# How to Measure the $Q\overline{Q}$ Potential in a Light-Front Calculation

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**Abstract**

A scheme is developed which shows how one would, given a light-front Hamiltonian for QCD, extract the $Q\overline{Q}$ potential, i.e. the quantity which corresponds to the potential between two infinitely heavy quarks in a rest frame, from a light-front calculation. The resulting potential will in general depend on the direction along which the infinitely heavy sources are separated and thus provides a direct probe of violations of rotational invariance in a physical observable. Furthermore, easy comparison to data from $c\bar{c}$ and $b\bar{b}$ spectroscopy and to lattice data is possible. The scheme may thus be very helpful in constructing a light-front Hamiltonian through an iterative procedure.

## 1. Introduction

The central quantity to be discussed in these lectures is:

$V(\vec{R}) \equiv$ potential energy of an $\infty$-heavy $Q\overline{Q}$-pair at separation $\vec{R}$ in its rest frame.

There are several reasons to study this observable in the light-front (LF) framework

- The LF formalism lacks manifest rotational invariance. Therefore, if one starts with a *wrong* LF Hamiltonian for QCD, the result $V(\vec{R})$ depends on the orientation of $\vec{R}$ with respect to the 3-axis. [1] A measurement of $V(\vec{R})$ thus provides a direct probe of rotational invariance in a physical observable.

- Calculating $V(\vec{R})$ on the LF allows one to test both short (asymptotic freedom) and long distance (confinement) aspects of the theory within the same set of calculations by simply changing $|\vec{R}|$, and thus provides a stringent test for perturbative as well as nonperturbative aspects of the interactions.

- And most importantly, $V(\vec{R})$ in QCD is very well known over a large range of distances from the spectroscopy of heavy $Q\overline{Q}$ mesons as well as from nonperturbative Euclidean lattice calculations (at least in the absence of dynamical quarks — but it is easy to make the same approximation in a LF framework).

[1] We use the notation $A^{\pm} = A_{\mp} = (A^0 \pm A^3)/\sqrt{2}$, $\vec{A}_\perp = (A^1, A^2)$.

Before we embark on deriving the effective LF Hamiltonian for two infinitely heavy sources, it is instructive to understand physically what it means to have two fixed sources at rest from the LF point-of-view. As should be clear from Fig.1, fixed charges in a rest frame correspond to charges that move with constant velocity on the LF [$v^+ = v^- = 1/\sqrt{2}$ in the example in Fig.1]. Furthermore, if the longitudinal separation is $\Delta x^3$ in the rest frame, the charges have fixed separation $\Delta x^- = \sqrt{2}\Delta x^3$ in the longitudinal LF direction [in the more general case, where the charges are moving with constant four velocity $v^\mu$, where $\vec{v}_\perp = 0$ in the rest frame, one obtains $\Delta x^- = \Delta x^3/v^+$].

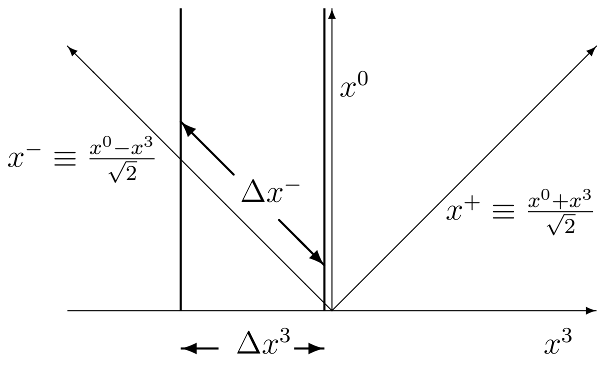


Fig.1: World lines for two charges with longitudinal separation $\Delta x^3$ in the rest frame.

The transverse separation is the same on the LF as it is in a rest frame description. Therefore, in order to understand the physics of fixed charges with separation $\vec{R} = (R^1, R^2, R^3)$ in a LF description, we must first understand how to describe a "dumbbell," with ends separated by $(\Delta x^1, \Delta x^2, \Delta x^-) = (R^1, R^2, R^3/v^+)$, that moves with constant velocity $v^+$.

## 2. One Heavy Quark on the LF

A pair of fixed sources can also be interpreted (and treated) as one *extended* source moving with constant velocity $v^\mu$ (for simplicity, we will keep $\vec{v}_\perp = \vec{0}$). This is reminiscent of heavy quarks and thus, as a warmup exercise, it is very instructive to consider one (pointlike) heavy quark on the LF first (see also Refs.[1,2]).

