Stochasticity of the BKL scenario

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Theory of Relativity Seminar

(IFT FUW and DBP NCBJ)

Based on:

[1] P. Goldstein and WP,
"Generic instability of the dynamics underlying
the Belinski-Khalatnikov-Lifshitz scenario",
Eur. Phys. J. C (2022) 82: 216.

[2] A. Góźdź, A. Pȩdrak, and WP,
"Quantum dynamics corresponding to chaotic BKL scenario",
Eur. Phys. J. C (2023), to be published,
arXiv:2204.11274 [gr-qc].

[3] V. Belinski and WP, collaboration since 2010.

OUTLINE

Introduction

- A. Friedmann model (1922)
  - assumes isotropy and homogeneity of space
  - solution includes gravitational singularity
- E. Lifshitz analysed Friedmann's solution (1946):
  isotropy is unstable in the evolution towards singularity

  E. M. Lifshitz, J. Phys., U. S. S. R. **10**, 116 (1946); E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963)
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- Dynamics of BVIII and BIX was analyzed to get insight into the dynamics of spacetime near the cosmological spacelike singularity

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  general relativity implies the existence of generic solution that is singular (incomplete geodesics and diverging invariants)
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  - is stable against perturbation of initial data
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BKL conjecture and singularity theorems

- The Penrose-Hawking singularity theorems (of 60-ties) concern possible existence of incomplete geodesics in spacetime, but incompleteness does not mean (in general) that the invariants diverge.
- These theorems say little about the dynamics of gravitational field near singularities so that are of little usefulness in the context of finding corresponding quantum dynamics.
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Models of the BKL scenario

The BKL scenario presents a very complicated dynamics so that to work with it one needs to use models.

There exist two convenient models:

- the vacuum model of the BKL scenario (called mixmaster); vacuum BIX model; exact model of the dynamics

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- the massive model of the BKL scenario; includes effectively some contribution from matter field; presents asymptotic dynamics near the singularity of BIX model

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Dynamics of massive model of the BKL scenario

In what follows we consider the dynamics of the massive model:

$$\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}, \tag{1}$$

where $a = a(t) > 0,\ b = b(t) > 0,\ c = c(t) > 0$ are effective directional scale factors, and $t \in \mathbb{R}$ is a monotonic function of proper time.

The solutions to (1) must satisfy the constraint

$$\frac{d \ln a}{dt}\frac{d \ln b}{dt} + \frac{d \ln a}{dt}\frac{d \ln c}{dt} + \frac{d \ln b}{dt}\frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{2}$$

Eqs (1)-(2) present the essence of the dynamics underlying the BKL scenario.

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Eqs (1)-(2) present the essence of the dynamics underlying the BKL scenario.

BKL scenario (cont)

Massive model of BKL scenario has numerical support (via considering general BIX):

- Simulations of the dynamics near the singularity confirm the asymptotic dynamics

  C. Kiefer, N. Kwidzinski, and W.P., Eur. Phys. J. C (2018) 78:691

- Kretschman's curvature invariant diverge in the evolution towards the singularity

  N. Kwidzinski and W.P., Eur. Phys. J. C (2019) 79:199

Solution to the BKL scenario

We have found exact solution to the dynamics (1)–(2):

P. Goldstein and W.P., Eur. Phys. J. C (2022) 82: 216

$$\tilde{a}(t) = \frac{3}{t - t_0}, \quad \tilde{b}(t) = \frac{30}{(t - t_0)^3}, \quad \tilde{c}(t) = \frac{120}{(t - t_0)^5}, \tag{3}$$

where $t > t_0$ and where $t_0 < 0$ is an arbitrary real number.

The solution (3) is unstable against small perturbation:

$$a(t) = \tilde{a}(t) + \epsilon\alpha(t), \tag{4a}$$

$$b(t) = \tilde{b}(t) + \epsilon\beta(t), \tag{4b}$$

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Solution to the BKL scenario (cont)

