On the spectral dimensionality of quantum space(time)s

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[1] M. Eckstein & **T. T.**, Phys. Rev. D **102**, 086003 (2020)

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M. Arzano & **T. T.**, Phys. Rev. D **89**, 124024 (2014)

Outline:

1. The spectral dimensionality from both sides
   - Spectral dimension
   - Dimension spectrum

2. Analysis of two kinds of examples
   - Quantum-deformed sphere
   - κ-Minkowski noncommutative spacetime

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Motivation

- Two crucial properties of the (semi)quantum spacetime are its effective dimension and the fate of relativistic symmetries
- It is conceivable that the (spectral) dimension $d_S(\sigma \approx 0) \neq 4$ due to some small-scale structure of spacetime
- Such results were indeed obtained in e.g. Dynamical Triangulations, Hořava-Lifschitz gravity, Asymptotic Safety and Causal Sets
- Almost always $d_S(\sigma \approx 0) < 4$ and most often $d_S(\sigma \approx 0) = 2$
- Similar behaviour has been observed for QG models in $d \neq 4$ topological dimensions
- In the context of (spectral) noncommutative geometry, the heat trace is instead characterized by the dimension spectrum
- Related issues include calculations of the vacuum energy density, Casimir effect and entanglement entropy

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Spectral dimension out of diffusion

On a Riemannian manifold (M, h) of dimension d, let us consider a (fictitious) diffusion process with the (auxiliary) time parameter σ:

$$\frac{\partial}{\partial \sigma} K(x, x_0; \sigma) = -\Delta K(x, x_0; \sigma)\,, \quad K(x, x_0; 0) = \frac{\delta^{(d)}(x - x_0)}{\sqrt{|\det h(x)|}}\,, \qquad (1)$$

where the Laplacian $\Delta = -h^{ij}\nabla_i\nabla_j$, $i, j = 1, \ldots, d$ or is a more general (pseudo)differential operator. The diffusion is characterized by the average return probability (the heat trace)

$$\mathcal{P}(\sigma) = \mathrm{Tr}_{V \subset M} e^{-\sigma\Delta} = V^{-1} \int_V d^d x \, \sqrt{|\det h(x)|}\, K(x, x; \sigma)\,. \qquad (2)$$

Then the spectral dimension of M can be extracted via the formula

$$d_S(\sigma) := -2\, \frac{d \log \mathcal{P}(\sigma)}{d \log \sigma}\,. \qquad (3)$$

In particular, for $\mathbb{R}^d$ with $\Delta = -\partial^i\partial_i$ we recover $d_S(\sigma) = d$.

Spectral dimension out of the heat operator

Heat trace definition extends from a Laplacian Δ acting on a manifold M to a closed operator T on a separable Hilbert space $\mathcal{H}$,

$$\mathcal{P}(\sigma) := \mathrm{Tr}_{\mathcal{H}} e^{-\sigma T} = \sum_{n=0}^{\infty} e^{-\sigma \lambda_n(T)}\,, \tag{4}$$

where λ_n are eigenvalues of T. To this end $e^{-\sigma T}$ needs to be trace-class, which is not always true for an abstract T.

- On a non-compact manifold M or for $\mathcal{H}$ with a non-compact algebra of observables, one has to introduce an IR cutoff F, so that

$$\mathcal{P}(\sigma, F) := \mathrm{Tr}_{\mathcal{H}} F\, e^{-\sigma T}\,; \tag{5}$$

  F may either factor out or lead to the IR/UV mixing.
- If the order of T is $\eta := \mathrm{ord}\, T \neq 2$, we should modify (3) to

$$d_S(\sigma) := -\eta \frac{d \log \mathcal{P}(\sigma)}{d \log \sigma} \tag{6}$$

  but $\mathrm{ord}\, T$ is ambiguous for an abstract T – cf. κ-Minkowski space.

