Quasi-local mass of weak gravitational field

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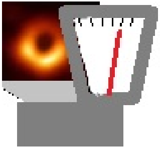
joint work with J. Jezierski and J. Kijowski

Theory of Relativity seminar, 20.11.2020

Plan of the talk

- Technical introduction
- Rigid Sphere condition
- Linearized Cauchy problem for the Einstein equation
- Quadratic approximation of Hawking mass



For an asymptotically flat spacetime:

$$g_{\mu\nu} = \text{Minkowski} + h_{\mu\nu}, \qquad h_{\mu\nu} \sim O(r^{-1}),$$

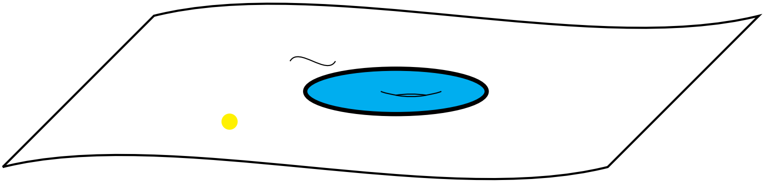
Arnowitt–Deser–Misner mass (1961)

$$\mathcal{H}_{ADM} = \lim_{r\to\infty} \frac{1}{16\pi} \oint_{S(r)} (h^j{}_{k,j} - h^j{}_{j,k}) dS^k$$

- "Measured" at spatial infinity
- It takes into account both the energy of gravity and that of matter fields.
- Positive definite (Schoen–Yau 1979)
- $\mathcal{M}_{ADM} = 0 \iff g_{\mu\nu} = \text{Minkowski}.$

Notable example of a non-asymptotically-flat spacetime : FLRW.

Quasi-local mass — assigned to an *extended*, but *finite* region of spacetime.



Hawking energy

$$\mathcal{H}_{Hawking} := \sqrt{\frac{Area\ S}{16\pi}}\left(1 - \frac{1}{16\pi}\int_S H_\mu H^\mu \mathrm{d}a\right)$$

Where H^μ is the extrinsic curvature of the 2-surface S in the enveloping spacetime.

Extrinsic curvature and torsion

Let S be a 2D submanifold in $\mathcal{M}$.

Extrinsic curvature

$$H^{\mu}{}_{AB}\, X^A Y^B := \left(\overset{4}{\nabla}_X Y \right)^{\perp} \qquad X, Y \in TS \qquad \sim \mathcal{L}_{\mathbf{n}} \overset{2}{g}_{AB}$$

Mean curvature

$$H^{\mu} := H^{\mu}{}_{AB}\, \overset{2}{g}{}^{AB} \in (TS)^{\perp} \qquad \sim \mathrm{grad} \sqrt{\det \overset{2}{g}_{AB}}$$

Extrinsic torsion

$$t_A\, X^A := \frac{T_{\mu}}{||T||} \overset{4}{\nabla}_X \left(\frac{H^{\mu}}{||H||} \right) \qquad X \in TS,\, T \in (TS)^{\perp},\, T_{\mu} H^{\mu} = 0$$

Spherical harmonics

The Laplace–Beltrami operator on a unit sphere $\mathbb{S}^2$:

$$\overset{\circ}{\Delta} := \overset{\circ}{\sigma}{}^{AB}\overset{\circ}{\nabla}_A\overset{\circ}{\nabla}_B$$

defines a decomposition of functions into **spherical harmonics**. (eigenfunctions of $\overset{\circ}{\Delta}$, eigenvalues: $-l(l+1),\ l \in \mathbb{N}$)

S — 2D spacelike topological sphere.

$$\mathbb{S}^2 \xrightarrow{\phi} \mathcal{S}$$

$$\phi^* \overset{2}{g}_{AB} = p \cdot \overset{\circ}{\sigma}_{AB}$$

The operator $\overset{\circ}{\Delta}\phi^*$ defines spherical harmonics on $\mathcal{S}$. Condition $\mathrm{dip}(p) = 0$ makes the definition unique.

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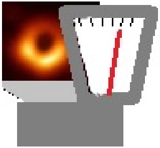
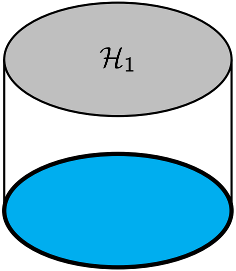


Interpretation?