Inserting (4) into (1)–(2) leads, in the first order in ϵ, to the following exact solution of the resulting equations:

$$\alpha(t) = \exp(-\theta/2)\big[K_1 \cos(\omega_1\theta+\varphi_1)+K_2 \cos(\omega_2\theta+\varphi_2)\big]+K_3 \exp(-2\theta), \tag{5a}$$

$$\beta(t) = \exp(-5\theta/2)\big[(4+6\sqrt{6})K_1 \cos(\omega_1\theta+\varphi_1) \tag{5b}$$

$$+ (4-6\sqrt{6})K_2 \cos(\omega_2\theta+\varphi_2)\big] + 30K_3 \exp(-4\theta), \tag{5c}$$

$$\gamma(t) = -4\exp(-9\theta/2)\big[(26+9\sqrt{6})K_1 \cos(\omega_1\theta+\varphi_1) \tag{5d}$$

$$+ (26-9\sqrt{6})K_2 \cos(\omega_2\theta+\varphi_2)\big] + 200K_3 \exp(-6\theta)\,, \tag{5e}$$

where $\theta = \ln(t-t_0)$. The two frequencies read

$$\omega_1 = \frac{1}{2}\sqrt{95-24\sqrt{6}}, \qquad \omega_2 = \frac{1}{2}\sqrt{95+24\sqrt{6}}\,,$$

where $K_1, K_2, K_3, \varphi_1, \varphi_2$ are arbitrary (to some extent) constants.

Chaotic phase of the BKL scenario

- The manifold $\mathcal{M}$ defined by $\{K_1, K_2, K_3, \varphi_1, \varphi_2\}$ is a submanifold of $\mathbb{R}^5$. Thus, (5) presents generic solution as the measure of $\mathcal{M}$ is nonzero.
- The relative perturbations $\alpha/a, \beta/b$, and γ/c grow as $\exp(\frac{1}{2}\theta)$.
  - The multiplier $1/2$ plays the role of a Lyapunov exponent, describing the rate of divergences.
  - Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) is chaotic.
- Chaos results from strong nonlinearity of the dynamics and growing curvature of spacetime (increasing effectively the nonlinearity) in the evolution towards the singularity.

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Chaotic phase of BKL scenario (cont)

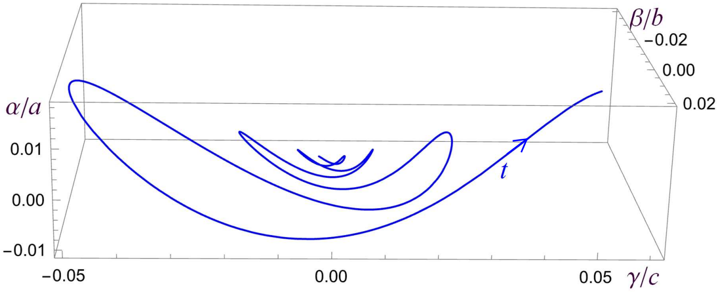

Figure: Linear instability of the special solution (3). The graph presents the parametric curve defined by the time dependence of α/a, β/b, and γ/c.

Quantization of the BKL scenario

In what follows, we quantize the BKL scenario by making use of the integral quantization method (IQM).

We have already quantized Hamilton's dynamics of that scenario ignoring its chaotic phase

- quantum singularity turns into quantum bounce
- quantum evolution is unitary across quantum bounce

A. Góźdź, W.P., and G. Plewa, Eur. Phys. J. C **79**, 45 (2019); A. Góźdź and W.P., Eur. Phys. J. C **80**, 142 (2020)

Quantization of the chaotic phase of the BKL scenario:

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Quantization of the BKL scenario (cont)

Two novelties:

- direct quantization of the solution to the classical dynamics instead of physical Hamiltonian
- quantization of time variable on the same footing as spatial variables

In what follows we apply the IQM to the chaotic BKL scenario.

For details of IQM, see Appendix A.

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Configuration space

Definition of the configuration space:

$$T = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) : \xi \in (\mathbb{R}\times\mathbb{R}_+)\times(\mathbb{R}\times\mathbb{R}_+)\times(\mathbb{R}\times\mathbb{R}_+)\}\,, \quad (6)$$

where every pair (ξ_k, ξ_{k+1}), $(k = 1, 3, 5)$, parameterizes the affine group $\mathrm{Aff}(\mathbb{R})$.

ξ_1, ξ_3, and ξ_5 denote 3 time variable;
scale factors are denoted as follows: $\xi_2 = a$, $\xi_4 = b$, $\xi_6 = c$.
Because $a, b, c > 0$ and $\xi_1, \xi_3, \xi_5 \in \mathbb{R}$, the configuration space parameterizes the simple product of 3 affine groups $\mathrm{Aff}(\mathbb{R}) \times \mathrm{Aff}(\mathbb{R}) \times \mathrm{Aff}(\mathbb{R}) =: G$ to be used in quantization.

As the observational data are parameterized by a single time parameter, the variables $\{\xi_1, \xi_3, \xi_5\}$ should be mapped onto a single variable representing time.

