Fermion coupling to loop quantum gravity

Based on: ArXiv: 2112.08865

Cong Zhang

University of Warsaw

In collaboration with Jerzy Lewandowski

Introduction

Over the last decades, loop quantum gravity (LQG) has been well developed

—Canonical approach: [Ashtekar & Lewandowski 2004, Han, Ma, Huang 2007, Thiemann 2008, et.al.]

—Spinfoam Model: [Perez 2003, Rovelli & Vidotto 2015, et.al.]

—Group field theory: [Fredel 2005, et.al.]

Some achievements of LQG [Ashtekar, Alesci, Assanioussi, Bodendofer, Dapor, Domagala, Giesel, Han, Kaminski, Liegener, Lewandowki, Liu, Ma, Makinen, Okolow, Pwalowski, Rovelli, Simolin, Sahlmann, Thiemann, Yang, Zhang, et.al.]:

—a well defined kinematic Hilbert space,

—solving the Gauss and diffeomorphism constraint explicitly,

—a family of operators representing geometric observables: area, volume, length, curvature et.al. ,

—the dynamics: the Hamiltonian constraint operator, transition amplitude, the attempt to analyze the dynamics et.al.

—semiclassical analysis: coherent state system, large j limit of spinfoam model et.al.

—cosmology & BH model: big bounce, BH-WH transition, discreteness of BH mass spectrum et.al.

…….

Introduction

Spacetime tells matter how to move; matter tells spacetime how to curve

—J. A. Wheeler

LQG sets a stage for incorporating matters into quantum spacetime

—massless Klein-Gordon field, dust field coupled to canonical LQG [Rovelli & Lee 1994, Brown & Karel1995, Giesel &Thiemann 2010, Domagala et. al. 2010, Lewandowski et. al. 2011, Han & Rovelli 2013, Bianchi et. al. 2013…]

—minimal coupling of fermions and Yang-Mills fields to covariant LQG [Han & Rovelli 2013, Bianchi et. al. 2013…]

We are concerning about the model of LQG coupled to fermion field

By employing the procedure proposed by [Thiemann 1998], we:

—solve the Gauss constraint explicitly

—regularize and quantize the Hamiltonian constraint by introducing the vertex Hilbert space.

Introduction

A vertex Hilbert space is a Hilbert space group averaged with diffeomorphisms preserving some specific vertices

The vertex Hilbert space was introduced for the model of gravity coupled to scalar field
[Alesci, Assanioussi, Lewandowski & Mäkinen 2015]

—a graph-changing Hamiltonian operator is usually defined as the limit of some regularized Hamiltonian operators as the regulator approaches 0,

—introduce the so-called vertex Hilbert space to remove regulators and define limit

—the operator on the vertex Hilbert space carries the diffeomorphism-covariance feature

Classical phase space

Classical model of gravity coupled to the fermion field:

—First order formulation $S[\omega, e, \Psi]$: fermion couples to the connection ω directly.

—Second order formulation $S[e, \Psi]$: fermion couples to the spin connection Γ compatible with e, where there is no torsion involved.

—Regular Hamiltonian analysis tells $\Pi = \sqrt{q}\Psi^\dagger$.

—In our model, $\sqrt{q}$ will become an operator: $\hat{\Pi}^\dagger = \widehat{\sqrt{q}}\,\hat{\Psi}$.

—Contradiction: $0 = [\hat{\Pi}, \widehat{f(A)}\,]^\dagger = [\widehat{f(A)}\,, \hat{\Pi}^\dagger] = [\widehat{f(A)}\,, \widehat{\sqrt{q}}\,\hat{\Psi}] \neq 0$

One proposed the half-density $\widetilde{\Psi} := \sqrt[4]{q}\Psi$ and $\widetilde{\Pi} = \widetilde{\Psi}^\dagger$ for quantization [Thiemann et.al. QSD]

Classical phase space

The classical phase space: $(A_a^i, E_j^b, \xi, \xi^\dagger, \nu, \nu^\dagger)$,

A_a^i: an SU(2) connection on spatial manifold

$E_j^b = |\det(e_a^i)| e_j^b$ densitized triad

$\xi := \sqrt[4]{q}\Psi_-$, $\nu := \sqrt[4]{q}\Psi_+$

(anti-)Poisson brackets: for $A, B = \pm 1/2$

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^b \delta_j^i \delta(x, y)$$

$$\{\xi_A(x), \xi_B^\dagger(y)\}_+ = -i\delta_{AB}\delta(x, y)$$

$$\{\nu_A(x), \nu_B^\dagger(y)\}_+ = -i\delta_{AB}\delta(x, y)$$

Classical phase space

Gauss constraint G_m:

$$G_m = \left(\frac{1}{\kappa\beta}D_a E^a_m + \frac{1}{2}\xi^\dagger\sigma_m\xi\right),$$

Diffeomorphism constraint H_a :

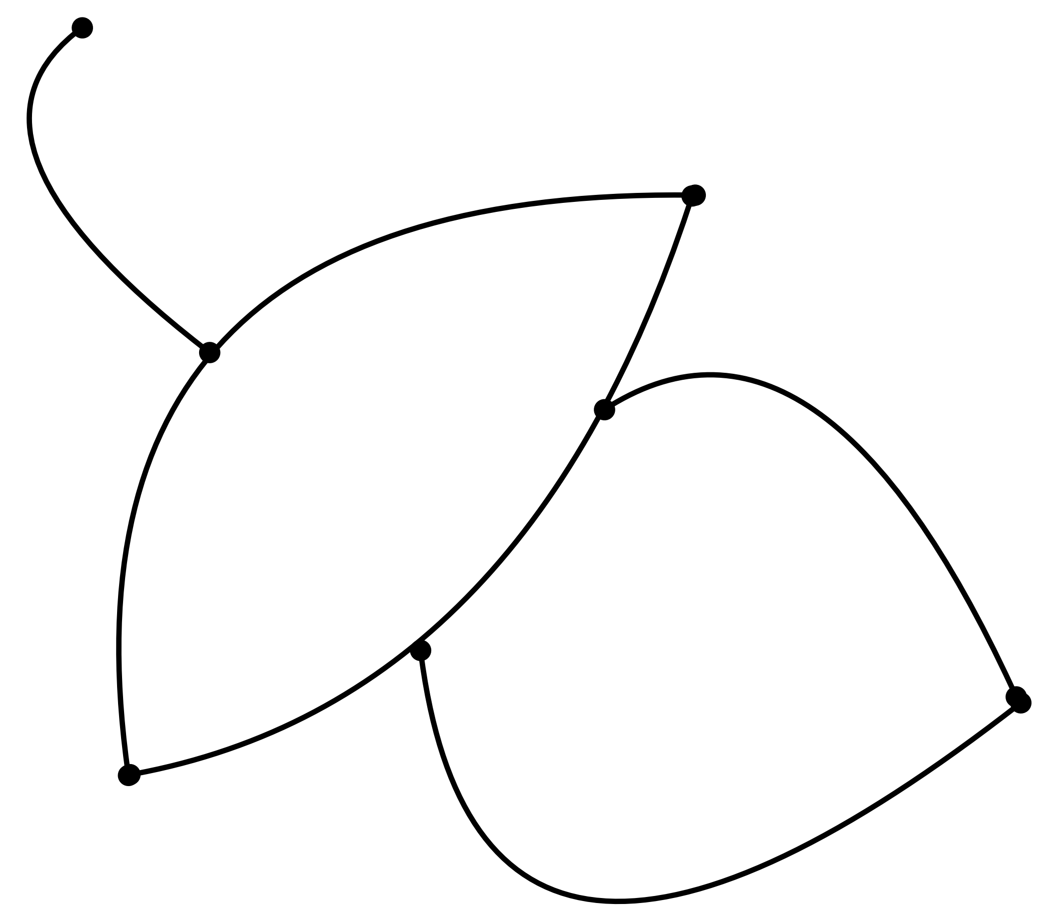
$$H_a = \frac{1}{\kappa\beta}E^b_i F^i_{ab} + \frac{i}{2}\left\{\xi^\dagger D_a\xi - (D_a\xi)^\dagger\xi + \nu^\dagger D_a\nu - (D_a\nu)^\dagger\nu\right\} + \beta K^m_a G_m,$$

Hamiltonian constraint H:

$$H = H_G + \frac{1}{\sqrt{q}}\Big[i(\xi^\dagger E^a_i\sigma^i D_a\xi - (D_a\xi)^\dagger E^a_i\sigma^i\xi) - \beta E^a_i K^i_a\xi^\dagger\xi - \frac{1}{\beta}(1+\beta^2)D_a E^a_i\xi^\dagger\sigma^i\xi - \beta E^a_i D_a\left(\xi^\dagger\sigma^i\xi\right)$$

$$-i(\nu^\dagger E^a_i\sigma^i D_a\nu - (D_a\nu)^\dagger E^a_i\sigma^i\nu) + \beta E^a_i K^i_a\nu^\dagger\nu - \frac{1}{\beta}(1+\beta^2)D_a E^a_i\nu^\dagger\sigma^i\nu - \beta\frac{1}{\sqrt{q}}E^a_i D_a\left(\nu^\dagger\sigma^i\nu\right)\Big].$$

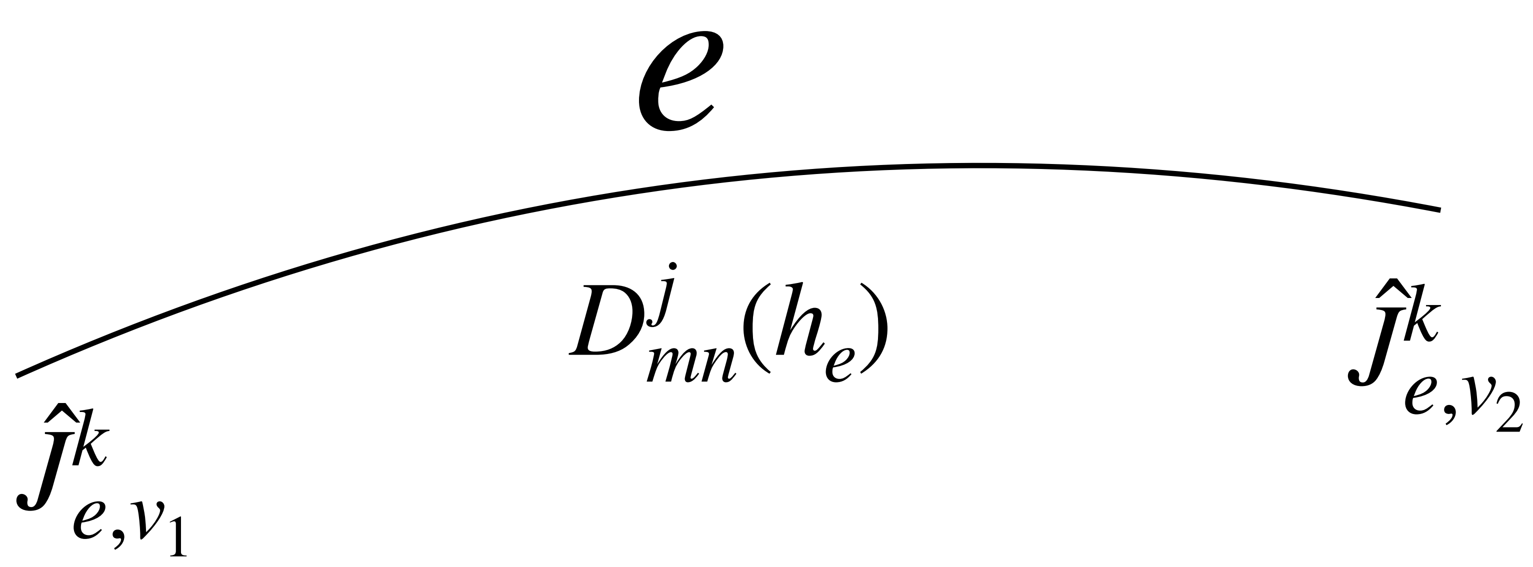
Quantization: Gravity



$$\mathscr{H}_\gamma^G = L^2(\mathrm{SU}(2)^{|E(\gamma)|}, \mathrm{d}\mu_H)$$

Multiplication operator: $D^j_{mn}(h_e)$

Derivative operator: $\hat{J}^k_{e,v}$ (left or right vector field on SU(2))



