}_{AB}, l \leq k+1\rangle\ , \quad k \geq 0\ .$$

$\tilde{R}_{1A} = 0$

$$q_A = -\frac{1}{2} n^B{}_{A|B}\ , \quad k = 0$$

$$(k-1)(k+2) g_{0A}^{(k+2)} = (k+1)(g_{AB}^{(k+1)})^{|B} + \langle g_{0A}^{(k+1)},\ g_{\mu\nu}^{(l)}, l \leq k \rangle\ , \quad k \geq 1$$

where $\hat{g}_{AB} = -s_{AB}$ defines covariant derivative $|A$ and raises indices.

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For $k = 1$

$$p^B{}_{A|B} - \frac{1}{8}(n_{CD} n^{CD})_{,A} = 0 \ \Rightarrow \ p_{AB} = \frac{1}{8}(n_{CD} n^{CD}) \hat{g}_{AB} + \tilde{p}_{AB} \ ,$$

where $\tilde{p}_{AB}$ is a symmetric TT-tensor on the sphere

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$$\tilde{p}^A_A = 0 \ , \quad \tilde{p}^B_{A|B} = 0 \ .$$

Hence $\tilde{p}_{AB} = 0$ and

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$$\check{g}^{(k+1)}_{AB,0} = \langle M,\ L_A,\ g^{(l)}_{\mu\nu},\ l \leq k\rangle \ , \ k \geq 2$$

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$\tilde{R}^{(2)}_{00} = 0$ and $\tilde{R}^{(2)}_{0A} = 0$

$$M_{,0} = \langle n_{AB} \rangle$$

$$L_{A,0} = -\frac{1}{3} M_{,A} + \langle n_{AB} \rangle$$

Recursive solving (equivalent to Bondi, Sachs ...)

Physical metric

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Free data: n_{AB} (traceless), initial values of $M, \ L_A, \ \check{g}_{AB}^{(k)}$ with $k \geq 3$.
Metric components given algebraically or in quadratures.

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The Bondi mass aspect: $M - \frac{1}{16}(n_{AB}n^{AB})_{,0}$
The Bondi news function: $n_{AB,0}$

Stationary metrics with scri $\mathscr{I}^+$

- Choose a null surface Σ_0 intersecting $\mathscr{I}^+$ along S_2.
- Use the 1-parameter group of motion generated by the Killing vector K to construct a null foliation $u =$const.
- Propagate coordinates r, x^A from Σ_0 along $K = \partial_u$.
- Modify r to get $\tilde{g}_{01} = 1$.
- Assure nonsingularity of $\tilde{R}_{\mu\nu}$.

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Metric transforms to the Bondi-Sachs form with asymptotics

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Conclusion: $\tilde{g}$ is transparently asymptotically flat in coordinates adapted to the symmetry $K = \partial_u$.

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Equation $\tilde{R}^{(1)}_{1A} = 0$ yields

$$n^{B}{}_{A|B} = -2q_{,A}$$

Since the r. h. s. is not general we postulate

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No counterpart for $q = c^m Y_{1m}$, but in this case $q_{,A}$ cannot be obtained as $n^B{}_{A|B}$ (no regular solutions for n_{AB}).

$n_{AB} = -f_{|AB} + \frac{1}{2}(\hat{\Delta}_s f)s_{AB}$ can be gauged away by means of transformation induced by $u' = u + f$ on $\mathscr{I}^+$ and $r' = r - \frac{1}{4}\Delta_s f$.

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Conclusion: there are coordinates such that $n_{AB} = 0$ and

$$\tilde{g}_{AB} = -r^2 s_{AB} + O(\frac{1}{r}) \ , \ \ \tilde{g}_{0A} = \frac{2L_A}{r} + O(\frac{1}{r^2}) \ .$$

They are given up to translation of u and r with $f = b^m Y_{1m} + c$.

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$$L_A dx^A = \bar{A}(\bar{r} \times d\bar{r}) + \bar{B} d\bar{r} \ , \ \ \bar{r} \in S_2 \subset R^3 \ .$$

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Residual transformations allow to remove $\bar{B}$ if $M \neq 0$ since $L'_A = L_A + M f_{,A}$. This and rotation of $\bar{A}$ leads to Kerr-like expression

$$L_A dx^A = 2aM \sin^2\theta d\varphi \ , \ \ M \neq 0 \ .$$

$g_{00}^{(k)}$, $g_{0A}^{(k)}$ with $k \geq 4$ and $s^{AB} g_{AB}^{(k)}$ with $k \geq 3$ can be recursively found from $\tilde{R}_{11} = \tilde{R}_{1A} = s^{AB} \tilde{R}_{AB} = 0$.

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Solutions are not unique (otherwise we would end up with the Kerr metric only).

Lemma. A traceless tensor N_{AB} on S_2 can be decomposed as follows

$$N_{AB} = f_{|AB} - \frac{1}{2} f_{|C}{}^{C} s_{AB} + \eta^{C}{}_{(A} h_{|B)C} \,.$$

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$$N_{A}{}^{B}{}_{|B} = F_A$$

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Counterimage of F we know from $n_{A}{}^{B}{}_{|B} = q_{,A}$

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If $F = 0$ we have equation

$${}^{*}N_{A}{}^{B}{}_{|B} = H_{,A}$$

where ${}^{*}N_{AB} = \eta_{A}{}^{C} N_{CB}$ is also traceless and symmetric.

Hence

$$^{*}N_{AB} = h_{|AB} - \frac{1}{2} h_{|C}{}^{C} s_{AB} \,, \quad \Delta_s h + 2h = 2H + c \,.$$

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Thus, $f^{(k)}$ and $h^{(k)}$ exist if l. o. terms do not contain $Y_{k-1,m}$
If they exist they are given up to $b^m Y_{k-1,m}$

Conclusions

Every stationary metric with smooth scri and nonvanishing energy (also negative) tends to

$$\tilde{g} = du((1-\frac{2M}{r})du+2dr+\frac{2aM}{r}\sin^2\theta d\varphi)-(r^2 s_{AB}+\frac{Q_{AB}}{r})dx^A dx^B+O(r^{-2})$$

At each order r^{-k} ($k \geq 1$) in $\tilde{g}_{AB}$ up to $4k+6$ new parameters can appear (in combination with derivatives of $Y_{k+1,m}$). Q_{AB} denotes quadrupole term.

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$$Q = c_1[(\sin^2\theta - \frac{1}{3})d\theta^2 - \frac{1}{3}\sin^2\theta d\varphi^2] + c_2 \sin^3\theta d\theta d\varphi$$

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