Energy is the generator of time evolution. To define it, we need a notion of time direction — a frame of reference!



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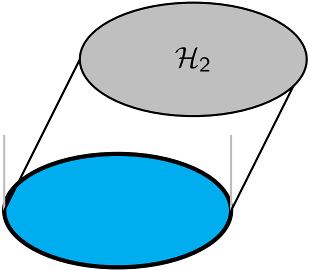

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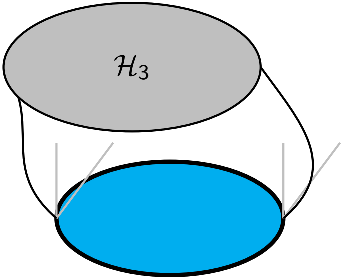
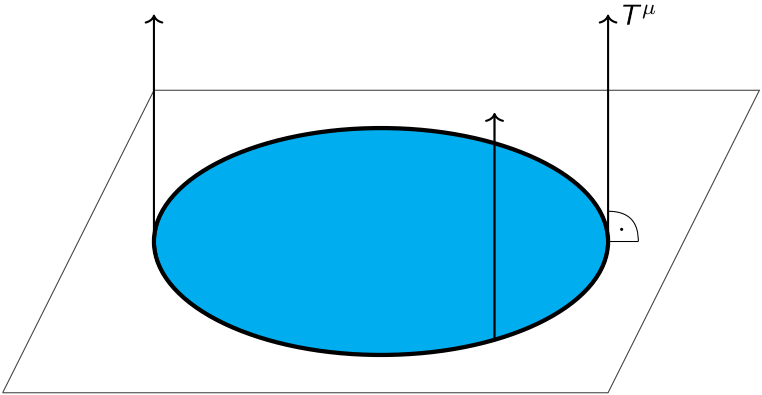


In search of lost time



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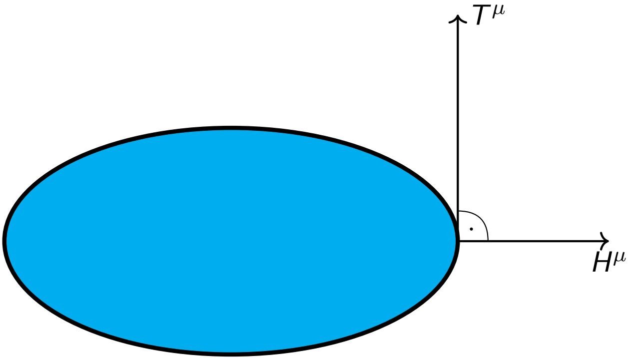

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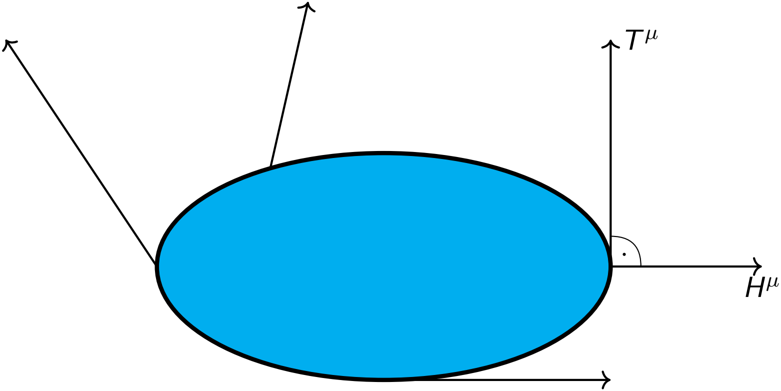

Perhaps asking for a well-behaved quasi-local mass for *any* region is just too greedy?

Rigid spheres (H.P.Gittel, J.Jezierski, J.Kijowski)

We call S a *rigid sphere* if:

1. H^μ is spacelike
2. $\mathring{\Delta}(\mathring{\Delta}+2)\,||H||=0$
3. $\mathring{\Delta}(\mathring{\Delta}+2)\,t_A{}^{||A}=0$

A general spacetime sufficiently close to the Minkowski metric allows an eight-parameter family of rigid spheres.