$D^j_{mn}(h_e)$: parallel transpose from v_1 to v_2

$\hat{J}^k_{e,v}$: Area vector at the v

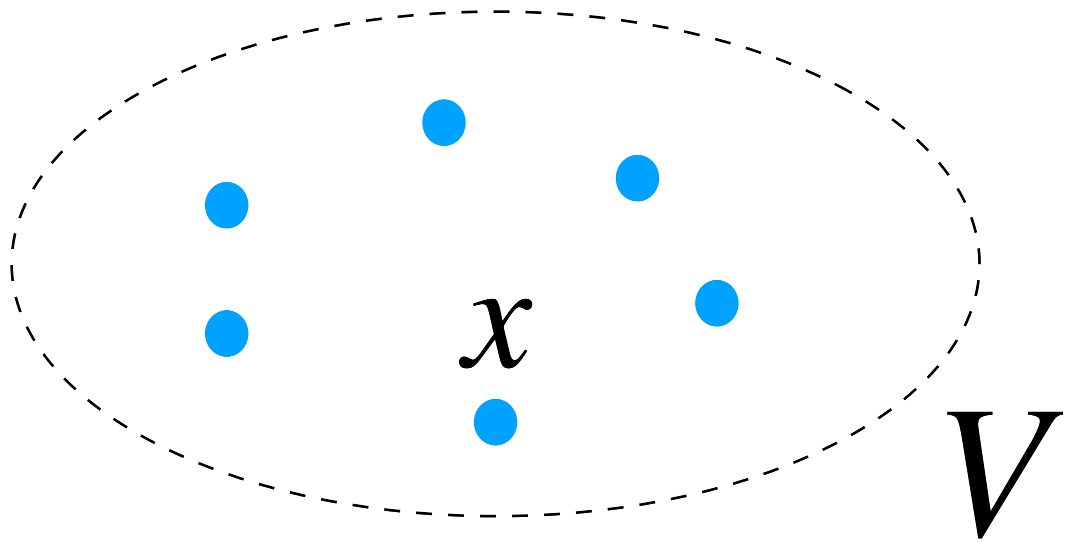
Quantization: Fermion

Canonical transformation :

$$\zeta_x = \frac{1}{\sqrt{\hbar}} \int_\Sigma \mathrm{d}^3y \sqrt{\frac{\chi_\epsilon(x,y)}{\epsilon^3}}\,\xi(y)$$

New anti-commutator relation:

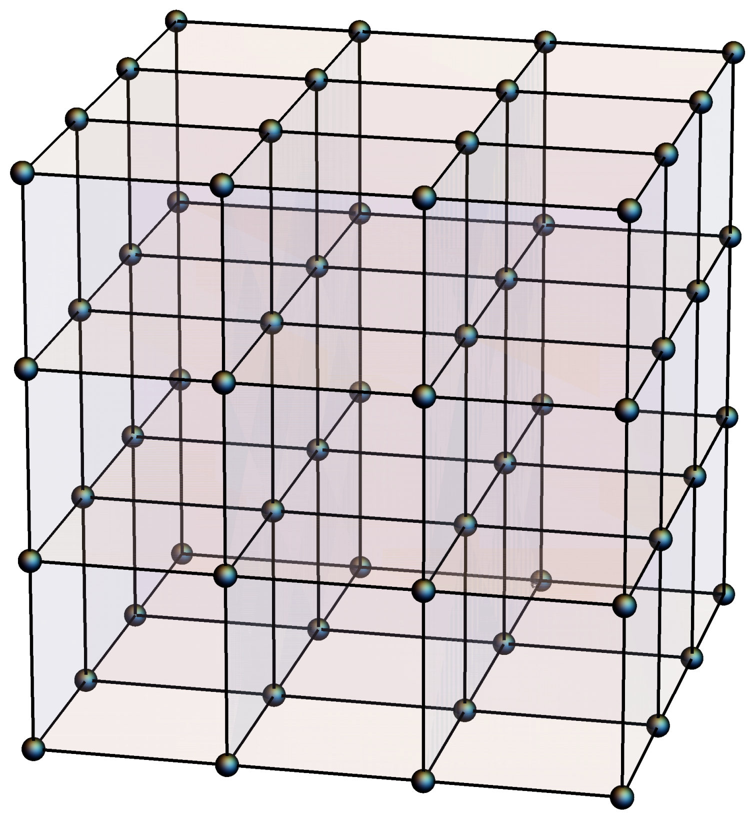
$$\{\zeta_{x,A}, \zeta^\dagger_{y,B}\}_+ = -\frac{i}{\hbar}\delta_{AB}\,\delta_{x,y}, \quad A, B = \pm\frac{1}{2}$$



$$\mathscr{H}^F_V = \bigotimes_{x\in V} \mathscr{H}^F_x \qquad \mathscr{H}^F_x = \mathrm{span}(|00\rangle_x, |01\rangle_x, |10\rangle_x, |11\rangle_x)$$

Ladder operator: $\hat{\zeta}_{x,A}$, $\hat{\zeta}^\dagger_{x,A}$ $A = \pm\frac{1}{2}$, for example: $\hat{\zeta}^\dagger_{x,\frac{1}{2}}|0,i_2\rangle_x = |1,i_2\rangle_x$, $\hat{\zeta}^\dagger_{x,-\frac{1}{2}}|i_1,0\rangle_x = (-1)^{i_1}|i_1,1\rangle_x$

Compare with Lattice QFT



γ

The Hilbert space: $\mathscr{H}_{\gamma}^{G} \otimes \mathscr{H}_{V(\gamma)}^{F}$

$$\hat{\zeta}_{x,A} = \sum_{k} \Theta_{A+}(k) \hat{\tilde{\zeta}}_{k,B} e^{ik\cdot x}$$

$$\hat{a}_k \sim \hat{\tilde{\zeta}}_{+,k} \qquad\qquad \hat{b}_k \sim \hat{\tilde{\zeta}}^{\dagger}_{-,k}$$

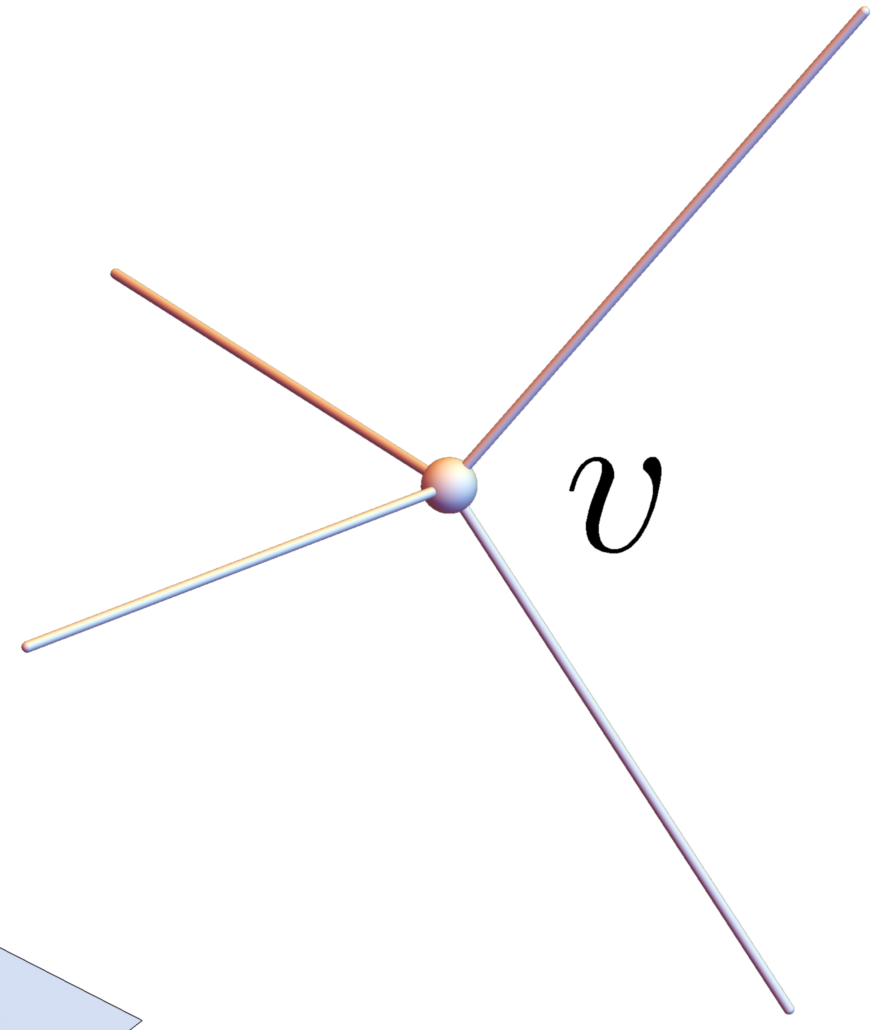
$\Theta_{A+}(k) e^{ik\cdot x}$ diagonalize the effective Hamiltonian:

$$\hat{H}^{F}_{\text{eff}} = \langle \text{background} \,|\, \hat{H}_F(\hat{\zeta}, \hat{\zeta}^{\dagger}, h_e, \hat{J}^{i}_{v,e}) \,|\, \text{background} \rangle$$

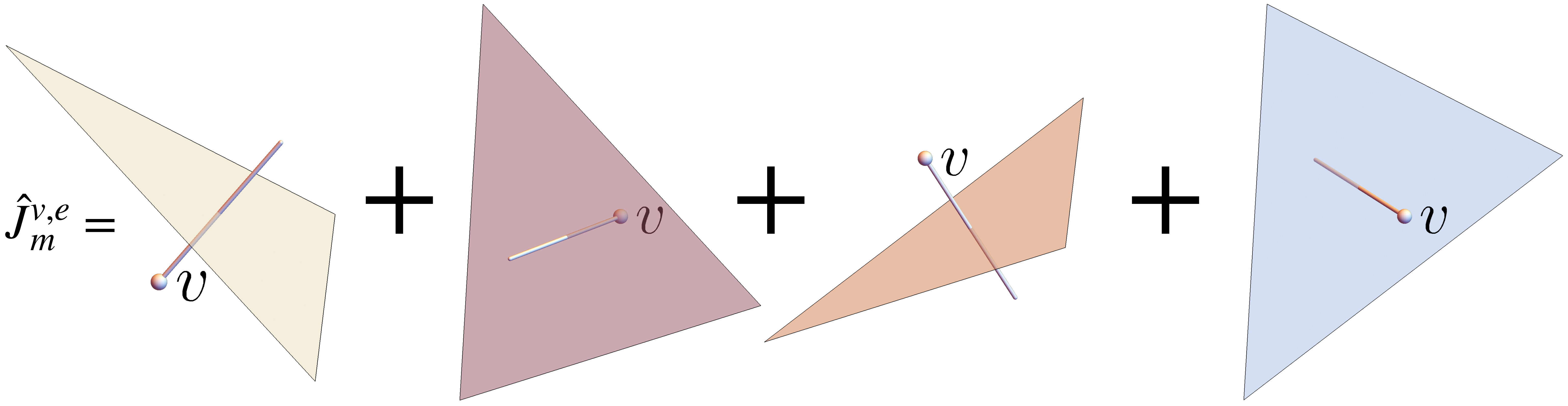
in $\mathscr{H}_{\gamma}^{G}$

The Gauss Constraint

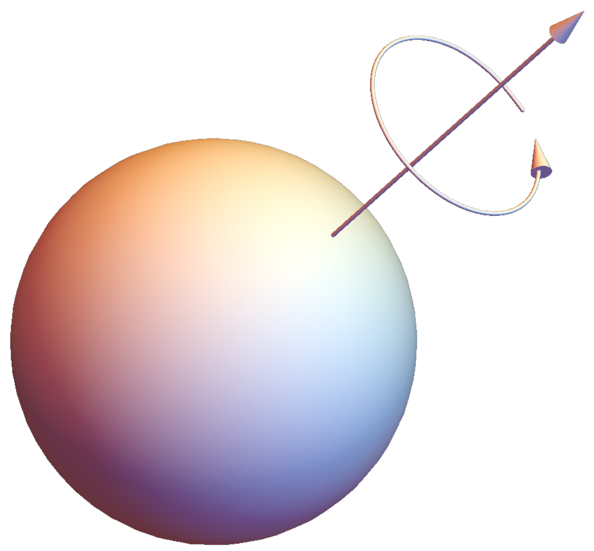
$$\hat{G}_{v,m} = \hbar \sum_{e} \hat{J}_{m}^{v,e} + \hbar \hat{\mathscr{F}}_{v,m}$$



$$\sum_{e} \hat{J}_{m}^{v,e} =$$



$$\hat{\mathscr{F}}_{v,m} =$$



$\hat{\mathscr{F}}_{v,m}$ performs like an angular momentum operator:

$$\hat{\mathscr{F}}_{v,m} |0,0\rangle_v = 0 = \hat{\mathscr{F}}_{v,m} |1,1\rangle_v$$