Further subtleties of the spectral dimension

- In the case of a compact or curved classical-limit spacetime
  - if the kernel of T is trivial, $d_S(\sigma) \to \infty$ in the IR and has to be supplemented with the classical profile;
  - otherwise, $d_S(\sigma) \to 0$ in the IR, which can be remedied by replacing $d_S(\sigma)$ with the spectral variance

$$v_S(\sigma) := d_S(\sigma) - \sigma \frac{d}{d\sigma} d_S(\sigma) \,. \tag{7}$$

- If the full spectrum of T is unknown, $d_S(\sigma)$ can be approximated using a heat trace expansion but only deep in the UV regime[2].
- In order to calculate $d_S(\sigma)$ in a pseudo-Riemannian case, one first has to perform the Wick rotation, which is generally cumbersome.

[2] M. Eckstein & **T. T.**, in preparation

Heat trace in the general setting

The heat trace of a (pseudo)differential operator T on a manifold M has the asymptotic expansion at $\sigma = 0$,

$$\mathcal{P}(\sigma) \underset{\sigma\downarrow 0}{\sim} \sum_{k=0}^{\infty} a_k(T)\,\sigma^{(k-d)/\eta} + \sum_{l=0}^{\infty} b_l(T)\,\sigma^l \log\sigma\,; \tag{8}$$

- if T is differential, coefficients $a_k(T)$ are given by integrals of the geometric invariants of (the bundle over) M, while all $b_l(T) = 0$;
- in the case of a non-compact M, the expansion coefficients will generally depend on an IR cutoff F.

More generally, the asymptotic expansion of the heat trace of an unbounded operator T on a separable Hilbert space $\mathcal{H}$ is

$$\mathcal{P}(\sigma) = \mathrm{Tr}_{\mathcal{H}} e^{-\sigma T} \underset{\sigma\downarrow 0}{\sim} \sum_{k=0}^{\infty} \sum_{m\in\mathbb{Z}} \sum_{n=0}^{p} a_{z(k,m),n}\,(\log\sigma)^n\,\sigma^{-z(k,m)}\,. \tag{9}$$

The dimension spectrum of an operator

The dimension spectrum of an operator T is the set of exponents

$$\mathrm{Sd}(T) := \bigcup_{k,m} z(k,m) \subset \mathbb{C} \tag{10}$$

and $(p+1)$ is called the order of $\mathrm{Sd}(T)$.

- If we define the maximal real dimension

  $$d_{\mathrm{Sd}} := \sup_{z \in \mathrm{Sd}} \mathrm{Re}(z)\,, \tag{11}$$

  then the UV limit of the spectral dimension $\lim_{\sigma \to 0} d_S(\sigma) = \eta\, d_{\mathrm{Sd}}$.
- Dimensions $z(k,m) \not\subset \mathbb{R}$ correspond to oscillations of $\mathcal{P}(\sigma)$ at small scales, leading to oscillations of $d_S(\sigma)$ – cf. quantum sphere.
- Sd does not tell about the dimensional flow or the IR limit.

Other properties of the dimension spectrum

Existence of the asymptotic heat trace expansion is not proven in general. Moreover, sometimes it is easier to apply the Mellin transform

$$\int_0^\infty \mathrm{Tr} e^{-\sigma T}\, \sigma^{s-1}\, d\sigma = \Gamma(s)\, \zeta_T(s) \tag{12}$$

and consider the associated spectral zeta function

$$\zeta_T(s) := \mathrm{Tr}\, T^{-s}\,, \quad \mathrm{Re}(s) \gg 0\,; \tag{13}$$

poles of $\Gamma \cdot \zeta_T$ correspond to elements of Sd.

At a higher level, the dimension spectrum is defined for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is an algebra of observables represented on a Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is an unbounded operator acting on $\mathcal{H}$.

Topology of the (Podleś) quantum sphere

The quantum sphere is a homogeneous space of the q-deformed group $\mathrm{SU}_q(2)$, described by a $*$-algebra with the generators A, B and B^*,

$$AB = q^2 BA\,, \qquad BB^* = q^{-2} A(1-A)\,,$$
$$AB^* = q^{-2} B^* A\,, \qquad B^* B = A(1-q^2 A)\,, \tag{14}$$

where $q \in (0,1)$. In the classical limit $q \to 1$ we recover the algebra of continuous functions on S^2. The algebra (14) can be represented on either of the $\mathrm{SU}(2)$ Hilbert spaces that are spanned by vectors:

$$|j,m\rangle\,, \quad m \in \{-j, -j+1, \dots, j\}\,,\ j \in \mathbb{N}\,;$$
$$|l,m\rangle_\pm\,, \quad m \in \{-l, -l+1, \dots, l\}\,,\ l \in \mathbb{N} + \tfrac{1}{2}\,. \tag{15}$$