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Hilbert space

The direct product of three affine groups G has the unitary irreducible representation in the following Hilbert space:
$\mathcal{H} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2} \otimes \mathcal{H}_{x_3} = L^2(\mathbb{R}_+^3, d\nu(x_1, x_2, x_3))$,
where $d\nu(x_1, x_2, x_3) = d\nu(x_1)d\nu(x_2)d\nu(x_3)$.

It enables defining in $\mathcal{H}$ the continuous family of affine coherent states
$\langle x_1, x_2, x_3 | \xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6 \rangle := \langle x_1 | \xi_1, \xi_2 \rangle \langle x_2 | \xi_3, \xi_4 \rangle \langle x_3 | \xi_5, \xi_6 \rangle$,
as follows

$$\mathcal{H} \ni \langle x_1, x_2, x_3 | \xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6 \rangle := U(\xi)\Phi_0(x_1, x_2, x_3)\,, \tag{7}$$

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Quantization of observables

The resolution of the identity in $\mathcal{H}$ can be used for mapping a classical observable $f : T \to \mathbb{R}$ into an operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ as follows

$$\begin{aligned} \hat{f} &:= \frac{1}{A_\phi} \int_G d\mu(\xi) |\xi\rangle f(\xi) \langle\xi| \\ &= \frac{1}{A_{\Phi_1} A_{\Phi_3} A_{\Phi_5}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_1, \xi_2) \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_3, \xi_4) \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_5, \xi_6) \\ &|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle f(\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6) \langle \xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6| \,. \qquad (9) \end{aligned}$$

There exist two important characteristics of quantum observables:

- expectation values and variances; they allow to compare quantum and classical worlds
- they correspond to classical values of measured quantities; variances describe quantum smearing of observables.

For definition of variance of observable, see Appendix B.

Quantization of observables

The resolution of the identity in $\mathcal{H}$ can be used for mapping a classical observable $f : T \to \mathbb{R}$ into an operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ as follows

$$\begin{aligned} \hat{f} &:= \frac{1}{A_\phi} \int_G d\mu(\xi) |\xi\rangle f(\xi) \langle\xi| \\ &= \frac{1}{A_{\Phi_1} A_{\Phi_3} A_{\Phi_5}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_1, \xi_2) \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_3, \xi_4) \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_5, \xi_6) \\ & |\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle f(\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6) \langle \xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6| \,. \end{aligned} \tag{9}$$

There exist two important characteristics of quantum observables:

- expectation values and variances; they allow to compare quantum and classical worlds
- they correspond to classical values of measured quantities; variances describe quantum smearing of observables.

For definition of variance of observable, see Appendix B.

Equations of motion

Classical observables should be related to their quantum counterparts by the corresponding expectation values.

This idea leads directly to the conditions for a family of states $\{\Psi_\eta(x_1, x_2, x_3) = \langle x_1, x_2, x_3|\Psi_\eta\rangle, \eta \in \mathbb{R}^s\}$ parameterized by a set of evolution parameters $\eta = (\eta_1, \eta_2, \dots \eta_s)$ enumerating the set of trial functions. We require the states $|\Psi_\eta\rangle$ to satisfy:

$$\langle\Psi_\eta|\hat{\xi}_k|\Psi_\eta\rangle = t, \quad k = 1, 3, 5 \tag{10}$$

$$\langle\Psi_\eta|\hat{\xi}_2|\Psi_\eta\rangle = a(t)\ , \tag{11}$$

$$\langle\Psi_\eta|\hat{\xi}_4|\Psi_\eta\rangle = b(t)\ , \tag{12}$$

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Eq. (10) represents the single time constraint. The parameter η labels the family of states to be found, and it should be a function of t as the r.h.s. of (10)–(13) depends on t.The solution of Eqs. (10)–(13) allows to construct the vector state dependent on classical time, $|\Psi_{\eta(t)}\rangle \in \mathcal{H}$. Eqs. (10)–(13) define quantum equations of motion.

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Eqs. (10)–(13) define quantum equations of motion.

Evolving wave packets

Example 1:
The coherent states

$$|CS_\epsilon; t\rangle := |t, a(t), b(t), c(t)\rangle \tag{14}$$

satisfy the equations of motions (10)–(13) with $\tau = t$.

