Light-Front Quantisation of Gauge Theories in a Finite Volume

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Abstract

We discuss the light front formulation of $SU(2)$ Yang-Mills theory on a torus.

1 Introduction

In 1949, Dirac, when formulating his rules of relativistic dynamics[1] suggested that one might use the variable $x^+ = t + x^3/c$ as a new time parameter instead of the usual Galileian time t. This new formulation, replacing the standard "instant form" of dynamics, he called "front form" referring to the fact that light fronts (LFs) $x^+ = const$ are now surfaces of equal time. Time evolution is accordingly governed by a LF Hamiltonian $P^- \equiv P^0 - P^3$. In the late sixties, Dirac's idea was extended to field theory in the context of current algebra and the quark parton model. This development culminated in the formulation of LF QCD a decade later[2].

The basic ingredient of canonical LF QCD is the use of the light cone (LC) gauge $A^+ \equiv A_- = 0$, which is physical (i.e. free of ghosts and negative metric states) and leads to a simple elimination of redundant degrees of freedom. The LF Hamiltonian becomes thus a function of the transverse gauge potentials A_i and the "good" fermion components ψ_+ only. The other important feature of canonical LF QCD is the triviality of the vacuum, which allows for a Fock expansion of hadrons in terms of only a few constituents. This has led to the method of discretised light cone quantisation (DLCQ)[3], which has been used to solve (low-dimensional) quantum field theories on a longitudinal momentum lattice.

In the meantime, however, it has become clear that the canonical formulation of LF QCD suffers from several problems: the LC gauge shares all the troubles known from axial gauges, in particular infrared singularities, $1/(k^+)^n$, related to residual gauge invariance. These induce symmmetry violating counterterms which obscure the renormalisability of the theory. Furthermore, it has become apparent that the vacuum is trivial only if there are no dynamical modes present with LF momenta $k^+ = \mathbf{k}_\perp = 0$, called zero modes (ZMs)[4].

To regulate and resolve the infrared problems we will work in a finite spatial volume, which allows a clear separation between ZMs and normal modes. Our final goal is to find a LF Hamiltonian with the ZMs included that might serve as a basis for DLCQ calculations. This contribu-

tion describes the first few steps of this program.

2   The LC Gauge on a Torus

We compactify by the restriction $-L \leq x^-$, x_1, $x_2 \leq L$ and by imposing periodic boundary conditions for the gauge fields. Space thus topologically becomes a torus. Consider the Wilson loop winding around the torus in x^--direction,

$$W[A] = \frac{1}{N}\mathrm{tr}\,\mathbf{P}\exp\left[-ig\int_{-L}^{L} dx^- A_-\right] . \tag{1}$$

Choosing the LC gauge $A_- = 0$ amounts to prescribing the value $W[A] = 1$ for this gauge invariant dynamical quantity which is clearly forbidden. The LC gauge on a torus, therefore, cannot be attained. A possible gauge choice, however, is $A_- = Y(\mathbf{x})$, where a ZM $Y(\mathbf{x})$ with respect to x^- is retained[5].

3   Complete Abelian Gauge Fixing

As a warm-up excercise we treat pure electromagnetism coupled to external sources J_μ in $d = 3 + 1$ including all ZMs. First we need some notation. We write any phase space variable f (gauge potentials and conjugate momenta) as a sum of four components f_r, $f = f_0 + f_1 + f_2 + f_3$, which are defined as

$$
\begin{aligned}
f_3 &\equiv f(x) - \frac{1}{2L}\int dx_3\, f(x) \\
f_2 &\equiv \frac{1}{2L}\int dx_3\, f(x) - \frac{1}{(2L)^2}\int dx_3 dx_2\, f(x) \\
f_1 &\equiv \frac{1}{(2L)^2}\int dx_3 dx_2\, f(x) - \frac{1}{(2L)^3}\int d^3x\, f(x) \\
f_0 &\equiv \frac{1}{(2L)^3}\int d^3x\, f(x) .
\end{aligned}
$$

We have denoted $x_3 \equiv x^-$; all integrations extend from $-L$ to L. The index r counts on how many spatial variables f_r depends; f_0, for example, is thus constant in all three spatial directions (a global ZM). The components f_r obey

$$\int_{-L}^{L} dx_r f_r = 0 , \tag{2}$$

which states the absence of a ZM with respect to x_r in f_r.

Before any elimination of variables the canonical pairs (in the Poisson bracket sense) are $(A_-, \Pi_+) \equiv (A_3, \Pi_3)$, (A_i, Π_i); $i = 1, 2$. After decomposition, Gauss's law reads

$$\begin{aligned}
\partial_3 \Pi_{33} + \partial_2 \Pi_{23} + \partial_1 \Pi_{13} &= J_{-3} \\
\partial_2 \Pi_{22} + \partial_1 \Pi_{12} &= J_{-2} \\
\partial_1 \Pi_{11} &= J_{-1} \\
0 &= J_{-0} \ .
\end{aligned} \tag{3}$$

The components Π_{rr}, $r = 1, 2, 3$ are to be eliminated. This can be done without ambiguities due to condition (2). The inverse operators ∂_r^{-1} are given by periodic sign functions[6].

