SSB and effective interactions

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Abstract

Elimination of constraints leads to new effective interactions for scalar fields with SSB and to gauge invariant Lagrangean in QED.

1. Introduction

It often happens in field theory that a dynamical system has constraints, eg. gauge theories, or fields quantized on the light front, where we observe the halving of degrees of freedom as compared with t=0 quantization. The rôle of constant fields (so called zero modes) also needs elucidation: on one hand there are to be considered as nondynamical constraints and on the other hand they describe normally the non vanishing vacuum expectation values of the fields. I developed recently a method to deal simply and explicitely with constraints[1], which I will illustrate here on the example of scalar field theory with "mexican hat" potential. As will be shown elimination of zero mode in this case has far reaching consequences: new interactions apear which lead to Bose-Einstein condensation. Perturbative vacuum is no longer the lowest energy state of the theory. The physical vacuum is populated by condensates of particles and their number increase with decreasing x^+. The same procedure is then applied to QED, where there are no condensates.

2. Real scalar field

Constraint eleimination[1] is simply done by integrating out the variables lacking time derivative in the Feynman path integral. We will apply our procedure to the "mexican hat" scalar field theory with the Lagrangean density

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4 \tag{2.1}$$

Notice the "wrong" mass term sign.

We see immediatly that for fields constant along direction x^- the time derivative is multiplied by 0, i.e. these fields are constraints. It is therefore judicious to extract explicitely this constant in view of integrating over it. Let us therefore put (ϕ_0 is the constant field i.e. zero mode)

$$\phi = \phi_0 + \tilde{\phi} \quad \Rightarrow \quad \int \tilde{\phi} d\xi = 0 \tag{2.2}$$

Introducing the useful notation and concentrating, for simplicity, on one dimensional case, we have

$$\Sigma_n = \int \tilde{\phi}^n \tag{2.3}$$

our Lagrangean takes form

$$L = \int \dot{\tilde{\phi}}\tilde{\phi}' + \frac{m^2}{2}(\phi_0^2 + \Sigma_2) - \frac{\lambda}{24}\left[\phi_0^4 + 6\phi_0^2\Sigma_2 + 4\phi_0\Sigma_3 + \Sigma_4\right] \tag{2.4}$$

We must now perform the path integral with this Lagrangean. At each point in time the integration over ϕ_0 is doable but since in this variable the exponent is quartic this leads to complicated cylinder functions. Therefore it is better to do the integrals by stationary phase approximation; this procedure is justified by the fact that in any case we have to assume small λ. The stationary ϕ_0 is, to the lowest nontrivial order,

$$\phi_0^2 = \frac{6m^2}{\lambda} - 3\Sigma_2 \tag{2.5}$$

and this gives us the final effective Lagrangean

$$L = \int \dot{\tilde{\phi}}\tilde{\phi}' - \frac{3m^4}{2\lambda} - \frac{2m^2}{2}\Sigma_2 - \frac{\lambda}{24}\Sigma_4 - \frac{m\sqrt{\lambda}}{\sqrt{6}}\Sigma_3 + \frac{3\lambda}{8}\Sigma_2^2 \tag{2.6}$$

We recover in this way all the terms of the analysis done with $t = 0$ approach, with the exception of the last term, which is new. Its appearence is really astounding since it implies Bose-Einstein condensation in the system (notice the sign!). In order to see it more clearly let us write the new term explicitely

$$\frac{3\lambda}{8}\Sigma_2^2 = \frac{3\lambda}{8V}\int \tilde{\phi}^2 d\xi \int \tilde{\phi}^2 d\xi \tag{2.7}$$

where V is the volume of the system (we assume quantization in a box). It is easy to see that, for each momentum k^+ its action lowers the P^- and hence the energy of the system by an amount wchich is quadratic in the number of particles: a close analogy to Gross-Pitayevsky equation. Recall that in terms of cration and anihilation operators $a_k^\dagger, a_k$ we have

$$\int \tilde{\phi}^2 d\xi = \int_0^\infty \frac{dk^+}{2k^+} a_k^\dagger a_k \tag{2.8}$$

hence its action on a state containing n particles of momentum k^+ is

$$\Sigma_2^2 \mid n\ particles\ with\ k^+\ > \sim \frac{n^2}{k^{+2}} \mid n\ particles\ with\ k^+\ > \tag{2.9}$$

The new term therefore destabilizes the perturbative vacuum and its action is stronger for small k^+. The Σ_4 is not able to stabilize the condensate and to get precise numerical estimates we need to go beyond our simple approximations.

3. Electromagnetic field on the light front

We will quantize the electromagnetic field interacting with a general conserved current.