Realization of (14) as a wave packet constructed in the space of square integrable functions $L^2(\mathbb{R}^3_+, d\nu(x_1, x_2, x_3))$ reads

$$\Psi_{CS_\epsilon}(t, x_1, x_2, x_3) = \langle x_1, x_2, x_3 | CS_\epsilon; t\rangle$$
$$= e^{it(x_1+x_2+x_3)}\Phi_1(a(t)x_1)\Phi_2(b(t)x_2)\Phi_3(c(t)x_3)\,. \tag{15}$$

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Evolving wave packets (cont)

Example 2:

The Gaussian distribution wave packets:

$$\Psi_n(x;\tau,\gamma) = N x^n \exp\left[i\tau x - \frac{\gamma^2 x^2}{2}\right], \quad N^2 = \frac{2\gamma^n}{(n-1)!}, \tag{16}$$

which are dense in $L^2(\mathbb{R}_+, d\nu(x))$.

Expectation values and variances of $\hat{\xi}_k$ and $\hat{\xi}_{k+1}$ are:

$$\langle\Psi_n|\hat{\xi}_k|\Psi_n\rangle = \tau, \quad k = 1,3,5, \tag{17}$$

$$\langle\Psi_n|\hat{\xi}_{k+1}|\Psi_n\rangle = \frac{1}{A_\Phi}\frac{\Gamma(n-\frac{1}{2})}{(n-1)!}\gamma, \tag{18}$$

$$var(\hat{\xi}_k;\Psi_n) = \frac{4n-3}{4(n-1)}\gamma^2, \tag{19}$$

$$var(\hat{\xi}_{k+1};\Psi_n) = \frac{1}{A_\Phi^2}\Big(\frac{1}{n-1} - \frac{\Gamma(n-\frac{1}{2})^2}{(n-1)!^2}\Big)\gamma^2. \tag{20}$$

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Evolving wave packets (cont)

In the space $L^2(\mathbb{R}^3_+, d\nu(x_1, x_2, x_3))$, we take the corresponding wave packets:

$$\Psi_{n_1,n_3,n_5}(x_1, x_2, x_3;\ \tau_1, \tau_3, \tau_5, \gamma_1, \gamma_3, \gamma_5) = \Psi_{n_1}(x_1; \tau_1, \gamma_1)\Psi_{n_3}(x_2; \tau_3, \gamma_3)\Psi_{n_5}(x_3; \tau_5, \gamma_5)\,. \tag{21}$$

To meet the properties (10)–(13) for the wave packets Ψ_{n_1,n_3,n_5}, we choose the parameters τ_k and γ_k as follows:

$$\tau_1 = \tau_3 = \tau_5 = t\,, \tag{22}$$

$$\gamma_k = A_{\Phi_k} \frac{(n_k - 1)!}{\Gamma\left(n_k - \frac{1}{2}\right)} \cdot f_k(t)\,, \quad k = 1, 3, 5\,, \tag{23}$$

where

$$f_k(t) = \begin{cases} \tilde{a}(t) + \epsilon\alpha(t)\,, & k = 1 \\ \tilde{b}(t) + \epsilon\beta(t)\,, & k = 3 \\ \tilde{c}(t) + \epsilon\gamma(t)\,, & k = 5 \end{cases}\,. \tag{24}$$

Evolving wave packets (cont)

Variances in the Hilbert space $\mathcal{H}$ for the Gaussian wave packets read:

$$var(\hat{\xi}_k; \Psi_{n_1,n_3,n_5}) = \mathcal{A}_k f_k(t)^2 \,, \tag{25}$$
$$var(\hat{\xi}_{k+1}; \Psi_{n_1,n_3,n_5}) = \mathcal{B}_k f_k(t)^2 \,, \tag{26}$$

where

$$\mathcal{A}_k = A^2_{\Phi_k} \frac{(4n_k - 3)(n_k - 1)!(n_k - 2)!}{4\Gamma\left(n_k - \frac{1}{2}\right)^2} \,, \tag{27}$$
$$\mathcal{B}_k = \frac{(n_k - 1)!(n_k - 2)!}{\Gamma\left(n_k - \frac{1}{2}\right)^2} - 1 \,. \tag{28}$$

These results show that all positions of our system in time and space are smeared owing to nonzero variances. It is an important fact about possibility of avoiding singularities in this dynamics.