The classical scalar and spinorial Laplacians act in these spaces as

$$\Delta^{\mathrm{sc}} |j,m\rangle = j\,(j+1)\,|j,m\rangle\,,$$
$$\Delta^{\mathrm{sp}} |l,m\rangle_\pm = (l+\tfrac{1}{2})^2 |l,m\rangle_\pm\,. \tag{16}$$

Laplacians on the quantum sphere

The simplified Laplacian is the square of the so-called simplified Dirac operator, acting on basis states as (ill-defined for $q \to 1$)

$$\Delta_q^{\mathrm{sm}}|l,m\rangle_\pm = \frac{1}{(q^{-1}-q)^2}\, q^{-(2l+1)}|l,m\rangle_\pm\,. \tag{17}$$

The spinor Laplacian is given by the square of the full Dirac operator and acts on basis states as

$$\Delta_q^{\mathrm{sp}}|l,m\rangle_\pm = \frac{1}{(q^{-1}-q)^2}\left(q^{-(l+1/2)} - q^{l+1/2}\right)^2|l,m\rangle_\pm\,. \tag{18}$$

The scalar Laplacian is defined by the first Casimir of the Hopf algebra $\mathcal{U}_q(\mathfrak{su}(2))$ that acts on basis states as

$$\Delta_q^{\mathrm{sc}}|j,m\rangle = \frac{q^{1/2}}{(1-q)^2}\left(q^{-j} - 1 - q + q^{j+1}\right)|j,m\rangle\,. \tag{19}$$

Calculating the spectral dimension

The spectral dimension for the Laplacian $\Delta_q^{\rm sm}$ is given exactly by

$$d_S^{q,{\rm sm}}(\sigma) = -\,2\,\frac{\left[G'\left(\log(u\sigma)\right)+4\right]\log(u\sigma)+F'\left(\log(u\sigma)\right)}{2\log^2(u\sigma)+G\left(\log(u\sigma)\right)\log(u\sigma)+F\left(\log(u\sigma)\right)+R(u\sigma)}$$

$$\frac{+G\left(\log(u\sigma)\right)+u\sigma R'(u\sigma)}{}\,, \qquad (20)$$

where G, F are certain bounded, periodic functions and R is a convergent series. There are no exact formulae for d_S for other Laplacians but in the UV they can be expressed via (20) as

$$d_S^{q,{\rm sp}}(\sigma) = d_S^{q,{\rm sm}}(\sigma) + \mathcal{O}(\sigma)\,,$$
$$d_S^{q,{\rm sc}}(\sigma) = d_S^{\sqrt{q},{\rm sm}}(q^{-1/2}\sigma) + \mathcal{O}\left((\log\sigma)^{-2}\right). \qquad (21)$$

It justifies our choice of $\eta = 2$ in all cases. $d_S^{q,{\rm sm}}$ and $d_S^{q,{\rm sp}}$ both diverge in the IR, hence they little differ in general.

Spectral dims. for different Laplacians and varying q

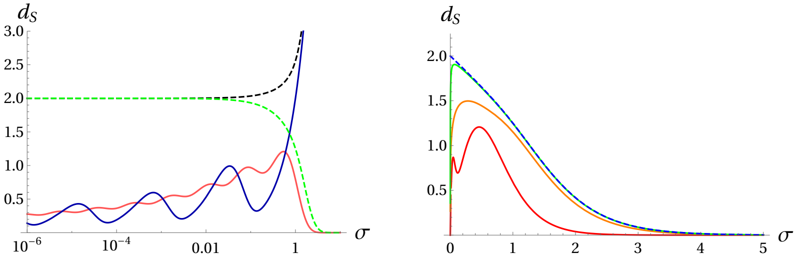

Figure: (left) spectral dim. for Δ_q^{sc} (red) and Δ_q^{sp} (blue) with $q = 0.15$, and for classical 2-sphere with Δ^{sc} (green) and Δ^{sp} (black) Laplacians; (right) spectral dim. for Δ_q^{sc}, with $q = 0.9$ (green), $q = 0.5$ (yellow) and $q = 0.1$ (red), and for classical 2-sphere with Δ^{sc} Laplacian (blue)

The amplitude of oscillations rapidly decreases with growing q.