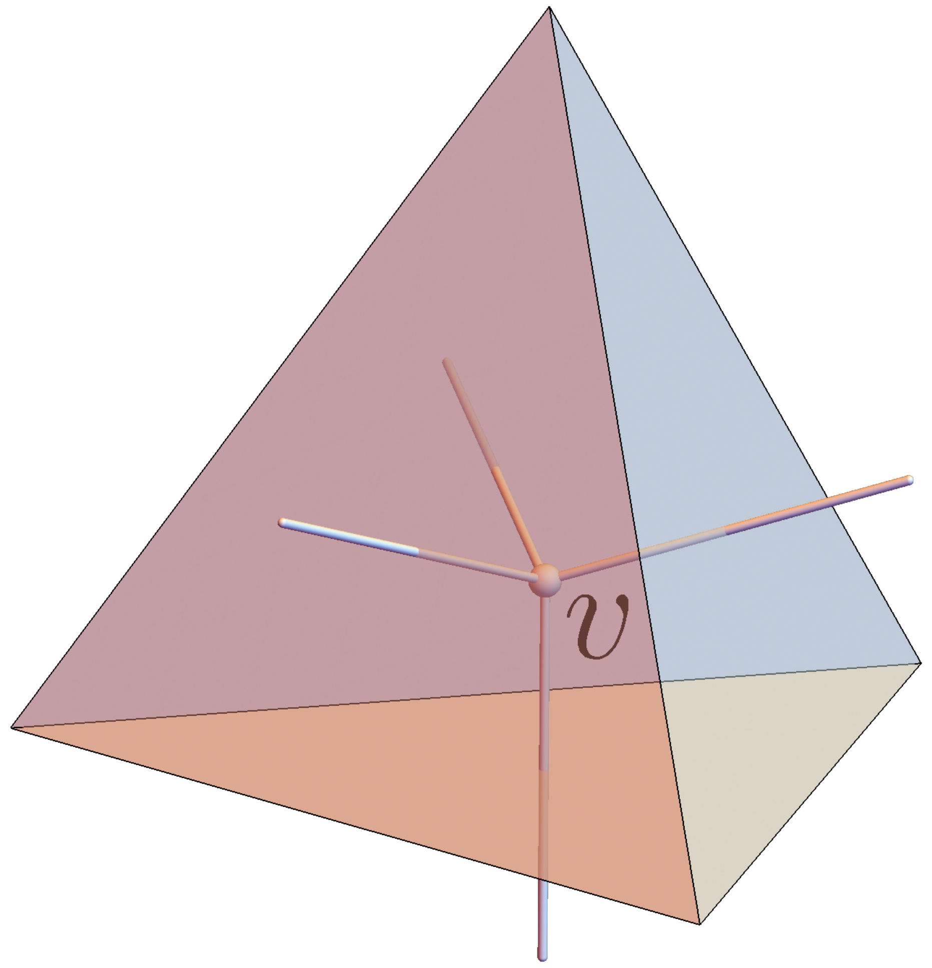
$$\hat{\mathscr{F}}_{v,m} \left(|1,0\rangle_v, |0,1\rangle_v \right) = \left(|1,0\rangle_v, |0,1\rangle_v \right) \frac{\sigma_m}{2}$$

The Gauss Constraint

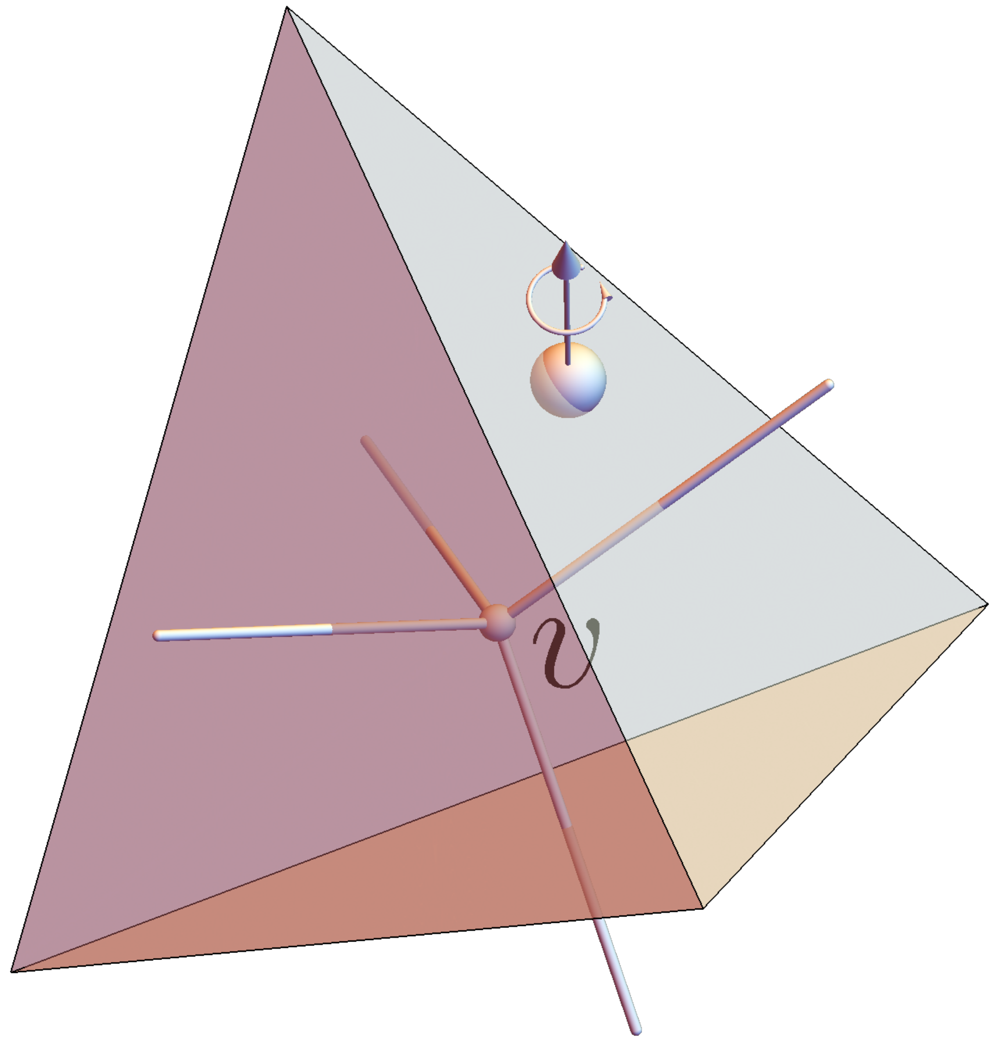
$$\hat{G}_{v,m} = \hbar \sum_e \hat{J}^{v,e}_m + \hbar \hat{\mathscr{F}}_{v,m}$$

$$\left(\text{Inv}\left(\mathscr{H}^G_v\right) \otimes |0,0\rangle_v\right) \oplus$$

$$\left(\text{Inv}\left(\mathscr{H}^G_v\right) \otimes |1,1\rangle_v\right)$$



$$\text{Inv}\left(\mathscr{H}^G_v \otimes \mathscr{H}^F_v\right)$$



$$\mathscr{H}^F_v \equiv \mathscr{H}_0 \oplus \mathscr{H}_0 \oplus \mathscr{H}_{1/2}$$

$$|\hat{\mathscr{F}}_v|^2 \mathscr{H}_j = j(j+1)\mathscr{H}_j$$

$\hat{\mathscr{F}}_{v,m}$ performs like an angular momentum operator:

$$\hat{\mathscr{F}}_{v,m}|0,0\rangle_v = 0 = \hat{\mathscr{F}}_{v,m}|1,1\rangle_v$$

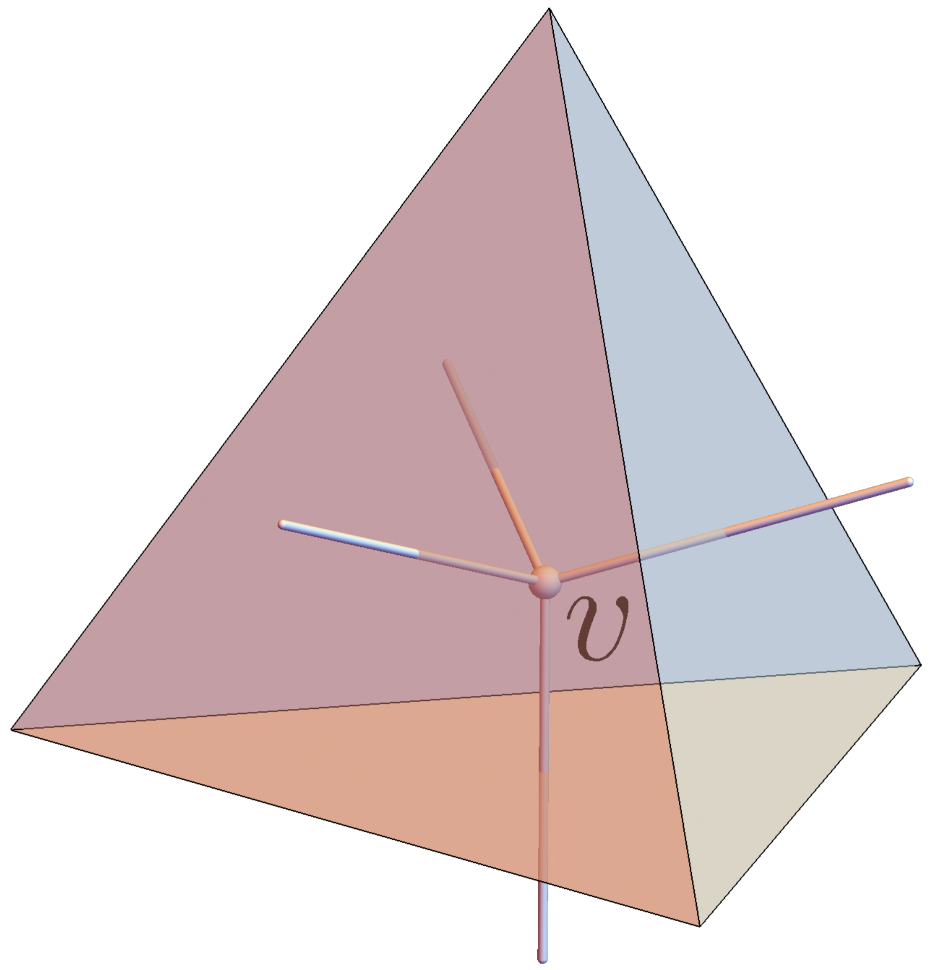
$$\hat{\mathscr{F}}_{v,m}\left(|1,0\rangle_v, |0,1\rangle_v\right) = \left(|1,0\rangle_v, |0,1\rangle_v\right)\frac{\sigma_m}{2}$$