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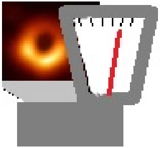

Einstein field equations

$$2\overset{4}{\mathrm{R}}_{\mu\nu} - \overset{4}{\mathrm{R}}\mathrm{g}_{\mu\nu} + 2\Lambda \mathrm{g}_{\mu\nu} = 16\pi \mathrm{T}_{\mu\nu}$$

$(g^2_{kl}, \mathrm{P}^2_{kl})$ + constraints

$\Uparrow$ EOM

$(g^1_{kl}, \mathrm{P}^1_{kl})$ + constraints

$$\mathrm{P}^{kl} := \sqrt{\mathrm{g}}(\mathrm{g}^{kl} K - K^{kl})$$

Einstein field equations

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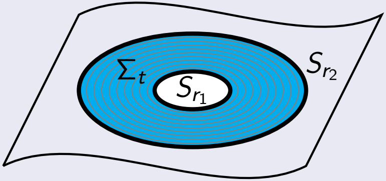
$(g^1_{kl}, \mathrm{P}^1_{kl})$ + constraints

$$\mathrm{P}^{kl} := \sqrt{\mathrm{g}}(\mathrm{g}^{kl} K - K^{kl})$$

1+1+2 splitting

$$\Sigma_t = \{x^0 = t, r_1 \leq x^3 \leq r_2\} = \bigcup_{r\in[r_1,r_2]} S_r$$

$$S_r = \{x \in \Sigma_{t_0} : x^3 = r\}$$



$$(x^0, x^1, x^2, x^3) = (\, t \,,\, \vartheta \,,\, \varphi \,,\, r\,)$$

Indices and covariant derivatives associated with geometry levels:

- α, β, γ - Whole spacetime (0, 1, 2, 3) ;
- a, b, c - Cauchy surface (1, 2, 3) |
- A, B, C - 2D Spheres (1, 2) ||

Background metric: spherically symmetric solutions

Generalized Birkhoff's Theorem

- $T_{\mu\nu} = 0$
- Spherical symmetry

Imply a locally unique solution, the Kottler metric:

$$\eta_{\mu\nu} = -f \mathrm{d}t^2 + \frac{1}{f}\mathrm{d}r^2 + r^2 \left[\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right]$$

$$f(r) = 1 - \frac{2m}{r} - \frac{r^2}{3}\Lambda$$

Perturbative expansion

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{background}} + \underbrace{h_{\mu\nu}}_{\text{perturbation}}$$

ADM data for the Kottler metric: $(\eta_{\mu\nu}, 0)$

First-order perturbation of ADM data on a Kottler background

$$(h_{kl}, P^{kl}) \qquad h_{kl} = g_{kl} - \eta_{kl} \qquad P^{kl} = \mathrm{P}^{kl}$$

- 4 constraints (linearized Gauss–Codazzi constraints)
- 4-parameter family of gauge transformations

$$h_{\mu\nu} \quad \to \quad h_{\mu\nu} + \mathcal{L}_\xi \eta_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

The formalism of reduced linear Cauchy data

$$(h_{kl}, P^{kl}) \Rightarrow (\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y})$$

Axial invariants

$$\mathbf{y} := 2\Pi^{-1} r^2 P^{3A||B} \varepsilon_{AB}$$

$$\mathbf{Y} := \Pi(\overset{\circ}{\Delta} + 2) h_{3A||B} \varepsilon^{AB} - \Pi(r^2 h^C{}_{A||CB} \varepsilon^{AB}),_3$$

Polar invariants

$$\mathbf{x} := r^2 h_{AB}{}^{||AB} - (\overset{\circ}{\Delta} + 1) H + \mathcal{B}\left[2h^{33} + 2rh^{3C}{}_{||C} - rfH,_3\right],$$

$$\mathbf{X} := 2r^2 P^{AB}{}_{||AB} - \overset{\circ}{\Delta}(P^{AB}\eta_{AB}) + \mathcal{B}\left[2rP^{3A}{}_{||A} + \overset{\circ}{\Delta} P^3{}_3\right].$$

$$\mathcal{B} := (\overset{\circ}{\Delta} + 2)\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r}\right)^{-1} \text{ — A quasi-local operator}$$