For simplicity, we will first take the heavy quark limit for the canonical Hamiltonian, which can be written in the form

$$P_B^- = \frac{M_b^2 + \vec{k}_{b\perp}^2}{2p_b^+} + P_{HL}^- + P_{LL}^-, \tag{1}$$

where $B$ represents the hadron, $b$ is the heavy quark, $P_{HL}^-$ contains the interactions between heavy ($b$) and light degrees of freedom and $P_{LL}^-$ contains all terms involving light degrees of freedom only. The heavy quark limit is obtained by making an expansion in inverse powers of the $b$-quark mass. For this purpose we write

$$p_b^+ = P_B^+ - p_L^+ = M_B v^+ - p_L^+, \tag{2}$$

where $p_L^+$ is defined to be the sum of the longitudinal (LF-) momenta of all light degrees of freedom. For the (total) LF-energy we write on the l.h.s. of Eq.(1)

$$P_B^- = M_B v^- = \frac{M_B}{2v^+} = \frac{M_b + \delta E}{2v^+}, \tag{3}$$

where $\delta E \equiv M_B - M_b$ is the "binding energy" of the hadron. After inserting Eqs.(2) and (3) into Eq.(1) and expanding one obtains

$$\begin{aligned} \frac{M_b+\delta E}{2v^+} &= \frac{M_b^2+\vec{k}_{b\perp}^2}{2(M_B v^+ - p_L^+)} + P_{HL}^- + P_{LL}^- = \frac{M_b^2+\vec{k}_{b\perp}^2}{2\left(M_b v^+ + \delta E v^+ - p_L^+\right)} + P_{HL}^- + P_{LL}^- \\ &= \frac{M_b}{2v^+} - \frac{\delta E}{2v^+} + \frac{p_L^+}{2v^{+2}} + \mathcal{O}(1/M_b) + P_{HL}^- + P_{LL}^-. \end{aligned} \tag{4}$$

Note that we assumed that $\vec{p}_\perp$ of the heavy quark is small compared to its mass which is justified in a frame where the transverse velocity of the heavy hadron vanishes. The term proportional to $M_b$ cancels between the l.h.s. and the r.h.s. of Eq.(4) and we are left with

$$\frac{\delta E}{v^+} = \frac{p_L^+}{2v^{+2}} + \mathcal{O}(1/M_b) + P_{HL}^- + P_{LL}^-. \tag{5}$$

The *brown muck* Hamiltonian $P_{LL}^-$ is the same as for light-light systems and will not be discussed here. The interaction term between the heavy quark and the brown muck ($P_{HL}^-$) is more tedious but straightforward. For example, heavy quark pair creation terms (via instantaneous gluons) are proportional to $1/(p_{b_1}^+ + p_{b_2}^+)^2 \propto 1/M_b^2$ and can thus be neglected. Similarly, pair creation of heavy quarks from virtual gluons is also suppressed by at least one power of $M_b$. This also justifies our omission of states containing more than one heavy quark from the start. Other terms that vanish in $P_{HL}^-$ include interactions that involve *instantaneous exchanges* of heavy quarks, which are typically proportional to the inverse $p^+$ of the exchanged quark and thus of the order $\mathcal{O}(M_b^{-1})$. Up to this point, all interaction terms that we have considered vanish in the heavy quark limit. The more interesting ones are of course those terms which survive. The simplest ones are the instantaneous gluon exchange interactions with light quarks or gluons, which are, respectively, $V_{Qq} \propto (p_q^+ - p_q^{\prime +})^{-2}$ and $V_{Qg} \propto (p_g^+ + p_g^{\prime +})(p_g^+ - p_g^{\prime +})^{-2}$ and remain unchanged in the limit $M_b \to \infty$. Terms which involve instantaneous gluon exchange and are off-diagonal in the brown muck Fock space behave in the same way.

The quark gluon vertex simplifies considerably. For finite quark mass one has for the matrix element for the emission of a gluon with momentum $k$, polarization $i$ and color $a$ between quarks of momentum $p_1$ and $p_2$

$$P_{QQg}^- = -igT^a \left\{ 2\frac{k^i}{k^+} - \frac{\vec{\sigma}_\perp \vec{p}_{2\perp} - iM_b}{p_2^+}\sigma^i - \sigma^i \frac{\vec{\sigma}_\perp \vec{p}_{1\perp} + iM_b}{p_1^+} \right\}, \tag{6}$$

where spinors as well as creation/destruction operators have been omitted for simplicity. In the heavy quark limit [note $1/p_1^+ - 1/p_2^+ = \mathcal{O}(M_b^{-2})$] the spin dependent terms drop out and one finds

$$P_{QQg}^- = -2igT^a \frac{k^i}{k^+}. \tag{7}$$

The spin of the heavy quark thus decouples completely, giving rise to the well known $SU(2N_f)$ symmetry in heavy quark systems.