The Gauss Constraint

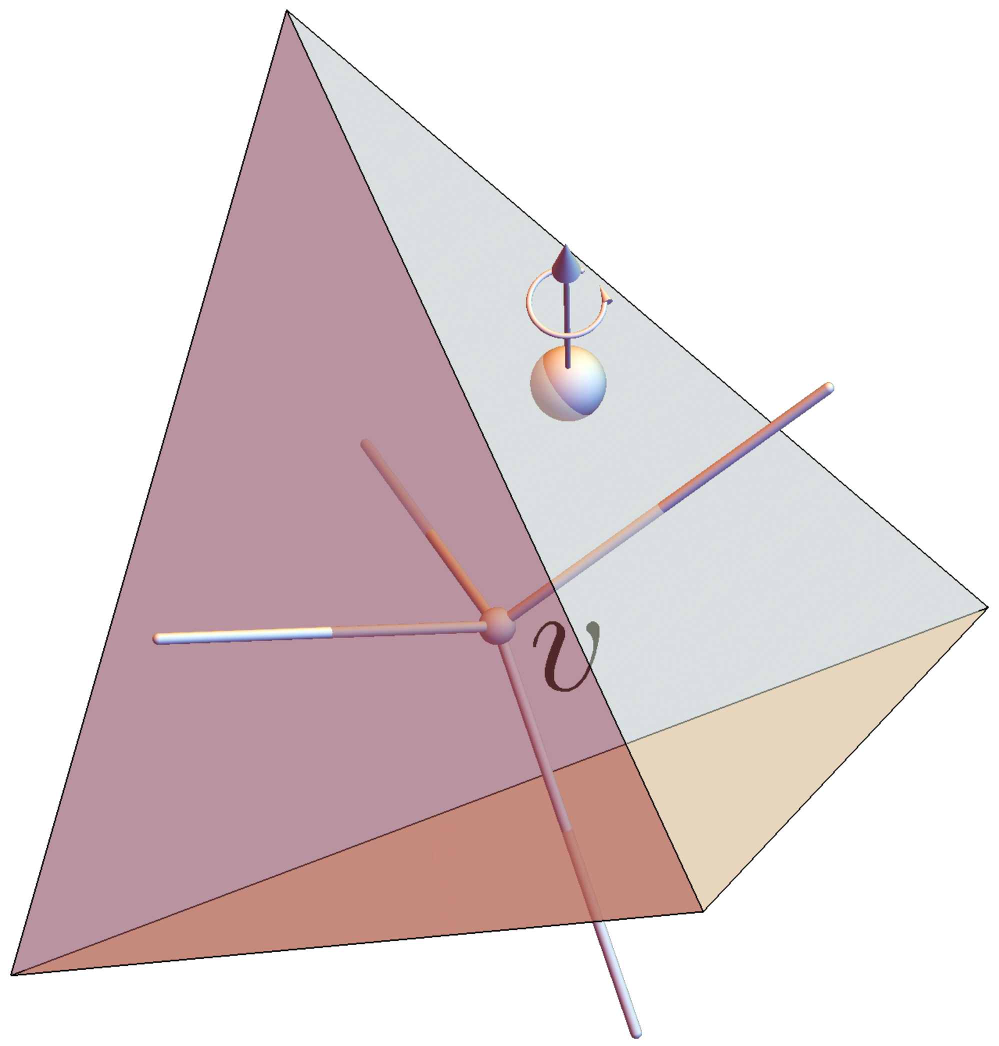
$$\hat{G}_{v,m} = \hbar \sum_{e} \hat{J}^{v,e}_{m} + \hbar \hat{\mathscr{F}}_{v,m}$$

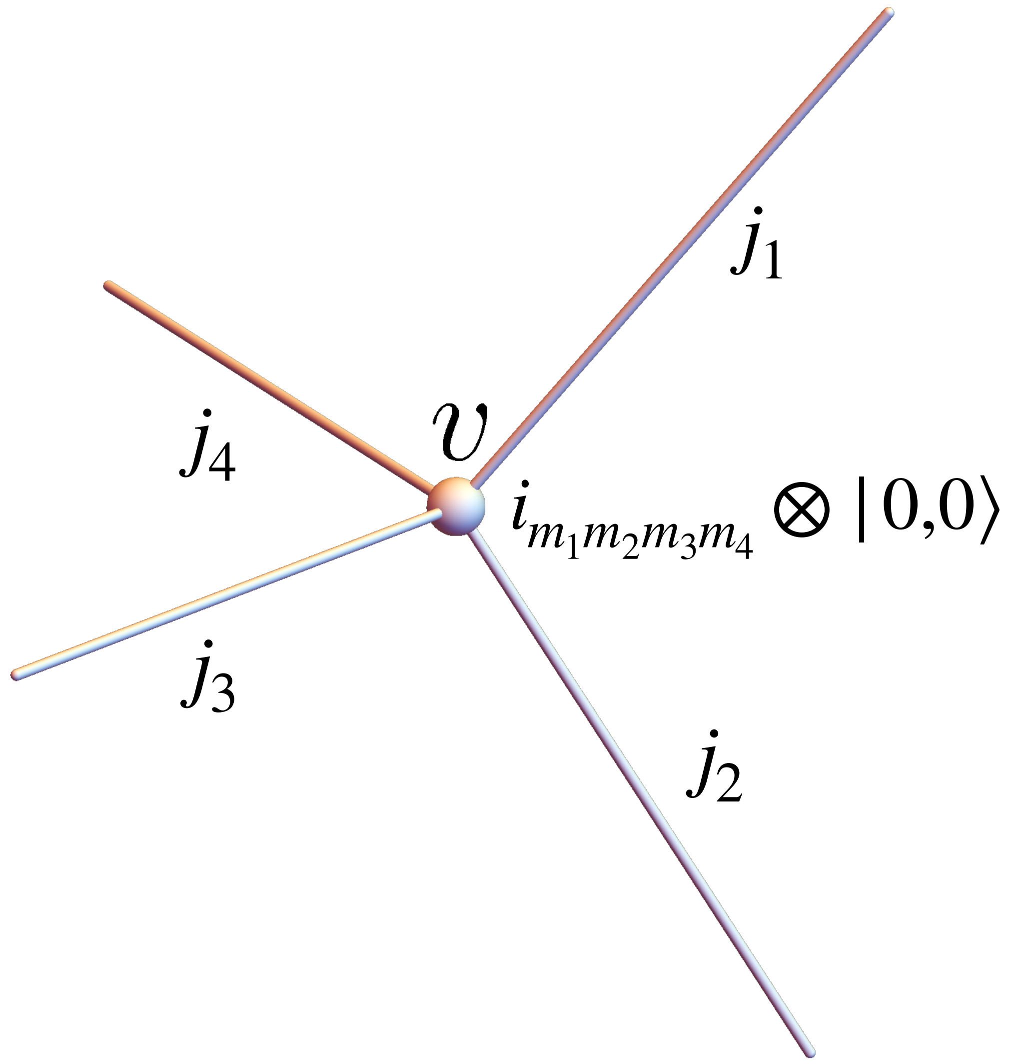
$$\left(\mathrm{Inv}\left(\mathscr{H}^{G}_{v}\right) \otimes |0,0\rangle_{v}\right) \oplus$$

$$\left(\mathrm{Inv}\left(\mathscr{H}^{G}_{v}\right) \otimes |1,1\rangle_{v}\right)$$

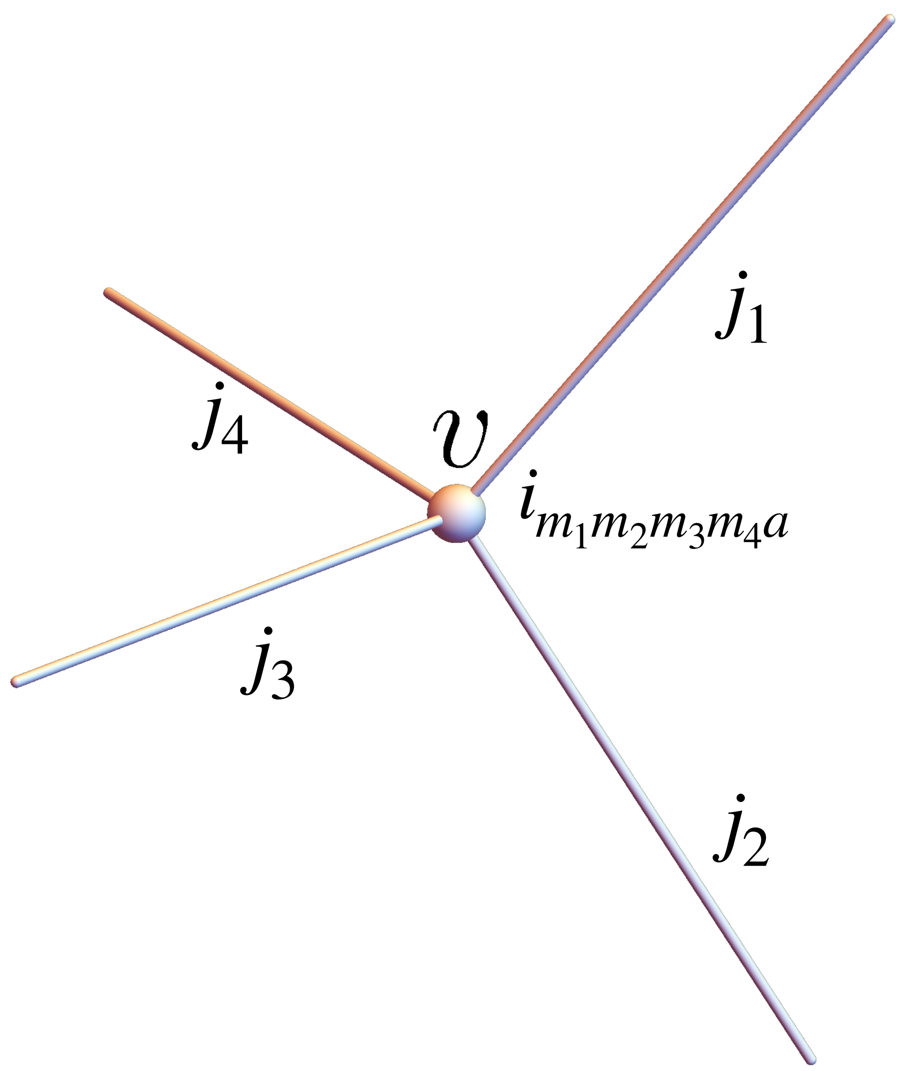


$$\mathrm{Inv}\left(\mathscr{H}^{G}_{v} \otimes \mathscr{H}^{F}_{v}\right)$$





$$\mathscr{H}^{G}_{v} = \mathscr{H}_{j_1} \otimes \mathscr{H}_{j_2} \otimes \mathscr{H}_{j_3} \otimes \mathscr{H}_{j_4}$$



The Hamiltonian Constraint

$$H[N] = \int \mathrm{d}^3x\ H[A_a^i(x), E_j^b(x), \xi_{v,A}, \xi_{v,A}^\dagger]$$

However, the basic operators in LQG are

- $h_e = \mathscr{P} \exp\left(\int_e A\right),$

- $\hat{J}^j_{v,e} = \frac{1}{\kappa\beta}\int_{S_{e,v}} \widehat{\mathrm{d}x^a \mathrm{d}x^b \epsilon_{abc} E_j^c},$

- $\hat{\zeta}_x = \frac{1}{\sqrt{\hbar}}\int_\Sigma \mathrm{d}^3y \widehat{\sqrt{\frac{\chi_\epsilon(x,y)}{\epsilon^3}}} \xi(y)$