M. Eckstein & **T. T.**, Phys. Rev. D **102**, 086003 (2020)

Dim. spectra in the classical and quantum cases

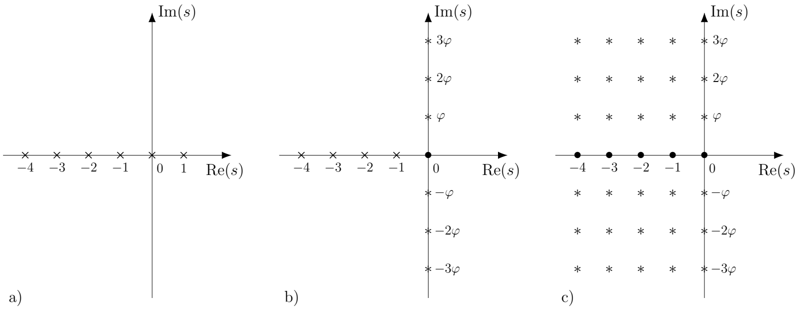

Figure: Dimension spectrum for different Laplacians (a) on classical 2-sphere $\mathrm{Sd}(\Delta^{\mathrm{sp}}) = \mathrm{Sd}(\Delta^{\mathrm{sc}})$ and quantum sphere (b) $\mathrm{Sd}(\Delta_q^{\mathrm{sm}})$, (c) $\mathrm{Sd}(\Delta_q^{\mathrm{sp}}) = \mathrm{Sd}(\Delta_q^{\mathrm{sc}})$; where $\varphi = \pi/\log q$ (and the symbols $\times$, $*$ and $\bullet$ denote elements of Sd corresponding to poles of the function $\Gamma \cdot \zeta$ of order 1, 2 and 3, respectively)

In particular, $d_{\mathrm{Sd}} = 0$ and $\mathrm{ord}\,\mathrm{Sd} = 3$ for all quantum Laplacians.

M. Eckstein, B. Iochum & A. Sitarz, Commun. in Math. Phys. **332**, 627 (2014)
M. Eckstein & **T. T.**, Phys. Rev. D **102**, 086003 (2020)

$n+1$-dimensional κ-Minkowski space

κ-Minkowski space is the spacetime covariant under the action of κ-Poincaré (Hopf) algebra. Its time and spatial coordinates satisfy

$$[X_0, X_a] = \frac{i}{\kappa} X_a\,, \qquad [X_a, X_b] = 0\,, \qquad a, b = 1, \ldots, n\,, \tag{22}$$

spanning the Lie algebra $\mathfrak{an}(n)$, which is a subalgebra of $\mathfrak{so}(n+1,1)$. In turn, $\mathfrak{an}(n)$ generates the group $\mathrm{AN}(n)$, whose elements are defined as the ordered exponentials of algebra elements, e.g. in the time-to-the-right ordering they have the form

$$g = e^{-ip^a X_a} e^{ip_0 X_0}\,, \quad p_0, p_a \in \mathbb{R}\,. \tag{23}$$

$\mathrm{AN}(n)$ (a subgroup of $\mathrm{SO}(n+1,1)$, with a $(n+2) \times (n+2)$ matrix representation) can be seen as the momentum space corresponding to κ-Minkowski space.

Calculating the heat trace

Calculations become simpler in classical coordinates

$$\begin{aligned} k_0 &= \kappa \sinh\left(\tfrac{p_0}{\kappa}\right) - \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,, \\ k_a &= e^{p_0/\kappa} p_a \,, \\ k_{-1} &= \kappa \cosh\left(\tfrac{p_0}{\kappa}\right) + \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,, \end{aligned} \tag{24}$$

satisfying $k_0^2 + k_a k^a - k_{-1}^2 = -\kappa^2$ and $k_{-1} > 0$. The heat kernel can be expressed, via the noncommutative Fourier transform, in the momentum space representation

$$K(x, x_0; \sigma) = \frac{1}{(2\pi)^d} \int d\mu(k)\; e^{-\sigma \mathcal{L}(k)} e^{ik(x - x_0)} \,, \tag{25}$$

where $\mathcal{L}(k)$ is the momentum-space version of a given Laplacian. κ-Minkowski space is actually non-compact but it has been shown that the IR regularization factorizes.