The formalism of reduced linear Cauchy data

$$(h_{kl}, P^{kl}) \Rightarrow (\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y})$$

Axial invariants

$$\dot{\mathbf{y}} = \frac{f}{\Pi}\mathbf{Y}$$

$$\dot{\mathbf{Y}} = \Pi \left\{ \partial_3 \left[\frac{f}{r^2}(r^2\mathbf{y})_{,3} \right] + \frac{1}{r^2}(\overset{\circ}{\Delta} + 2)\mathbf{y} \right\}$$

Polar invariants

$$\dot{\mathbf{x}} = \frac{f}{\Pi}\mathbf{X}$$

$$\dot{\mathbf{X}} = \frac{\Pi}{r^2} \left\{ \left(f r^2 \mathbf{x}_{,3} \right)_{,3} + \left[\overset{\circ}{\Delta} + f(1 - 2\mathcal{B}) + 1 - r^2\Lambda \right] \mathcal{B}\mathbf{x} \right\}$$

$\mathcal{B} := (\overset{\circ}{\Delta} + 2)\left(\overset{\circ}{\Delta} + 2 - \frac{6m}{r} \right)^{-1}$ — A quasi-local operator

The formalism of reduced linear Cauchy data

$$(h_{kl}, P^{kl}) \Rightarrow (\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y})$$

1. The four scalar functions $(\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y})$ are gauge-invariant.
2. They contain the *entire* physical information of (h_{kl}, P^{kl}).
3. They are no longer subject to constraints (save for the mono-dipole part which represents *conserved charges*).
4. They diagonalize the symplectic form:

$$\underline{\Omega} = \int_\Sigma \delta \underline{P}^{kl} \wedge \delta \underline{h}_{kl} = \int_\Sigma \delta \underline{\mathbf{X}} \wedge \mathcal{A}\, \delta \underline{\mathbf{x}} + \delta \underline{\mathbf{Y}} \wedge \mathcal{A}\, \delta \underline{\mathbf{y}} + \text{ boundary terms}$$

$$\mathcal{A} := \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} \text{ — a quasi-local operator}$$

The hamiltonian of linearized gravity

By evaluating the symplectic form with the time Killing vector field we obtain a gauge invariant Hamiltonian:

$$\underline{\Omega}(\frac{\partial}{\partial t}, \cdot) = -16\pi\delta\mathcal{H}_{Invariant} + \text{gauge dependent boundary terms}$$

$$\begin{aligned} 16\pi\mathcal{H}_{Invariant} &= \frac{1}{2}\int_\Sigma \frac{f}{\Pi}\left[\underline{\mathbf{X}}\mathcal{A}\underline{\mathbf{X}} + \underline{\mathbf{Y}}\mathcal{A}\underline{\mathbf{Y}}\right] + \\ &+ \frac{1}{2}\int_\Sigma \frac{\Pi}{r^2}\left[f(r\underline{\mathbf{x}})_{,3}\mathcal{A}(r\underline{\mathbf{x}})_{,3} + \underline{\mathbf{x}}\frac{r^2}{f}V^{(+)}\mathcal{A}\underline{\mathbf{x}}\right] + \\ &+ \frac{1}{2}\int_\Sigma \frac{\Pi}{r^2}\left[f(r\underline{\mathbf{y}})_{,3}\mathcal{A}(r\underline{\mathbf{y}})_{,3} + \underline{\mathbf{y}}\frac{r^2}{f}V^{(-)}\mathcal{A}\underline{\mathbf{y}}\right] \end{aligned}$$

$V^{(+)}$, $V^{(-)}$ — quasi-local, positive definite operators.

D. R. Brill and S. Deser (1968)

For a perturbed Minkowski spacetime:

$$\mathcal{H}_{ADM} \approx \mathcal{H}_{Invariant}$$

A sensible quasi-local mass candidate should possess a similar property!