To quantize the Hamiltonian

1, regularize the classical expression
$H_\delta[N] = \sum_{v,e} H[h_e, \overrightarrow{J}_{e,v}, \zeta_{v,A}, \zeta_{v,A}^\dagger]$

2, quantize the regularized expression

The Hamiltonian Constraint

In our model, the fermion Hamiltonian is given by:

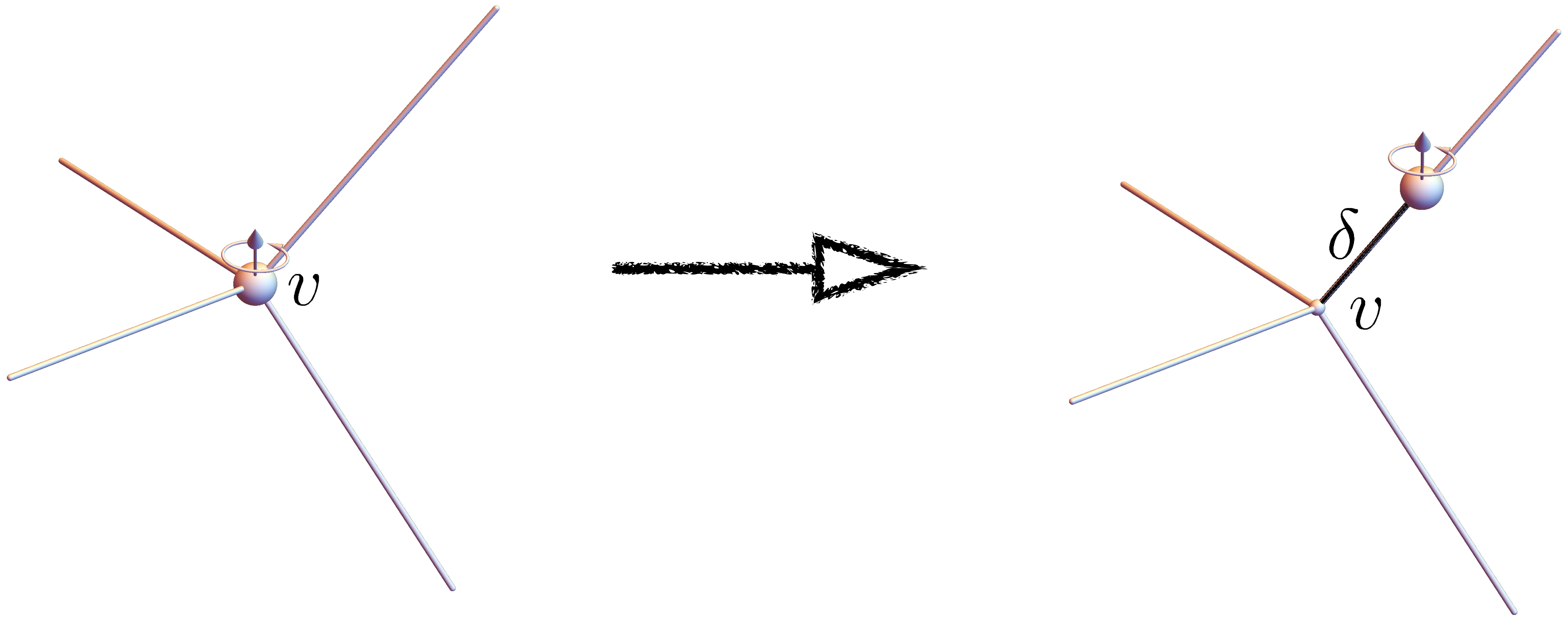
$$\widehat{H^F[N]} = \lim_{\delta\to 0} \widehat{H^F_\delta[N]} + \widehat{H^F_\delta[N]}^\dagger$$

$$\widehat{H_\delta(v)} \;:= \sum_{v\in V(\gamma)} i\widehat{H^{(1)}_\delta(v} + \frac{\beta}{2}\widehat{H^{(2)}_\delta(v} + \frac{1+\beta^2}{2\beta}\widehat{H^{(3)}_\delta(v)} + \beta\widehat{H^{(1)}_\delta(v)}$$

Consider the typical term: $\widehat{H^{(1)}_\delta(v)}$

The Hamiltonian Constraint

$\widehat{H_\delta^{(1)}(v)}$:



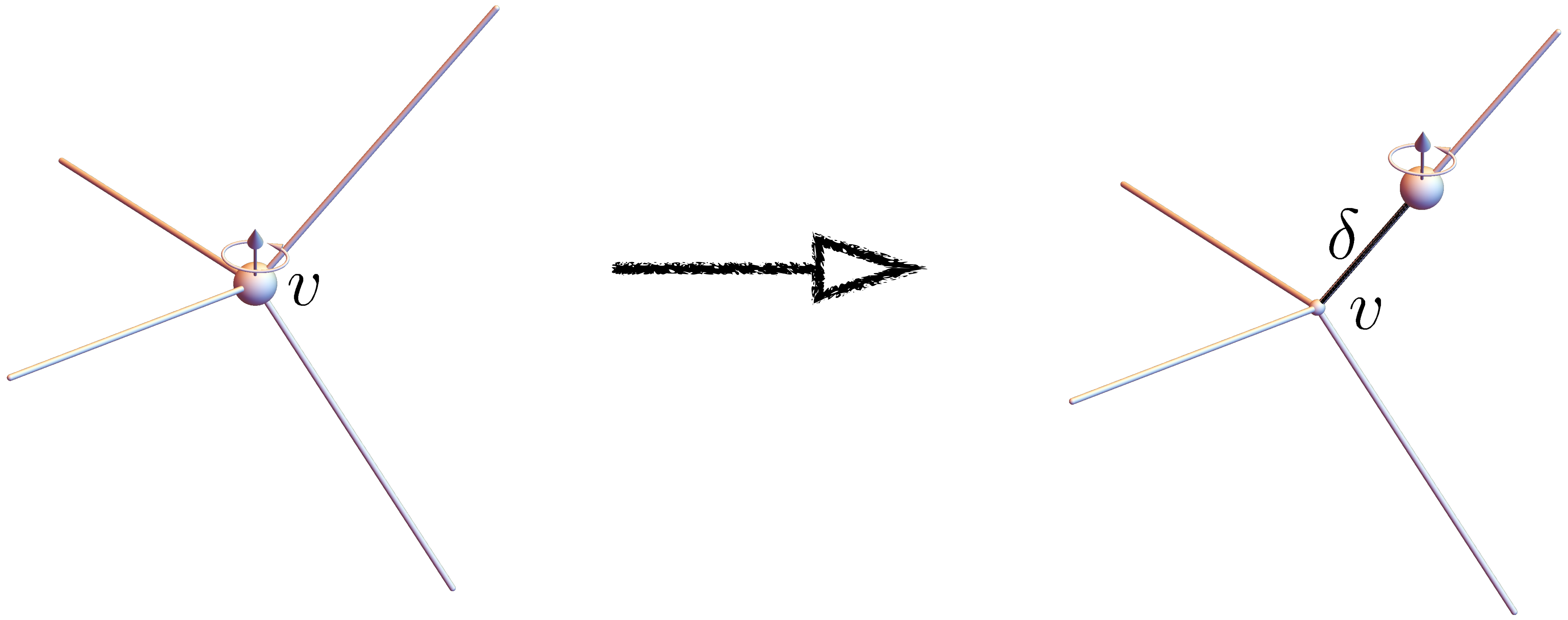
Two problems:

—to define the limit as $\delta \to 0$,

— $\widehat{H_\delta^{(1)}(v)}^\dagger$ is not gauge covariant.

The Hamiltonian Constraint

$\widehat{H_\delta^{(1)}(v)}$:

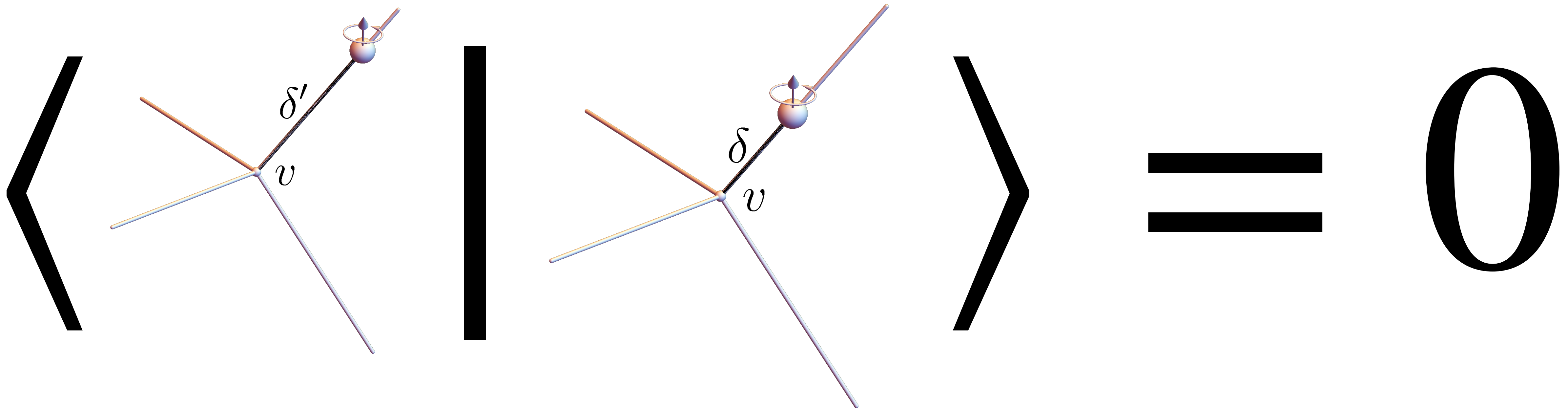


Two problems:

—to define the limit as $\delta \to 0$,

— $\widehat{H_\delta^{(1)}(v)}^\dagger$ is not gauge covariant.

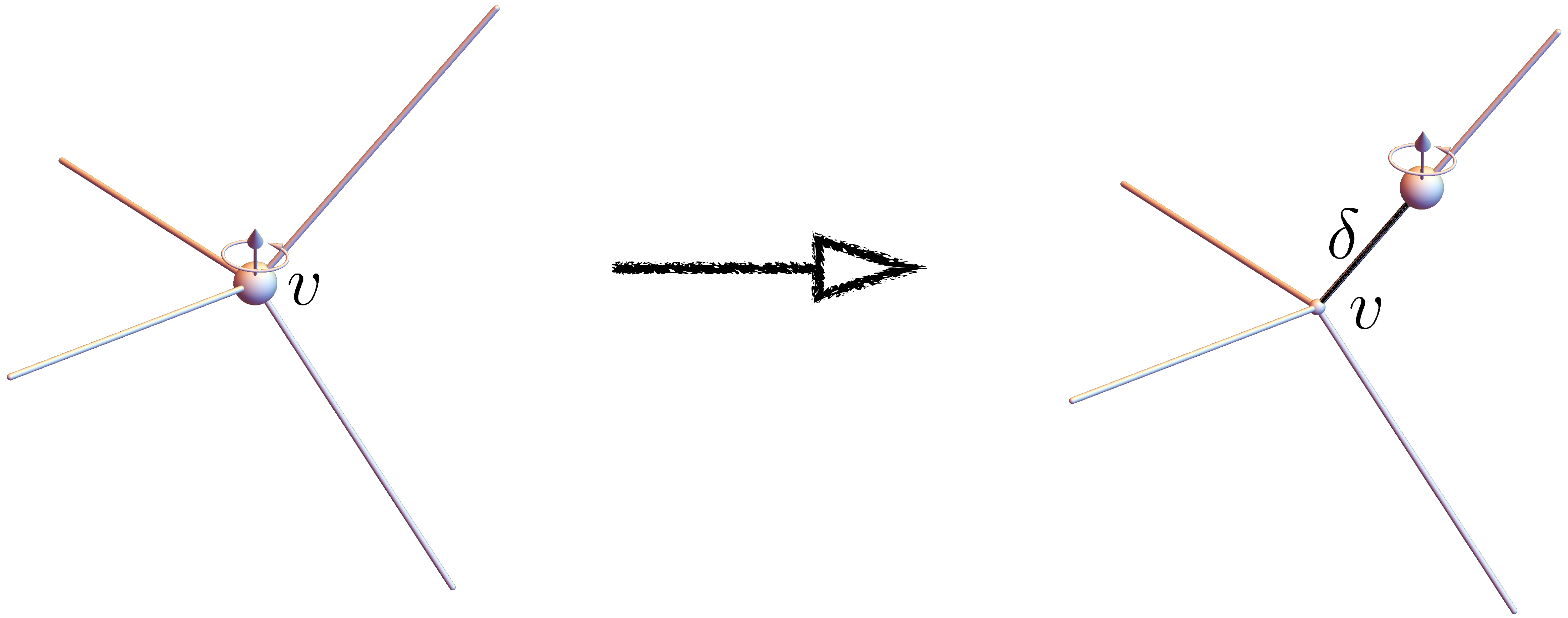
Problems 1:



$\lim_{\delta \to 0} \widehat{H_\delta^{(1)}(v)}$ cannot be well defined

The Hamiltonian Constraint

$\widehat{H^{(1)}_\delta(v)}$:

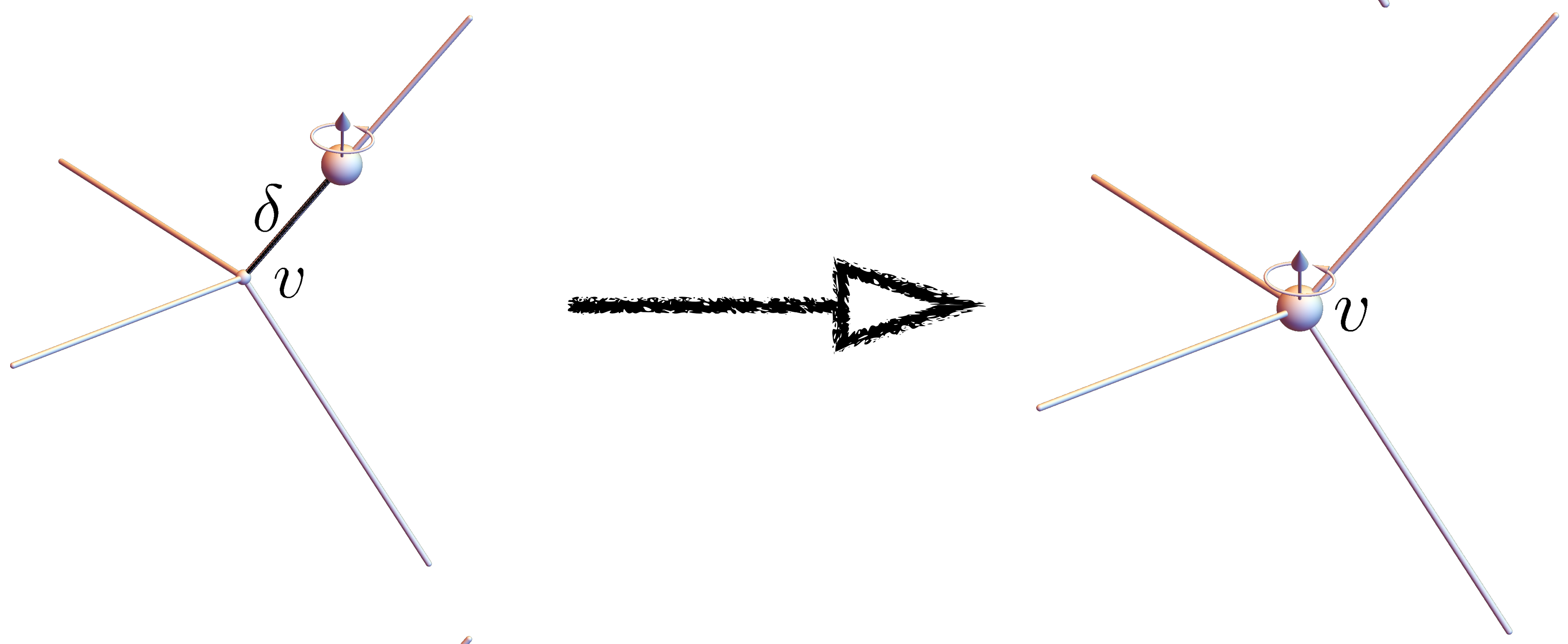


Two problems:

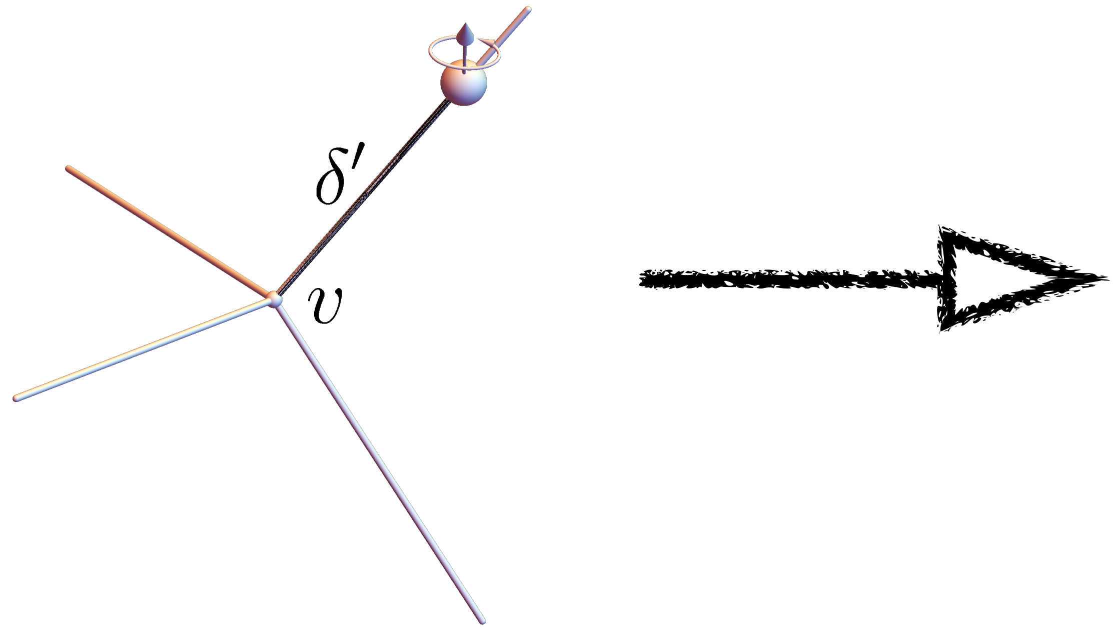
—to define the limit as $\delta \to 0$,

– $\widehat{H^{(1)}_\delta(v)}^\dagger$ is not gauge covariant.

Problems 2:



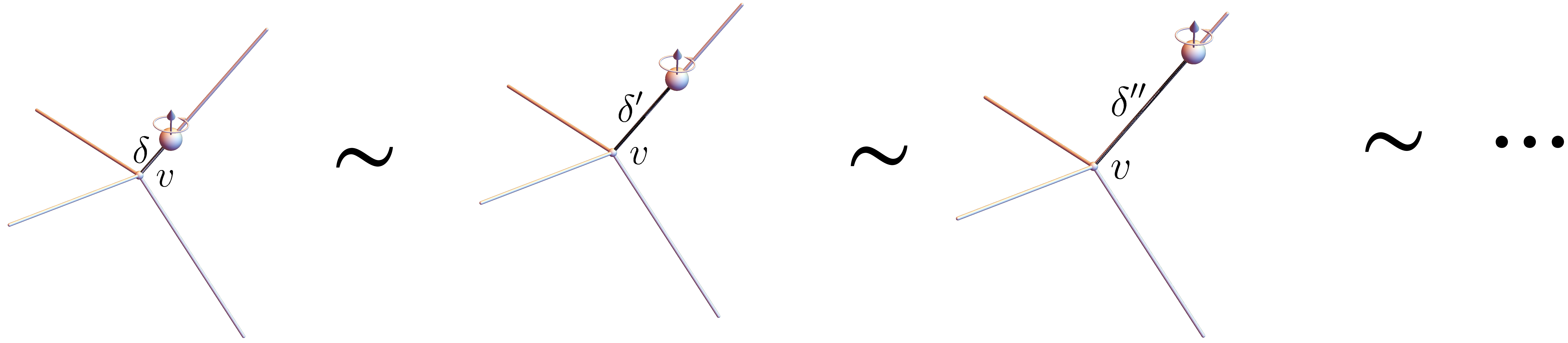
$\widehat{H^{(1)}_\delta(v)}^\dagger$:



0

The Hamiltonian Constraint

The vertex Hilbert is defined such that:

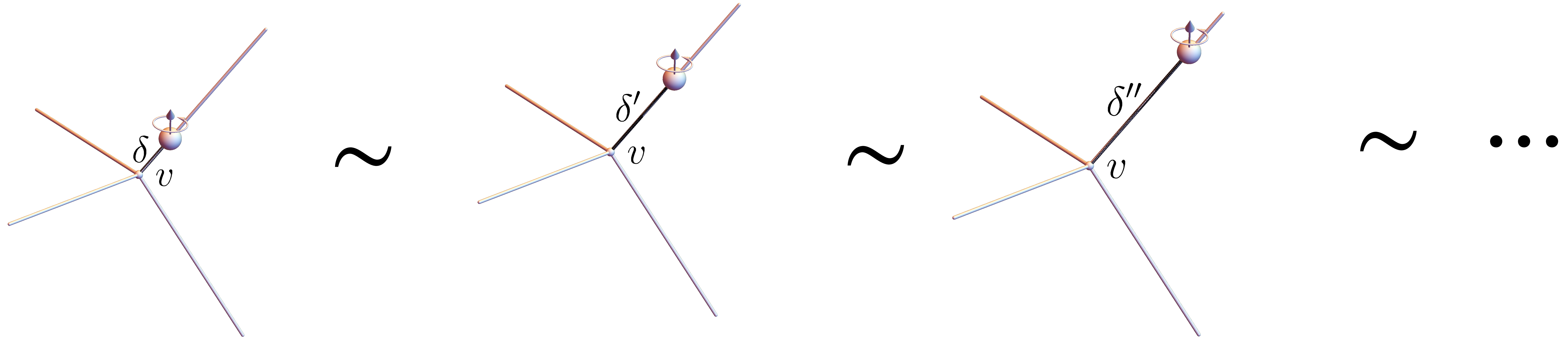


This can be done by group averaging with the diffeomorphisms preserving v.