Laplacians in the momentum representation

The bicovariant Laplacian, determined by the bicovariant differential calculus on κ-Minkowski space, has the form

$$\mathcal{L}_{\rm cv}(k_0, \{k_a\}) = k_0^2 + k_a k^a \,. \tag{26}$$

The bicrossproduct Laplacian is the Euclideanized simplest Casimir of the κ-Poincaré algebra (and satisfies the relation $\mathcal{L}_{\rm cv} = \mathcal{L}_{\rm cp} + \frac{1}{4\kappa^2}\mathcal{L}_{\rm cp}^2$)

$$\mathcal{L}_{\rm cp}(k_0, \{k_a\}) = 2\kappa \left(\sqrt{k_0^2 + k_a k^a + \kappa^2} - \kappa \right) . \tag{27}$$

The relative-locality Laplacian is given by the (squared) distance along geodesics in Euclidean momentum space

$$\mathcal{L}_{\rm rl}(k_0, \{k_a\}) = \kappa^2 {\rm arccosh}^2 \left(\frac{1}{\kappa} \sqrt{k_0^2 + k_a k^a + \kappa^2} \right) . \tag{28}$$

Results for the spectral dimension

The spectral dimensions for all Laplacians in 3+1, 2+1 and 1+1 dim can be calculated analytically (some earlier results were numerical[3]). In particular, in the case of the bicovariant Laplacian we obtain

$$
\begin{aligned}
d_S^{(3+1)}(\sigma) &= 3 + 2\kappa^2\sigma \frac{2\kappa\sqrt{\sigma} - \sqrt{\pi}\, e^{\kappa^2\sigma}(2\kappa^2\sigma + 1)(1 - \mathrm{erf}(\kappa\sqrt{\sigma}))}{-2\kappa\sqrt{\sigma} + \sqrt{\pi}\, e^{\kappa^2\sigma}(2\kappa^2\sigma - 1)(1 - \mathrm{erf}(\kappa\sqrt{\sigma}))}\,, \\
d_S^{(2+1)}(\sigma) &= 2 + \frac{\kappa^2\sigma\, U(\frac{3}{2}, 1, \kappa^2\sigma)}{U(\frac{1}{2}, 0, \kappa^2\sigma)}\,, \\
d_S^{(1+1)}(\sigma) &= 1 + 2\kappa^2\sigma \left(\frac{1}{\sqrt{\pi}\,\kappa\sqrt{\sigma}} \frac{e^{-\kappa^2\sigma}}{1 - \mathrm{erf}(\kappa\sqrt{\sigma})} - 1 \right)\,, \qquad (29)
\end{aligned}
$$

where $\mathrm{erf}(.)$ is the error function and $U(.,.,.)$ a Tricomi confluent hypergeometric function.
Formulae for the relative-locality Laplacian are similarly complicated.

[3] D. Benedetti, Phys. Rev. Lett. **102**, 111303 (2009)

Results for the spectral dimension – cont.

In the bicrossproduct Laplacian case, the expressions are simpler

$$
\begin{aligned}
d_S^{(3+1)}(\sigma) &= 6 - \frac{4\kappa^2\sigma}{2\kappa^2\sigma + 1}\,, \\
d_S^{(2+1)}(\sigma) &= 4 - 4\kappa^2\sigma\left(1 - \frac{K_0(2\kappa^2\sigma)}{K_1(2\kappa^2\sigma)}\right)\,, \\
d_S^{(1+1)}(\sigma) &= 2\,,
\end{aligned}
\tag{30}
$$

where $K_\alpha(.)$ is a modified Bessel function of the second kind. At small scales $\sigma\kappa^2 \approx 0$, we observe the dimensional drop for $\mathcal{L}_{\rm cv}$, dimensional rise for $\mathcal{L}_{\rm cp}$ and divergence for $\mathcal{L}_{\rm rl}$,

$$
\lim_{\sigma\to 0} d_S^{(n+1,{\rm cv})} = n\,, \qquad \lim_{\sigma\to 0} d_S^{(n+1,{\rm cp})} = 2n\,, \tag{31}
$$

while at large scales we always recover $\lim_{\sigma\to\infty} d_S^{(n+1)} = n+1$. In the above it was assumed that $\eta = 2$ for all Laplacians.