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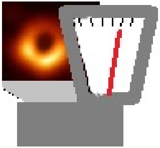

Hawking energy

$$\mathcal{H}_{\text{Hawking}} := \sqrt{\frac{\text{Area } \mathcal{S}}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\mathcal{S}} (\mathrm{H}_\mu \mathrm{H}^\mu + \frac{4}{3}\Lambda) \mathrm{d}a\right)$$

By using the scalar Gauss–Codazzi constraint:

$$\mathrm{g}\overset{3}{\mathrm{R}} - 2\Lambda \mathrm{g} = \mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}\mathrm{P}^2,$$

along with Gauss–Codazzi geometric identities for the embedding of $\partial\Sigma$ in Σ, we can relate the Hawking energy to a volume integral of ADM data:

$$16\pi \mathcal{H}_{Hawking} \approx \int_{\partial\Sigma} \mathcal{F}_1[\overset{2}{R}, \underbrace{k, K_{AB}}_{H^\mu{}_{AB}}] = \int_{\Sigma} \mathcal{F}_2[\mathrm{g}_{kl}, \mathrm{P}^{kl}]$$

$$\mathrm{g}\overset{3}{\mathrm{R}} - 2\Lambda\mathrm{g} = \mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}\mathrm{P}^2$$

$$\mathrm{g}\overset{3}{\mathrm{R}} - 2\Lambda \mathrm{g} = \mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}\mathrm{P}^2$$

$$\Downarrow$$

$$2(w^k k)_{,k} - 2\partial_A \left[\lambda \tilde{g}^{AB} \left(\frac{1}{\sqrt{g^{33}}} \right)_{,B} \right] + \frac{\lambda}{\sqrt{g^{33}}} \left(\overset{2}{R} + \frac{1}{2} k^2 - 2\Lambda \right) =$$

$$= \frac{\sqrt{g^{33}}}{\lambda} \left(P_{kl} P^{kl} - \frac{1}{2} P^2 \right) + \frac{\lambda}{\sqrt{g^{33}}} \left(k_{AB} k^{AB} - \frac{1}{2} k^2 \right)$$

$$\lambda = \sqrt{\det g_{AB}} \qquad w^a = \lambda \frac{g^{3a}}{g^{33}}$$

$$\mathrm{g}\overset{3}{\mathrm{R}} - 2\Lambda \mathrm{g} = \mathrm{P}^{kl}\mathrm{P}_{kl} - \frac{1}{2}\mathrm{P}^2$$

$$\Downarrow$$

$$2(w^k k),_k - 2\partial_A\left[\lambda \tilde{g}^{AB}\left(\frac{1}{\sqrt{g^{33}}}\right),_B\right] + \frac{\lambda}{\sqrt{g^{33}}}\left(\overset{2}{R} + \frac{1}{2}k^2 - 2\Lambda\right) =$$

$$= \frac{\sqrt{g^{33}}}{\lambda}\left(P_{kl}P^{kl} - \frac{1}{2}P^2\right) + \frac{\lambda}{\sqrt{g^{33}}}\left(k_{AB}k^{AB} - \frac{1}{2}k^2\right)$$

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$$\Downarrow$$

$$\int_{\partial\Sigma} r\left(\lambda R - \frac{1}{2}\lambda k^2 - \frac{2}{3}\lambda\Lambda\right) = -\int_\Sigma \left(\frac{\sqrt{g^{33}}}{k} + \frac{r}{2}\right)\left[(k^2 + \frac{4}{3}\Lambda)w^a\right]_{,a} +$$
$$\int_\Sigma \frac{g^{33}}{\lambda}\left(P_{kl}P^{kl} - \frac{1}{2}P^2\right) + \int_\Sigma \lambda\left(k_{AB}k^{AB} - \frac{1}{2}k^2 + \frac{1}{2}\tilde{g}^{AB}(\log g^{33}),_A(\log g^{33}),_B\right)$$