## 3. Divergences

Throughout the above discussion we have tacitly assumed that all momentum scales other than the heavy quark longitudinal momentum are small compared to the heavy quark mass (or longitudinal momentum). To justify this assumption we impose the following cutoffs:

- a small momentum cutoff $\Theta(k_i^+ - \varepsilon v^+)$ on all light constituents and on all momenta exchanged in instantaneous interactions.
- a cutoff on the maximum allowed difference in kinetic energy at each vertex $\Theta\left(\frac{\Lambda}{v^+} - \left|P_{kin}^-(in) - P_{kin}^-(out)\right|\right)$

Notice that the cutoffs have been chosen such that the spectrum (with the regulators present) is manifestly independent of $v^+$. Notice further that the above cutoffs guarantee that the heavy quark limit commutes with loop integrations. In other words, for loop integrals it does not matter whether one first computes the amplitudes using the full Hamiltonian and then takes the heavy quark limit or whether one takes the heavy quark limit first and computes loops using the heavy quark effective LF Hamiltonian. This guarantees that the heavy quark limit is nonsingular.

It should be emphasized that, while the above conditions are sufficient to guarantee a meaningful heavy quark limit, other, less stringent, ways to cutoff the divergences (e.g. using a transverse lattice in combination with some low-$k^+$ regulator) are conceivable.

In Section 2, we illustrated the heavy quark limit starting from the *canonical* LF Hamiltonian. However, most regularization schemes, particularly momentum cutoffs such as the one described above, require non-canonical counterterms, including "counterterm functions" [4]. How does this influence the heavy quark limit? The counterterm functions typically depend on ratios of momenta and one might be worried that this provides some "hidden" dependence on the heavy quark mass, because heavy quark momenta are of the order of the heavy quark mass. It turns out that, with the above or similar cutoffs, heavy quark momenta typically enter the counterterm functions through ratios between two heavy quark momenta, but not ratios of heavy quark to brown muck momenta.[2] Upon expanding the heavy quark momentum in powers of $1/M_b$ one thus finds that log's of $M_b$ cancel in the effective LF Hamiltonian and only $\mathcal{O}(1/M_b)$ corrections remain from the dependence of the counterterm functions on the heavy quark momentum. This is just another example of the fact that, with the above cutoffs in place, the heavy quark limit

[2] I was not able to give a strict proof for this intuitively obvious result but could not find a counterexample either.

and loop integrations commute. Therefore, as long as one keeps $\Lambda \ll M_b$:

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| The conterterm functions, which arise from renormalizing the LF-Hamiltonian using noncovariant cutoffs, depend, to leading order in $1/M_b$, only on ratios of brown muck momenta but not on the heavy quark momentum or its mass. |

By restricting all momenta (except the longitudinal momentum of the heavy quark) and momentum transfers to the heavy quark to be small compared to the heavy quark mass, we have implicitly limited the applications of the above formalism to physical processes which satisfy these conditions. In particular, we cannot use the above formalism directly for calculating Isgur-Wise functions at large recoil or decay constants for heavy mesons because these processes typically involve a large momentum transfer. The formalism is applicable, however, for calculating $B$-meson spectra as well as the $Q\bar{Q}$-potential at distances that are large compared to the inverse heavy quark mass.

For finite $M_Q$, the total momentum that can be carried by the brown muck is bounded from above by the total momentum of the hadron. Since the total momentum of the hadron in the heavy quark limit is infinite, this natural cutoff no longer exists. However, this does *not* lead to any new divergences: One can solve analytically the problem of photons coupled to an infinitely heavy electron, where one obtains a photon distribution that falls of at large $k^+$ like $\rho(k^+) \equiv \int dk_\perp^2 \rho(k^+, k_\perp^2) \sim 1/\left(v^+ \Lambda k^{+4}\right) + 1/k^{+5} \log\left(\Lambda v^+/k^+\right)$. In QCD, if one excludes the strict chiral limit, there are no massless excitations and one expects an even more rapid falloff at large $k^+$.

## 4. Two Heavy Sources

As we discussed in section 2, two heavy sources at fixed separation can be formally treated as one extended heavy source (just think of a dumbbell). Therefore, $H_{LF}$ for two sources is the same as for one source with two minor modifications:

- All vertices involving a heavy source get modified according to the rule

$$\left\{ \chi^{\dagger\prime} T^a \chi \times \hat{O}(brown\,muck) \right\} + h.c. \rightarrow$$
$$\left\{ \left[ \chi_Q^{\dagger\prime} T^a \chi_Q F_R(q) - \chi_{\bar{Q}}^\dagger T^a \chi_{\bar{Q}}{}' F_R(-q) \right] \times \hat{O}(brown\,muck) \right\} + h.c., \qquad (8)$$

  where the "form factor"

$$F_R(q) = \exp\left[ \frac{i}{2} \left( \frac{q^+ R^3}{v^+} - \vec{R}_\perp \vec{q}_\perp \right) \right] \qquad (9)$$

  arises from acting with the (kinematic !) displacement operator on the position of the heavy quark/antiquark (from $x^- = 0$, $\vec{x}_\perp = \vec{0}_\perp$ to $x^- = \pm R^3/2v^+$, $\vec{x}_\perp = \pm \vec{R}_\perp/2$) and $q$ is the *net* momentum transferred to the brown muck. This rule holds irrespective of the number of gluons involved in this process. $\hat{O}(brown\, muck)$ contains all the creation/destruction operators acting on the brown muck as well as the counterterms and, when necessary, the counterterm functions. Note that $\hat{O}(brown\, muck)$ is the same for one or two heavy sources.