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Stochastic aspects of BKL scenario

Having calculated the variances of quantum observables corresponding to perturbed $\{a, b, c\}$ and unperturbed $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ solutions, we describe the quantum instabilities as follows:

$$\kappa_k := \frac{var(\hat{\xi}_k; \Psi_{pert}) - var(\hat{\xi}_k; \Psi_{unpert})}{var(\hat{\xi}_k; \Psi_{unpert})}, \qquad k = 2, 4, 6 \tag{29}$$

where $\hat{\xi}_2 = \hat{a}$, $\hat{\xi}_4 = \hat{b}$, $\hat{\xi}_6 = \hat{c}$, and where Ψ_{pert} and Ψ_{unpert} denote perturbed and unperturbed wave packets, respectively.

Stochastic aspects of BKL scenario (cont)

Making use of

$$\begin{aligned} f_2(t)^2 &= (\tilde{a}(t) + \epsilon\alpha(t))^2 = \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t) + \epsilon^2\alpha(t)^2 \simeq \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t)\,, \\ f_4(t)^2 &= (\tilde{b}(t) + \epsilon\beta(t))^2 = \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t) + \epsilon^2\beta(t)^2 \simeq \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t)\,, \\ f_6(t)^2 &= (\tilde{c}(t) + \epsilon\gamma(t))^2 = \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t) + \epsilon^2\gamma(t)^2 \simeq \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t)\,. \end{aligned}$$

we obtain explicit form of (29), which in the 1-st order in ϵ, reads:

$$\kappa_a(t) := \kappa_2(t) = \frac{2\epsilon\tilde{a}(t)\alpha(t)}{\tilde{a}(t)^2} = 2\epsilon\frac{\alpha(t)}{\tilde{a}(t)}\,, \tag{30}$$

$$\kappa_b(t) := \kappa_4(t) = \frac{2\epsilon\tilde{b}(t)\beta(t)}{\tilde{b}(t)^2} = 2\epsilon\frac{\beta(t)}{\tilde{b}(t)}\,, \tag{31}$$

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The relative perturbations (30)–(32) are the same for the coherent states and the exponential wave packets.

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Stochastic aspects of quantum evolution (cont)

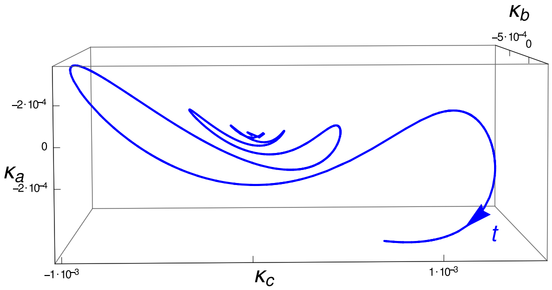
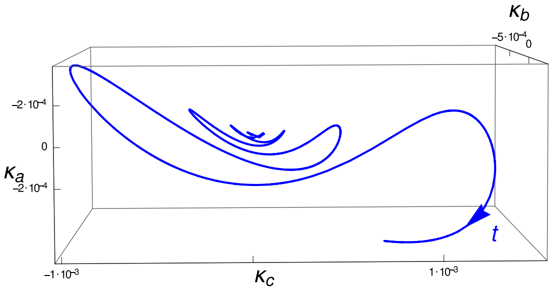


Figure: Parametric curve of relative quantum perturbations

Higher order approximations in ϵ would not change much the plot.

Stochastic aspects of quantum evolution (cont)



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Conclusions

- Since relative quantum and classical perturbations have quite similar time evolutions, we can say that quantization does not destroy classical chaos.
- Quantum chaos corresponds to classical chaos in the lowest order approximation.
- Nonlinearity of classical dynamics creates deterministic chaos. Non-vanishing variances of observables of the corresponding quantum dynamics create stochastic chaos.
- As calculated variances are always non-zero, the probability of obtaining divergencies of quantum observables corresponding to classical gravitational singularity is equal to zero.

A quantum system is in eigenstate of given operator iff its variance equals zero.

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Thank you!

Essence of integral quantization

If the configuration space Π is a half-plane,

$$\Pi := \{(p,q) \in \mathbb{R} \times \mathbb{R}_+\}, \quad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\},$$

it can be identified with the affine group Aff($\mathbb{R}$) =: G.
Multiplication law reads

$$(p_1, q_1) \cdot (p_2, q_2) := (p_1 + q_1 p_2, q_1 q_2) , \tag{33}$$

with the unity $(0,1)$ and the inverse $(p,q)^{-1} = (-p/q, 1/q)$.
This group has unitary irreducible representation realized in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x)) =: \mathcal{H}$, where $d\nu(x) = dx/x$, defined by

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in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x)) =: \mathcal{H}$, where $d\nu(x) = dx/x$,
defined by

$$U(p, q)\psi(x) = e^{ipx}\psi(qx), \quad \psi(x) \in \mathcal{H}\ . \tag{34}$$

Essence of integral quantization (cont)

Eq. (34) enables defining the continuous family of affine coherent states (ACS), denoted $\langle x|p,q\rangle \in \mathcal{H}$, as follows

$$\langle x|p,q\rangle = U(p,q)\langle x|\phi\rangle \ , \tag{35}$$

where $\langle x|\phi\rangle =: \phi(x) \in \mathcal{H}$ is the so-called fiducial vector, which is a free parameter (to some extent) of ACS quantization scheme.