$\mathscr{H}_{\rm vtx}$ is a dual space of the *cylindrical function* space, everything is well-defined by the duality

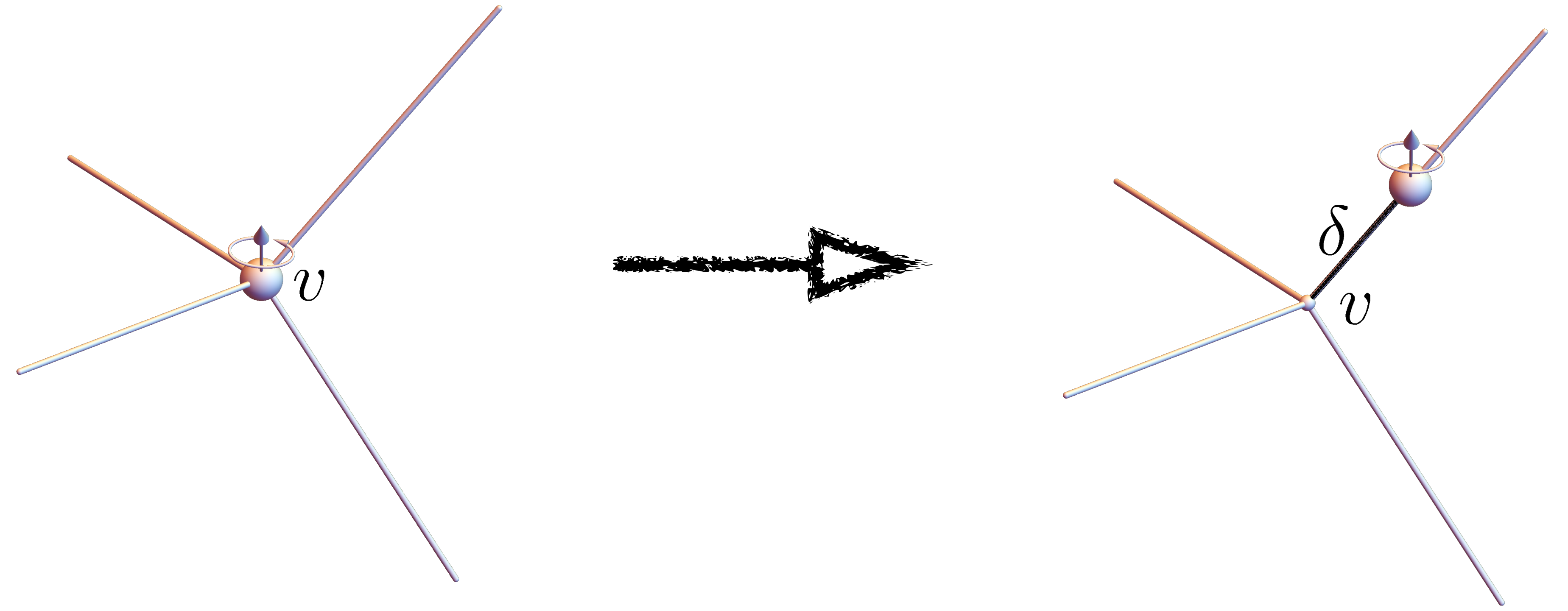
The Hamiltonian Constraint

The vertex Hilbert is defined such that:



$\lim_{\delta \to 0} \widehat{H_\delta^{(1)}(v)}'$ is well defined in $\mathscr{H}_{\text{vtx}}$, because $\widehat{H_\delta^{(1)}(v)}' = \widehat{H_{\delta'}^{(1)}(v)}' = \cdots$

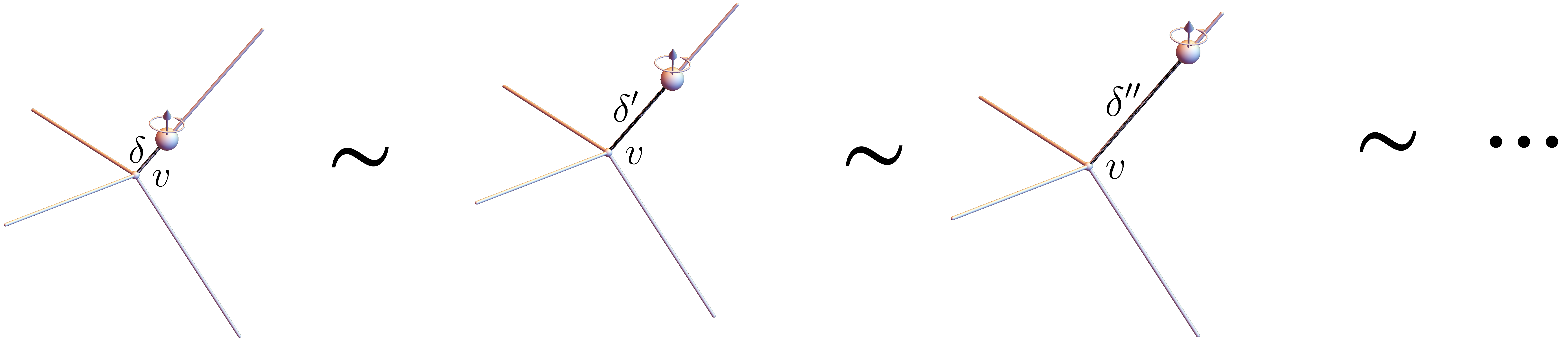
$\widehat{H_\delta^{(1)}(v)}$:



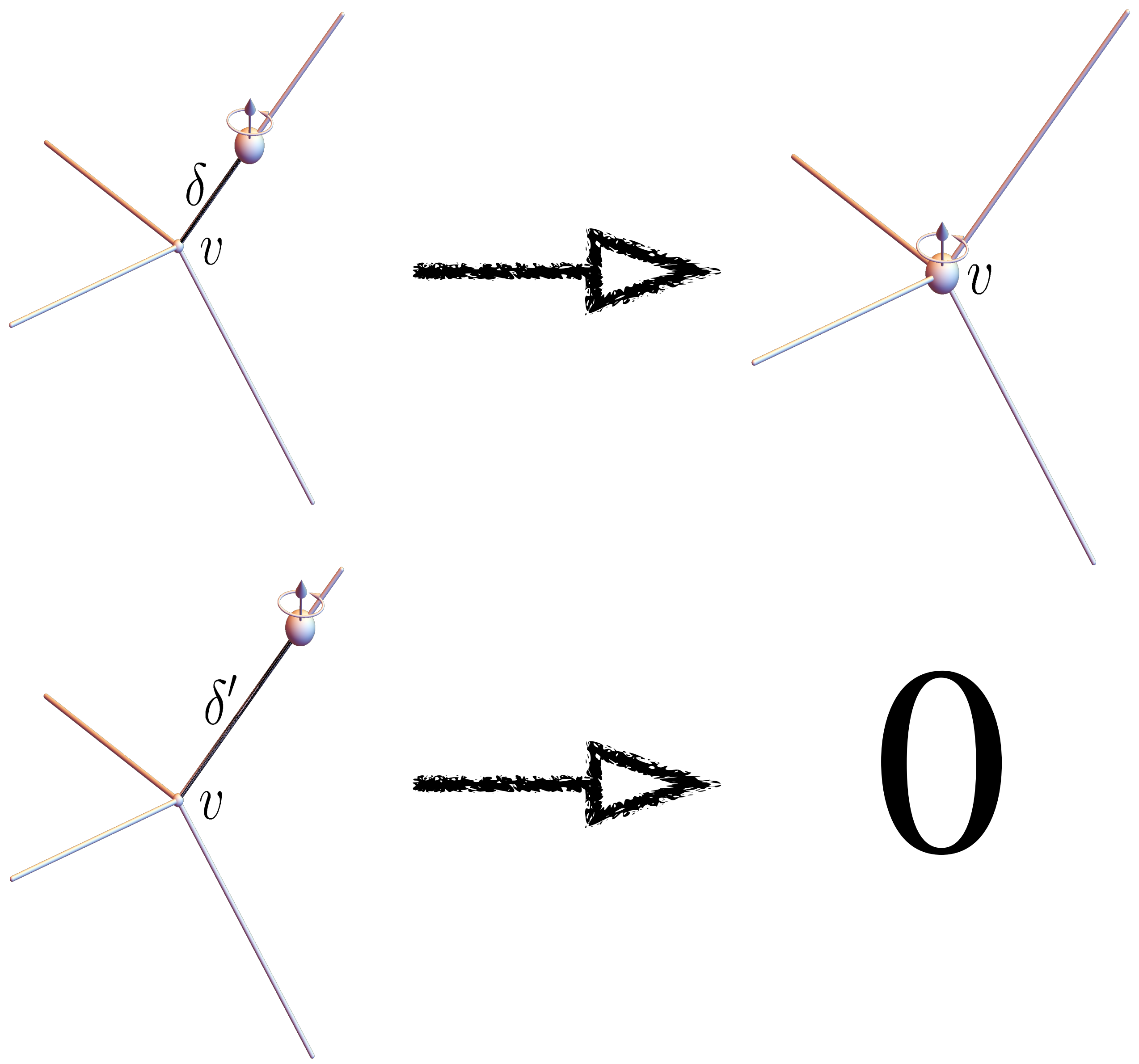
Problems 1 is solved

The Hamiltonian Constraint

The vertex Hilbert is defined such that:



$\widehat{H_\delta^{(1)}(v)}^\dagger$:



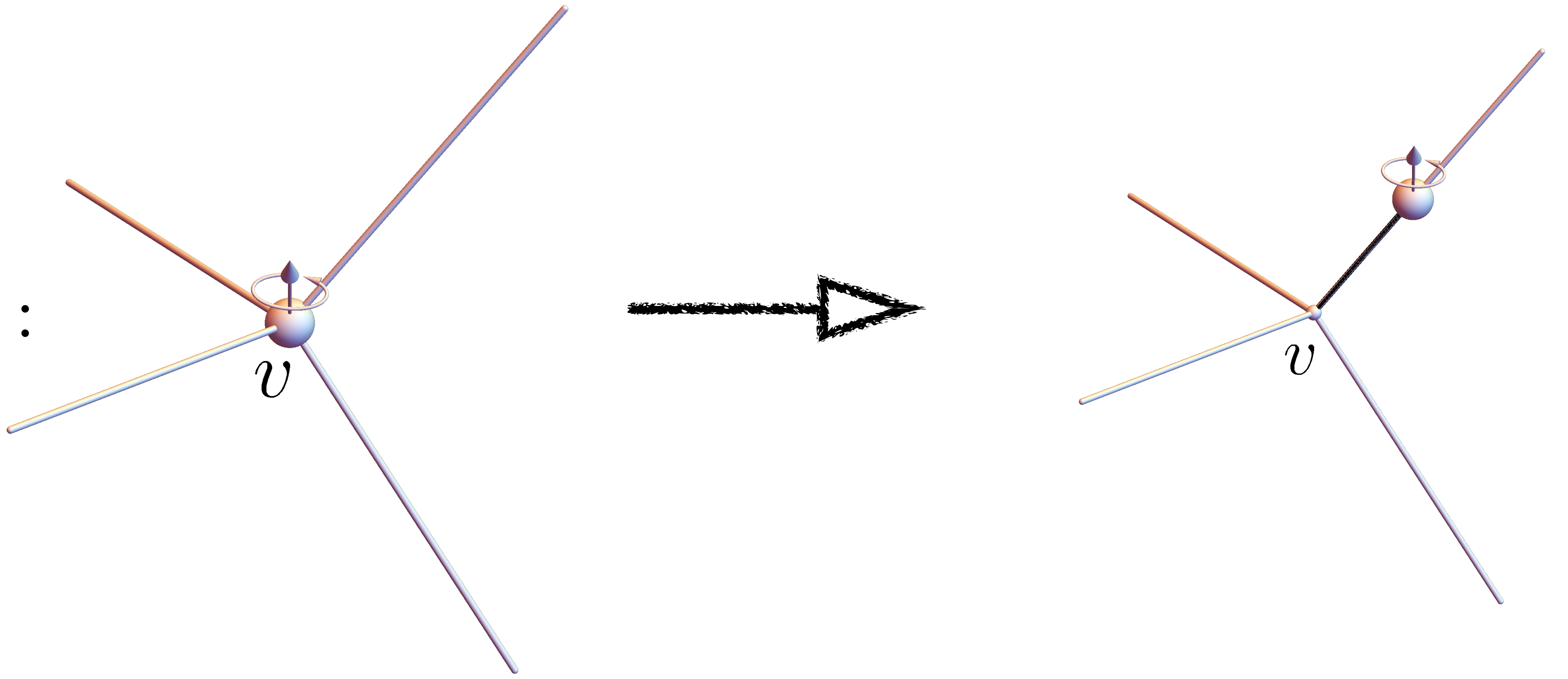
$\widehat{H_\delta^{(1)}(v)}^\dagger$ cannot be promoted as an operator in $\mathscr{H}_{\rm vtx}$.