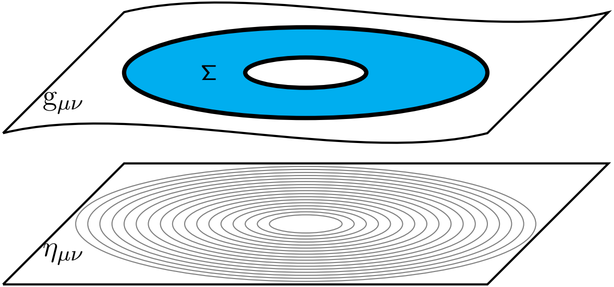
$$
\begin{aligned}
16\pi\mathcal{H}_{Hawking} \approx \quad & 16\pi\mathcal{H}_{Invariant} - \frac{2m}{r^2}\int_{\partial\Sigma}(\lambda - \Pi) \\
& + \frac{1}{2}\int_{\partial\Sigma} \frac{f\Pi}{r}\mathbf{y}\mathcal{A}\mathbf{y} + \frac{f\Pi}{r}\mathbf{x}(\mathcal{B}-1)\mathcal{A}\mathbf{x} \\
& + 2\int_{\partial\Sigma} r^3 f\Pi\, \delta({t_A}^{||A})\, \mathcal{A}\mathcal{B}\, \delta({t_A}^{||A}) \\
& - 2\int_{\partial\Sigma} r\Pi\, \delta(||H||)\ \mathring{\Delta}\mathcal{A}\mathcal{B}\left[\left(\frac{1}{4}\mathring{\Delta} - \frac{1}{2}f(\mathcal{B}-1)\right)\delta(||H||) - \frac{\sqrt{f}}{2r}\mathbf{x}\right]
\end{aligned}
$$

1. The coordinate r must be chosen to equal to the areal radius in both $\eta_{\mu\nu}$ and $g_{\mu\nu}$:

$$
r = \sqrt{\int_S \frac{\Pi}{4\pi}} = \sqrt{\int_S \frac{\lambda}{4\pi}}
$$

2. Quadratic boundary expressions in $\mathbf{x}$ and $\mathbf{y}$ represent the choice of control mode
3. By appropriate gauge transformation, we can set $\delta(||H||) = \delta({t_A}^{||A}) = 0$. This is a linearization of the **rigid sphere** condition.

$$\sqrt{\int_{S(r_{1,2})} \frac{\lambda}{4\pi}} = r_{1,2} = \sqrt{\int_{S(r_{1,2})} \frac{\Pi}{4\pi}}$$



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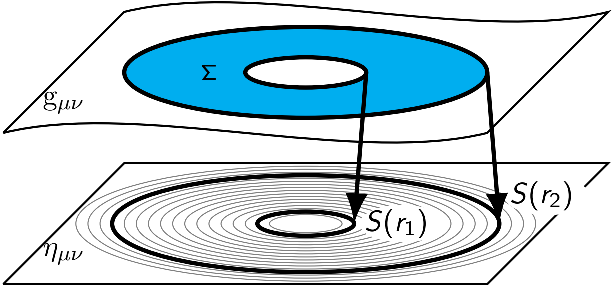

$$16\pi\mathcal{H}_{Hawking} \approx 16\pi\mathcal{H}_{Invariant} - \frac{2m}{r^2}\int_{\partial\Sigma}(\lambda - \Pi)$$
$$+\frac{1}{2}\int_{\partial\Sigma}\frac{f\Pi}{r}\mathbf{y}\mathcal{A}\mathbf{y} + \frac{f\Pi}{r}\mathbf{x}(\mathcal{B}-1)\mathcal{A}\mathbf{x}$$
$$+2\int_{\partial\Sigma} r^3 f\Pi\,\delta({t_A}^{||A})\,\mathcal{AB}\,\delta({t_A}^{||A})$$
$$-2\int_{\partial\Sigma} r\Pi\,\delta(||H||)\,\mathring{\Delta}\mathcal{AB}\left[\left(\frac{1}{4}\mathring{\Delta} - \frac{1}{2}f(\mathcal{B}-1)\right)\delta(||H||) - \frac{\sqrt{f}}{2r}\mathbf{x}\right]$$

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Conclusions

- The *boundary* of an extended spacelike region defines a distribution of preferred time directions on itself. The behaviour of such a distribution can be controlled by imposing some conditions on the shape of the boundary, *e.g.* the Rigid Sphere conditions.
- Quasi-local energy, as a generator of time evolution, should be approximated by the Hamiltonian of the linear theory.
- Hawking quasi-local mass satisfies this criterion with an appropriate choice of boundary spheres.
- Results suggest that the Q-L mass may be well-defined only for regions with "good" boundary (Rigid sphere condition).


Thank you for your attention!
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GRAVITATION