Comparing spectral dims. for different Laplacians

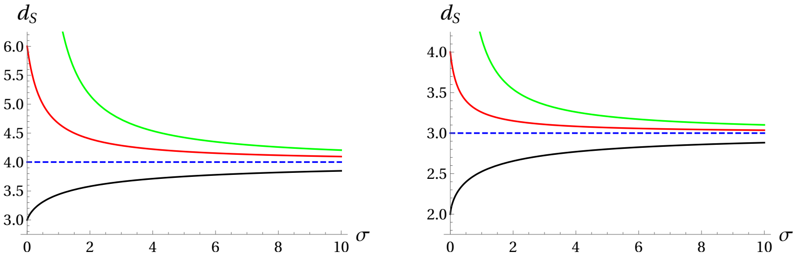

Figure: Spectral dims. for $\mathcal{L}_{\rm cv}$ (black), $\mathcal{L}_{\rm cp}$ (red) and $\mathcal{L}_{\rm rl}$ (green) Laplacians in 3+1 dim (left) and 2+1 dim (right)

Looking at (27), (28), one may argue that $\eta(\mathcal{L}_{\rm cp}) = 1$ and $\eta(\mathcal{L}_{\rm cp}) = 0$. Thus, all $d_S(\sigma)$ curves could in principle be superimposed by using $\eta = \eta(\kappa)$, such that $\lim_{\kappa \to \infty} \eta(\kappa) = 2$ and the appropriate $\eta(\kappa \approx 0)$.

M. Arzano & **T. T.**, Phys. Rev. D **89**, 124024 (2014)
M. Eckstein & **T. T.**, Phys. Rev. D **102**, 086003 (2020)

Dimension spectrum for different Laplacians

Expanding heat traces, we can read out the dimension spectra

$$\begin{aligned} \mathrm{Sd}_{(3+1)} &= \{\tfrac{3}{2}\} \cup \{\tfrac{1-n}{2} \,|\, n \in \mathbb{N}\} = \{\tfrac{3}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1, \ldots\}, \quad \mathrm{ord}\,\mathrm{Sd} = 1\,, \\ \mathrm{Sd}_{(2+1)} &= 1 - \mathbb{N} = \{1, 0, -1, -2, \ldots\}, \quad \mathrm{ord}\,\mathrm{Sd} = 2\,, \\ \mathrm{Sd}_{(1+1)} &= \tfrac{1}{2}(1 - \mathbb{N}) = \{\tfrac{1}{2}, 0, -\tfrac{1}{2}, -1, \ldots\}, \quad \mathrm{ord}\,\mathrm{Sd} = 1 \end{aligned} \tag{32}$$

for $\mathcal{L}_{\mathrm{cv}}$ and

$$\begin{aligned} \mathrm{Sd}_{(3+1)} &= \{3, 2\}, \quad \mathrm{ord}\,\mathrm{Sd} = 1\,, \\ \mathrm{Sd}_{(2+1)} &= 2 - \mathbb{N} = \{2, 1, 0, -1, -2, \ldots\}, \quad \mathrm{ord}\,\mathrm{Sd} = 2\,, \\ \mathrm{Sd}_{(1+1)} &= \{1\}, \quad \mathrm{ord}\,\mathrm{Sd} = 1 \end{aligned} \tag{33}$$

for $\mathcal{L}_{\mathrm{cp}}$. Here we assumed that $\eta = 2$ for both Laplacians, which is not necessarily accurate. In the $\mathcal{L}_{\mathrm{rl}}$ case the dimension spectra do not exist due to the divergent factor $e^{1/\sigma}$ in the heat traces.