Eq. (35) can be interpreted as the correspondence

$$(p,q) \longrightarrow |p,q\rangle\langle p,q| \tag{36}$$

between point of configuration space Π and quantum projection operator acting in $\mathcal{H}$.

The space of coherent states is strongly entangled:

$$\langle t,r|t',r'\rangle \neq 0 \quad \text{if} \quad t \neq t' \ \text{ or } \ r \neq r' \ , \tag{37}$$

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Essence of integral quantization (cont)

The irreducibility of the representation leads (due to Schur's lemma) to the resolution of the unity in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\frac{1}{A_\phi}\int_G d\mu(p,q)|p,q\rangle\langle p,q| = \mathbb{I}\,, \tag{39}$$

where $d\mu(p,q) := dp\, dq/q^2$ is the left invariant measure on G, and where $A_\phi := \int_0^\infty |\phi(x)|^2 \frac{dx}{x^2} < \infty$ is a constant.

Using (39), enables quantization of any observable $f : \Pi \to \mathbb{R}$

$$f \longrightarrow \hat{f} = \frac{1}{A_\phi}\int_G d\mu(p,q)|p,q\rangle f(p,q)\langle p,q|\,. \tag{40}$$

The operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ is symmetric by construction.
No ordering ambiguity occurs (notorious problem of canonical quantization).

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Variance of quantum observable

Variance is a stochastic deviation from expectation value of quantum observable; determines the value of smearing of quantum observable; can be used to define quantum fluctuations.
In the quantum state ψ, the variance is defined to be

$$var(\hat{A};\psi) := \langle(\hat{A} - \langle\hat{A};\psi\rangle)^2;\psi\rangle = \langle\hat{A}^2;\psi\rangle - \langle\hat{A};\psi\rangle^2 \,, \tag{41}$$

where $\langle\hat{B};\psi\rangle := \langle\psi|\hat{B}|\psi\rangle$.
If $\hat{A}$ is self-adjoint operator, we have important statement:

$$\Big(var(\hat{A};\psi) = 0\Big) \Longleftrightarrow \Big(\hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R}\Big)\,. \tag{42}$$

Using variance, one can construct the uncertainty principle:

$$var(\hat{A};\psi)var(\hat{B};\psi) \geq \frac{1}{4}\left|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right|^2$$

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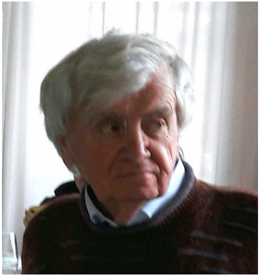
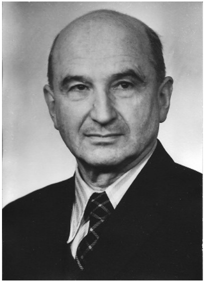
BKL conjecture





Vladimir Alekseevich Belinski   Isaak Markovich Khalatnikov   Evgeny Mikhailovich Lifshitz

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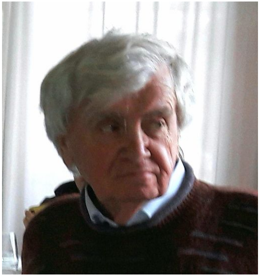

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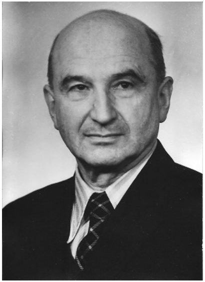

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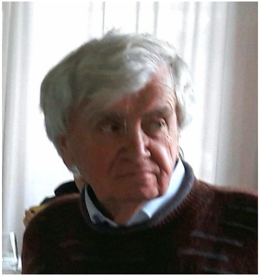

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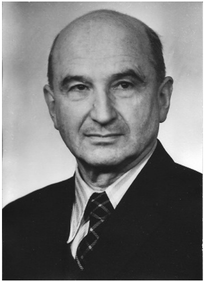

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