Given up $\widehat{H_\delta^{(1)}(v)}^\dagger$ but use $(\widehat{H_\delta^{(1)}(v)}')^\dagger$

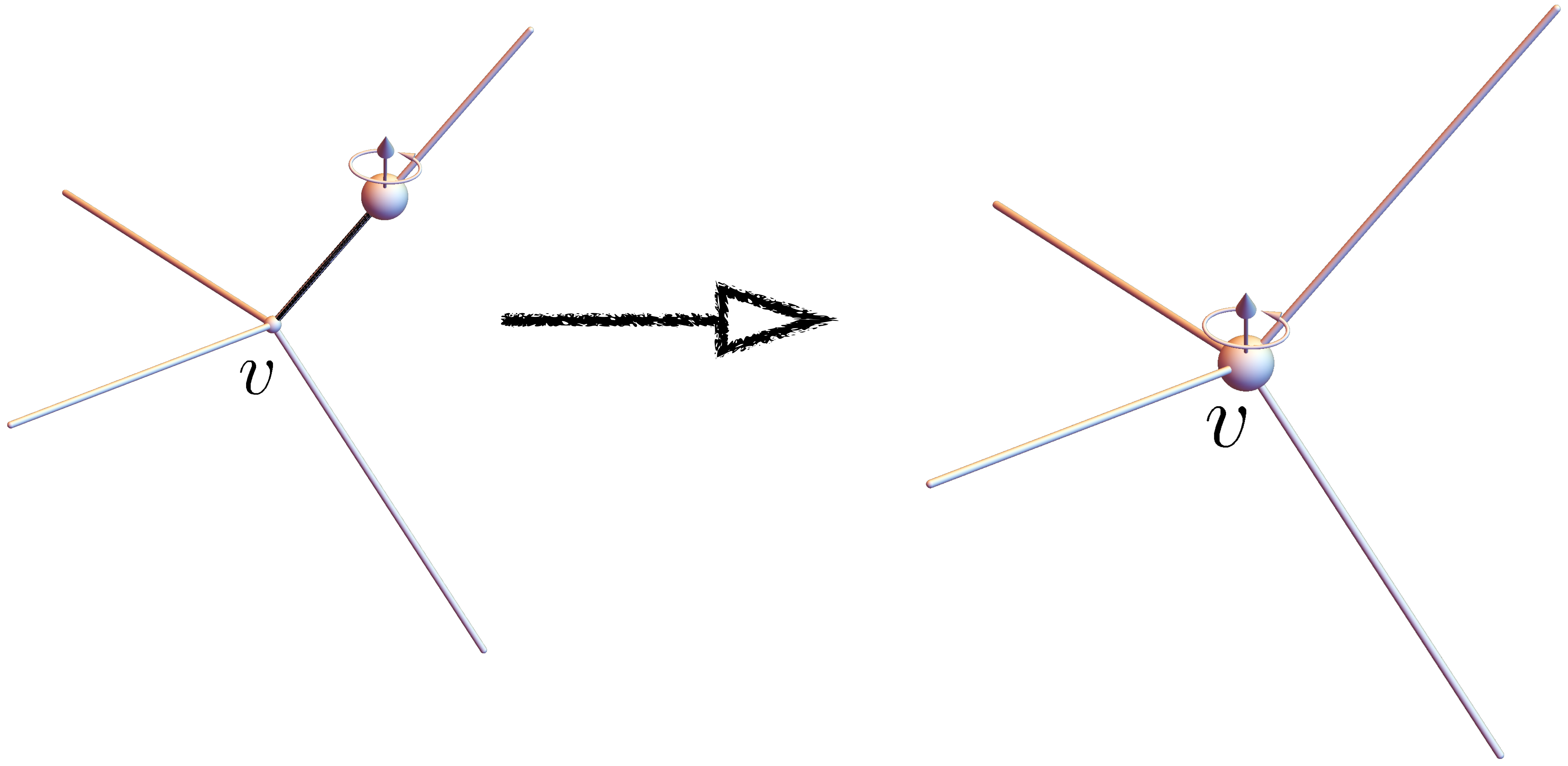
Problems 2 is solved

The Hamiltonian Constraint

$\lim_{\delta\to0}\widehat{H^{(1)}_\delta(v)}'$:

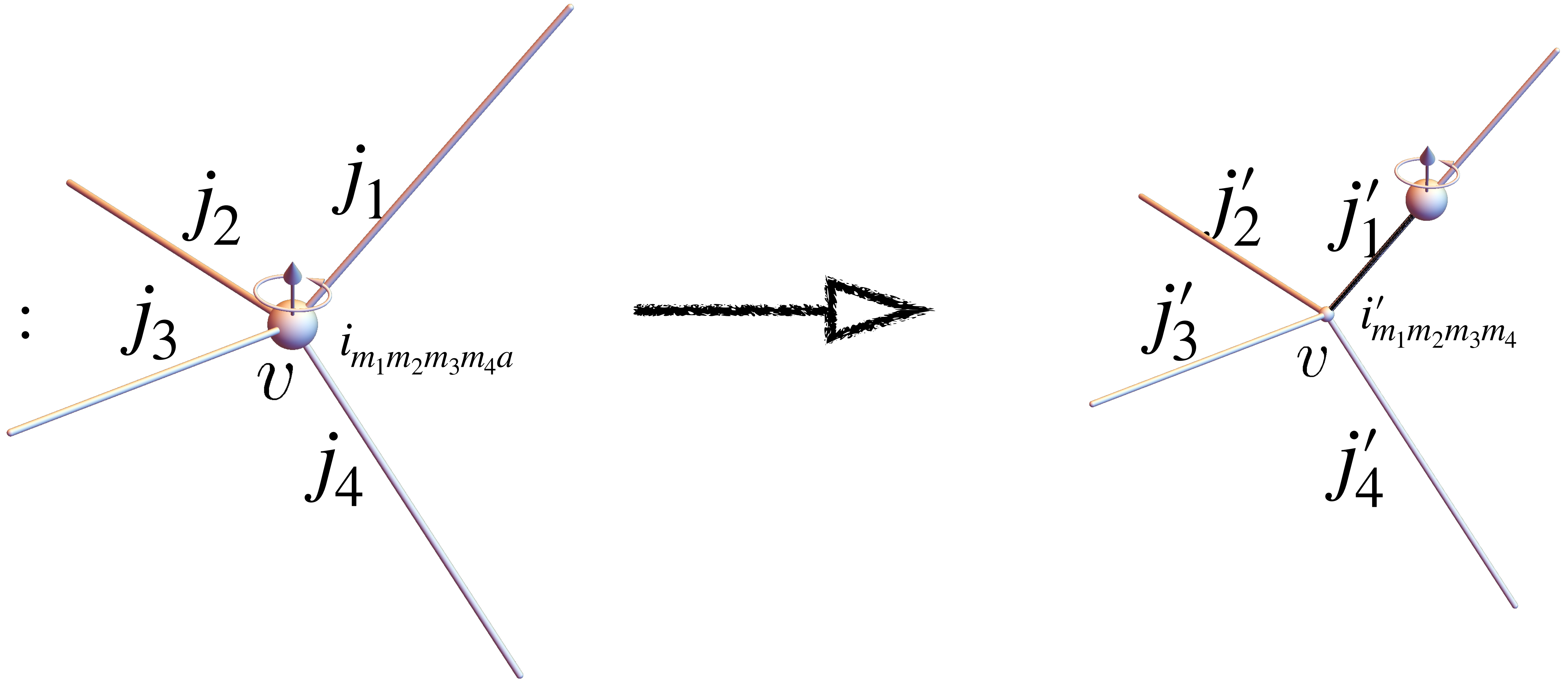


$\lim_{\delta\to0}(\widehat{H^{(1)}_\delta(v)}')^\dagger$:

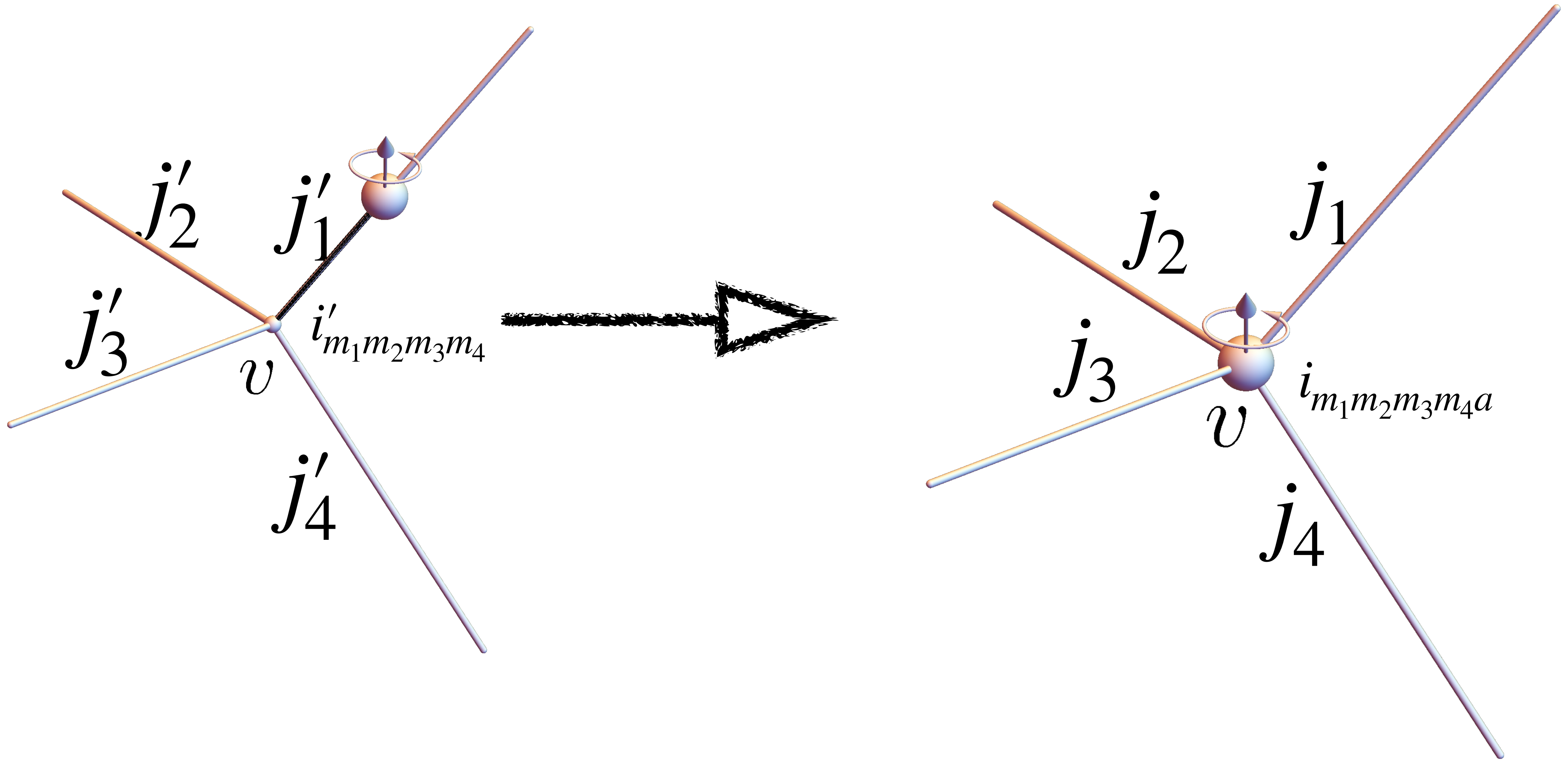


The Hamiltonian Constraint

$\lim_{\delta\to0}\widehat{H^{(1)}_\delta(v)}'$:



$\lim_{\delta\to0}(\widehat{H^{(1)}_\delta(v)'})^\dagger$:



Conclusion and outlook

Our work considers the coupling of fermion field to canonical LQG.

We investigate the Gauss and the Hamiltonian constraint in this model.

We solve the Gauss constraint explicitly, and regularize and quantize the Hamiltonian constraint by introducing the vertex Hilbert space.

This framework will be applied to recover the usual quantum field theory, and consider the backreaction between quantum matter and quantum spacetime.

Thank you