M. Eckstein & **T. T.**, Phys. Rev. D **102**, 086003 (2020)

Comparison of two quantum spaces

How differences in geometry are uncovered:

qS^2 $\operatorname{ord}\mathrm{Sd} = 3$ corresponds to $d_S(\sigma \approx 0) \sim -4/\log\sigma$

qS^2 Identical Sd's but different $d_S(\sigma)$'s for the Laplacians Δ_q^{sp} and Δ_q^{sc}

κM $\operatorname{ord}\mathrm{Sd} = 2$ corresponds to $d_S(\sigma \approx 0) \sim 2\alpha/(\alpha + \beta\,\sigma\log\sigma)$ for $\mathcal{L}_{\mathrm{cv}}$ and $d_S(\sigma \approx 0) \sim 2 + 2\alpha/(\alpha + \beta\,\sigma\log\sigma)$ for $\mathcal{L}_{\mathrm{cp}}$

κM Sd's cannot coincide even for the order $\eta = \eta(\kappa)$ defined so that $d_S^{(n+1)}(\sigma)$ would not depend on a Laplacian

Independent on a choice of Laplacian:

qS^2 The presence of oscillations in $d_S(\sigma)$ – IR/UV mixing?

qS^2 Third order poles in Sd – presence of singularities?

κM The lack of oscillations in $d_S(\sigma)$ – less fractal structure?

κM Second order poles in Sd in 2+1d – a distinguished case?

Summary

Conclusions and open questions

- It is much more informative to study all heat trace properties than only the spectral dimension or dimension spectrum
- The spectral dimension does not easily see the possible structure of complex exponents and oscillations
- The latter arise in systems with the discrete scale invariance
- The dimension spectrum does not capture the scale dependence, including the classical (IR) limit
- The latter may track the emergence of self-similarity in the UV
- The oscillations may possibly affect CMB, stochastic GW background, thermodynamics of photons...
- What is the reason for radical differences between our examples?
- Should the order of an operator be defined as scale-dependent?

κ-Poincaré (Hopf) algebra in 3+1 dim

The κ-Poincaré algebra is a particular deformation of the Poincaré algebra. In the so-called bicrossproduct basis, its Lorentz subalgebra is undeformed, to wit ($a = 1, 2, 3$, $\mu = 0, 1, 2, 3$)

$$[M_a, M_b] = i\epsilon_{abc}M^c\,, \quad [M_a, N_b] = i\epsilon_{abc}N^c\,, \quad [N_a, N_b] = -i\epsilon_{abc}M^c\,,$$
$$[M_a, P_0] = 0\,, \quad [M_a, P_b] = i\epsilon_{abc}P^c\,, \quad [P_\mu, P_\nu] = 0 \tag{34}$$

and the deformation occurs only for the brackets

$$[N_a, P_0] = iP_a\,,$$
$$[N_a, P_b] = i\delta_{ab}\left(\frac{\kappa}{2}\left(1 - e^{-2P_0/\kappa}\right) + \frac{1}{2\kappa}P_cP^c\right) - \frac{i}{\kappa}P_aP_b\,, \tag{35}$$

where $\kappa \in \mathbb{R}_+$, while the classical limit is given by $\kappa \to +\infty$. The κ-Poincaré algebra is also a non-trivial coalgebra with the antipode.

κ-Poincaré algebra – the coalgebra

The coproducts and antipodes for its Lorentz generators have the form

$$\Delta M_a = M_a \otimes \mathbf{1} + \mathbf{1} \otimes M_a \,, \qquad S(M_a) = -M_a \,,$$
$$\Delta N_a = N_a \otimes \mathbf{1} + e^{-K_0/\kappa} \otimes N_a + \frac{1}{\kappa}\epsilon_{abc} P^b \otimes M^c \,,$$
$$S(N_a) = -e^{P_0/\kappa} N_a + \frac{1}{\kappa}\epsilon_{abc} e^{P_0/\kappa} P^b M^c \,. \tag{36}$$

The κ-Poincaré algebra can be obtained from the q-deformed anti-de Sitter algebra $U_q(\mathfrak{so}(3,2))$ by taking the limit of the de Sitter radius $R \to \infty$ and the deformation parameter $q \to 1$, with the fixed ratio

$$R \log q \equiv \kappa^{-1} \,. \tag{37}$$

In the bicrossproduct basis used above, this Hopf algebra becomes $U(\mathfrak{so}(3,1)) \blacktriangleright\!\!\lhd \mathcal{T}$, where $\mathcal{T}$ is the enveloping algebra of translations.