With the introduction of the notation

$$x \equiv k^p k = \left[(k^1)^2 + (k^2)^2\right]^{1/2} \tag{3.1}$$

we have the mode Lagrangian

$$\begin{gathered} L_q = \frac{x^2}{2}(a_m^2 + b_m^2) + x(b_m \dot{a}_p - a_m \dot{b}_p) \\ +a_m(k^2 a_p - xka_\parallel) + b_m(k^2 b_p - xkb_\parallel) \\ +\frac{1}{2}(\dot{a}_p^2 + \dot{b}_p^2) - ka_p\dot{b}_\parallel + kb_p\dot{a}_\parallel \\ +x(a_\parallel \dot{b} - b_\parallel \dot{a}_\parallel) \\ -x(a_\perp \dot{b}_\perp - b_\perp \dot{a}_\perp) - \frac{k^2}{2}(a_\perp^2 + b_\perp^2) \\ +a_p f_m + b_p g_m + a_m f_p + b_m g_p \\ -a_\parallel f_\parallel - b_\parallel g_\parallel - a_\perp f_\perp - b_\perp g_\perp \end{gathered} \tag{3.2}$$

where we have used the Fourier components

$$A^\mu(k) = a_\mu + ib_\mu \qquad , \ k^p > 0 = a_\mu - ib_\mu \qquad , \ k^p < 0 \tag{3.3}$$

and similarely for the current

$$j^\mu(k) = f_\mu + ig_\mu \tag{3.4}$$

and introduced components parallel $(a_\parallel)$ and perpendicular $(a_\perp)$ to the 2 dimensional vector (k^1, k^2). The dot means the x^p derivative ("time" derivative); all indices are written below. The continuity equation for the current takes now form

$$\dot{f}_p = xg_m - kg_\parallel \quad , \dot{g}_p = -xf_m + kf_\parallel \quad . \tag{3.5}$$

The Lagrangian (3.2) looks and is considerably more complicated than its equal time counterpart. One component only is uncoupled from the others, the Lagrangian for this perpendicular component is

$$L_\perp = x(a_\perp \dot{b}_\perp - b_\perp \dot{a}_\perp) - \frac{k^2}{2}(a_\perp^2 + b_\perp^2) - a_\perp f_\perp - b_\perp g_\perp \tag{3.6}$$

This is simply the Lagrangian for the scalar field. It displays the halving of degrees of freedom ($a_\perp$ is conjugated to $b_\perp$). Note also that the continuity equations (3.5) do not restrict $f_\perp$ and $g_\perp$.

The remaining variables are coupled. The variables a_m and b_m lack the time derivative: those are constraint variables and the analogue of the Coulomb gauge at the light front is the gauge

$$A^m = 0 \tag{3.7}$$

a fact known[2] but seldom used. The variables $a_\|$ and $b_\|$ have their time derivatives entering linearly to the Lagrangian (3.2) as expected from light-front formalism. The variables a_p and b_p are the most surprising, since their time derivatives enter quadratically so, a priori, no halving of degrees of freedom is apparent. The canonical analysis would base on the gauge $A^m = 0$, which eliminates the constraint variables. However the resulting Lagrangian has still a large group of symmetries corresponding to time independent gauge transformations; the variables $a_p,\ b_p$ and $a_\|$ and $b_\|$ enter asymmetrically and, last not least, if the vector field is coupled to the Dirac field, important simplifications occur when $A^p = 0$, as noted earlier. Again integrating out the constraint variables clarifies the situation completely. After integrating out variables a_m and b_m we are simply left with

$$\tilde{L} = x(a_n \dot{b}_n - b_n \dot{a}_n) - \frac{k^2}{2}(a_n^2 + b_n^2) + a_n(f_\| - \frac{k}{x} f_p) + b_n(g_\| - \frac{k}{x} g_p) - \frac{f_p^2 + g_p^2}{2x^2} \quad , \tag{3.8}$$

where we have introduced new variables

$$a_n = a_\| + \frac{k}{x} a_p b_n = b_\| + \frac{k}{x} b_p \tag{3.9}$$

It is seen that a_n and b_n are the natural counterparts of $a_\perp,\ b_\perp$: they represent the other photon polarization. We have also obtained the light front Coulomb term: the last line of eq. 3.8. The most remarkable is the coupling of $a_n,\ b_n$ to the current: the coupling is only to the combination

$$j_\| - \frac{k}{x} j^p \tag{3.10}$$

There is no coupling to j^m components, and in the case of Dirac field all the simplifications of $A^p = 0$ gauge occur. Our procedure has therefore permitted us to identify the correct dynamical variables for quantization. We can express the projection operator on the physical degrees of freedom (the polarization sum)

$$d_{\mu\nu}(k) = \sum_{\lambda=1,2} e_\mu(\lambda) e_\nu(\lambda) = -g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{\eta_\alpha k^\alpha} \tag{3.11}$$

This is precisely what one uses to define Feynman rules. Equation (3.11) has a spurious singularity at $\eta_\alpha k^\alpha = 0$, i.e. when $k^p = 0$. At this point relations (3.9) become also singular and the light front energy becomes infinite. Therefore it is an infinite energy endpoint problem. In the variational approach to field theory on the light front[3] this point is excluded. From the point of view presented here, since all quantities are taken at *positive* k^p the singularity is of end point type. At the level of Feynman graphs it means that this singularity is to be regulated by *principal value prescription*.

In conclusion we see that our method of constraint elimination solves simply the liht front quantization. For tha main topic of interest: QCD zero modes will give new effective interaction, which, it is hoped, will give for the first time the possibility of calculating the condensates ab initio.

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References

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