- There is a static potential between the two heavy quarks. The canonical Hamiltonian yields $P^-_{HH} = g^2\, \chi^{\dagger\,\prime}_Q T^a \chi_Q\, \chi^\dagger_{\bar{Q}} T^a \chi_{\bar{Q}}{}'\, \delta^{(2)}(\vec{R}_\perp)|R^3|v^+$. In general, there will be a more complicated dependence on $\vec{R}$ which has to be determined by demanding self-consistency. For example, singularities arising from exchange of gluons with low $q^+$ between the two sources should cancel (nonperturbatively) with the IR behavior of the instantaneous potential in $P^-_{HH}$ [5].

Using Eq.(5), $V(\vec{R})$ can thus be extracted as follows:

1. For a given $\vec{R}$ and $v^+$, write down the effective LF-Hamiltonian for the heavy pair interacting with the brown muck (including the form factors Eq.(9) and including all the counterterms and counterterm functions which would also appear in a "heavy-light" system).
2. The lowest eigenvalue $\delta E^{(1)}$ from Eq.(5), i.e. the QCD-ground state in the presence of the two heavy sources, is then equal to $V(\vec{R})$.

The heavy quark potential thus calculated is equivalent to the potential which a lattice theorist would extract from an asymmetric rectangular Wilson loop. Since there is plenty of "quenched" lattice data around, it would make sense to omit light quarks completely in a first approach and to focus on the pure glue part of the brown muck. However, the formalism described above is so general that one could also use it in a LF calculation that includes (dynamical) light quarks.

## 5. Summary

I have set up a very general scheme for computing the potential energy between two heavy quarks from a LF calculation. The scheme has been deliberately left very general in order to make it easy for people using very different approaches to LF-QCD to adopt the scheme and to calculate this important observable. Calculating the heavy quark potential on the LF is useful for testing rotational invariance and for comparing to lattice data and fits to charmonium spectra. Compared to other LF calculations, the main advantage is that the calculation can be done in the pure glue sector (the two sources are static!) and one thus does not have to bother about issues such as kinetic versus vertex masses and zero modes of the quark. Of course, if the effective $Q\overline{Q}$ potential violates rotational invariance, this would perhaps also show up indirectly in the charmonium spectrum. However, a direct calculation of the $Q\overline{Q}$ potential, as I propose here, will reveal the source of the violation more clearly: instead of comparing charmonium states, which are polarized in various directions, one can just orient the "dumbbell" in different directions.

Conceivable applications of this scheme are, for example:

- helping to pin down the counterterm functions in $H_{LF}^{eff}$: one could imagine parameterizing the noncovariant counterterm functions in the Hamiltonian (basing the ansatz on some superposition of perturbation theory, power-counting and intuition) by a number of "coupling constants." Demanding rotational invariance of the $Q\bar{Q}$ potential at all distances provides constraints in this coupling constant space, thus complementing renormalization group studies of LF Hamiltonians.

- fixing the transverse scale and determining the renormalization constants for the various couplings which appear in the transverse lattice formulation of LF QCD.

Another potential use of the above formalism derives from a curious mathematical observation: usually, in a LF Hamiltonian, all energies scale like $(\text{momentum})^{-1}$ and thus an unconstrained variational calculation (total momentum not fixed) of the energy leads to nonsense: infinite LF momentum, zero energy for all states. This is quite cumbersone because this usually implies that one has to build momentum conservation into the variational ansatz — thus making it often difficult to calculate matrix elements of the interaction in the multiparticle ($> 3$) sectors of the wavefunction (unless one uses a plane wave basis). This is not the case for the $H_{LF}^{eff}$ for a heavy quark system (5), where, first of all, the brown muck momentum is *not* conserved, and second "runaway solutions" (momentum $\rightarrow \infty$) are prevented due to the term proportional to the brown muck momentum on the r.h.s. of Eq.(5), which arises from expanding the heavy quark kinetic term and acts like a Lagrangean multiplier. For heavy quark systems it is therefore easy to write down an ansatz for a orthonormalized set of many body wavefunctions that still allows simple evaluation of matrix elements. Many body techniques can thus more easily be applied to heavy quark systems on the LF than to hadrons with finite mass on the LF.

### References


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