Coalgebraic structure of momenta

The product of two plane waves $g = e^{-ip^a X_a} e^{ip_0 X_0}$, $h = e^{-iq^a X_a} e^{iq_0 X_0}$ is

$$g\, h = e^{-i(p^a \oplus q^a) X_a} e^{i(p_0 \oplus q_0) X_0} = e^{-i(p^a + e^{-p_0/\kappa} q^a) X_a} e^{i(p_0 + q_0) X_0} \,. \qquad (38)$$

The non-abelian addition $p_\mu \oplus q_\mu$ can be reconstructed by the translation generators P_μ acting as $P_\mu(p) = p_\mu$, $P_\mu(q) = q_\mu$ on a pair of points (p, q) in momentum space via the coproducts

$$\Delta P_0 = P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0 \,, \qquad \Delta P_a = P_a \otimes \mathbf{1} + e^{-P_0/\kappa} \otimes P_a \,. \qquad (39)$$

The inverse element $g^{-1} = e^{-i(\ominus p^a) X_a} e^{i(\ominus p_0) X_0} = e^{ie^{p_0/\kappa} p^a X_a} e^{-ip_0 X_0}$ is similarly given by the action of the antipodes

$$S(P_0) = -P_0 \,, \qquad S(P_a) = -e^{P_0/\kappa} P_a \,. \qquad (40)$$

Lorentzian mapping of momentum space

Acting with g on a spacelike vector $(0,\ldots,0,\kappa)$ one obtains $g \triangleright (0,\ldots,0,\kappa) = (k_0, \{k_a\}, k_{-1})$, where

$$\begin{aligned} k_0 &= \kappa \sinh\left(\tfrac{p_0}{\kappa}\right) + \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,, \\ k_a &= e^{p_0/\kappa} p_a \,, \\ k_{-1} &= \kappa \cosh\left(\tfrac{p_0}{\kappa}\right) - \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,. \end{aligned} \tag{41}$$

The coordinates obey $-k_0^2 + k_a k^a + k_{-1}^2 = \kappa^2$ and $k_0 + k_{-1} > 0$. In the classical limit $\kappa \to \infty$ we recover

$$\lim_{\kappa\to\infty} k_0 = p_0 \,, \qquad \lim_{\kappa\to\infty} k_a = p_a \,, \qquad \lim_{\kappa\to\infty} k_{-1} = \infty \,. \tag{42}$$

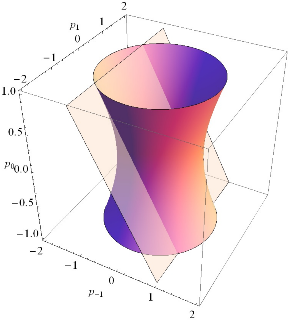

Figure: Lorentzian space of momenta

Euclidean mapping of momentum space

Acting with g on a timelike vector $(\kappa, 0, \ldots, 0)$ one obtains $g \triangleright (\kappa, 0, \ldots, 0) = (k_{-1}, \{k_a\}, k_0)$, where

$$\begin{aligned} k_0 &= \kappa \sinh\left(\frac{p_0}{\kappa}\right) - \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,, \\ k_a &= e^{p_0/\kappa} p_a \,, \\ k_{-1} &= \kappa \cosh\left(\frac{p_0}{\kappa}\right) + \frac{1}{2\kappa} e^{p_0/\kappa} p_a p^a \,. \end{aligned} \tag{43}$$

The coordinates obey $k_0^2 + k_a k^a - k_{-1}^2 = -\kappa^2$ and $k_{-1} > 0$. This can also be achieved via the Wick rotation ($\kappa \mapsto i\kappa, p_0 \mapsto ip_0$) and ($k_0 \mapsto ik_0, k_{-1} \mapsto ik_{-1}$).

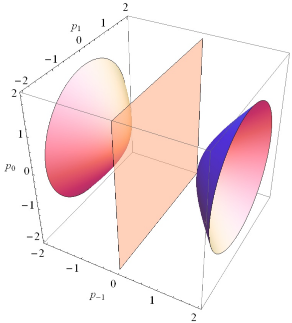

Figure: Euclidean space of momenta