A natural gauge choice for the case at hand is suggested by a method due to Faddeev and Jackiw[7]: set to zero those components of the gauge fields the conjugate momenta of which are eliminated via Gauss's law. Here, this leads to a gauge fixing first introduced by Palumbo in order to modify the axial gauge on a torus[8],

$$A_{33} = A_{22} = A_{11} = 0 \ . \tag{4}$$

This gauge will henceforth be called Palumbo gauge. Its actual virtues will become more clear in the non-abelian case. There still is a residual gauge freedom consisting of discrete shifts of the global ZMs A_{r0}. This can be eliminated by appropriately restricting the range of A_{r0} and will not be discussed here.

After solving additional, LF specific, second class constraints the Hamiltonian on the reduced phase space becomes

$$\begin{aligned}
\mathcal{H}_{red} &= -\frac{1}{2} A_{i3} \Delta_\perp A_{i3} - (\partial_i A_{i3})(\partial_3^{-1} J_{-3}) - J_{i3} A_{i3} - \frac{1}{2} J_{-3} \partial_3^{-2} J_{-3} + \\
&+ \frac{1}{2} \Pi_{30}^2 - J_{+0} A_{30} - J_{+2} \Delta_\perp^{-1} J_{-2} + \frac{1}{2} J_{12} \Delta_\perp^{-1} J_{12} - J_{+1} \partial_1^{-2} J_{-1} + \frac{1}{2} J_{21} \partial_1^{-2} J_{21} \ .
\end{aligned} \tag{5}$$

Note the appearance of additional Coulomb terms induced by ZMs in the last line. Eq. (5) should be compared with the naive version (without ZMs) given by

$$\mathcal{H}'_{red} = -\frac{1}{2} A_i \Delta_\perp A_i - (\partial_i A_i)(\partial_3^{-1} J_-) - J_i A_i - \frac{1}{2} J_- \partial_3^{-2} J_- \ , \tag{6}$$

where some ad hoc prescription of the inverse derivative $\partial_3^{-1,2}$ has to be used.

4 Complete Non-Abelian Gauge Fixing

We consider the simplest example, namely $SU(2)$ Yang-Mills theory. We mainly use matrix notation $f \equiv f^a \tau^a / 2$; $a = 1, 2, 3$, the τ^a being the standard Pauli matrices. The canonical

pairs before any elimination are the same as in the abelian case. With the help of the covariant derivative $D_r = \partial_r + ig[A_r, \quad]; r = 1, 2, 3$, Gauss's law can be written as

$$G \equiv \sum_{r=1}^{3} D_r \Pi_r = 0 \ . \tag{7}$$

Again we want to solve for the projections $\Pi_{rr}; r = 1, 2, 3$, and the corresponding gauge choice is the Palumbo gauge $A_{rr} = 0$. If we decompose Gauss's law according to the different phase space sectors we first find the global ZM $G_0 = 0$. This constraint will not be discussed here. In the other phase space sectors ($r > 0$) we obtain

$$\mathcal{D}_r \Pi_{rr} \equiv \partial_r \Pi_{rr} + ig[\mathcal{A}_r, \Pi_{rr}] = R_r \ , \tag{8}$$

where the field $\mathcal{A}_r = \sum_{s=0}^{r-1} A_{rs}$ is the sum of all ZMs of A_r with respect to x_r and therefore does not depend on x_r which will become important in a moment. The inhomogeneity R_r is independent of Π_{rr}. Thus, if we can invert the covariant derivative $\mathcal{D}_r$ (which essentially is the Faddeev-Popov (FP) matrix), we can solve Gauss's law for Π_{rr}. Note that in contrast to the naive LC gauge the FP matrix depends on the gauge fields, but only on the special configurations $\mathcal{A}_r$. As these are ZMs with respect to x_r, i.e. $\partial_r \mathcal{A}_r = 0$, and as $\mathcal{D}_r$ is an ordinary (and not a partial) differential operator, the eigenvalue problem of the covariant derivative factorises into space and color and becomes exactly solvable! This means that we can determine the spectral decomposition of the associated Green function $G_{(r)}$ and solve for the momenta $\Pi_{rr} = G_{(r)} * R_r$. For the axial gauge this has already been noted by Palumbo[8].

The eigenfunctions u_{n_r, α_r} are a product of plane waves $\exp(i\pi n_r x_r / L)$ and color matrices. The eigenvalues λ_{n_r, α_r} are given by

$$\lambda_{n_r, \alpha_r} = \frac{\pi n_r}{L} + g\alpha_r \ . \tag{9}$$

Here, the numbers n_r are integer and $\alpha_r = 0, \pm|\mathcal{A}_r|$, where $|\mathcal{A}_r| \equiv (\mathcal{A}_r^a \mathcal{A}_r^a)^{(1/2)}$ is the magnitude of the isovector with components $\mathcal{A}_r^a$. Thus, there would be ZMs of the FP matrix if $|\mathcal{A}_r| = \pi n_r / gL$. However, if we restrict the range of $\mathcal{A}_r$ to the fundamental modular domain[9], $|\mathcal{A}_r| < \pi n_r / gL$, the ZMs are absent and we can uniquely invert $\mathcal{D}_r$ which yields the Green function

$$G_{(r)}(x_r, y_r) = \sum_{n_r \neq 0; \alpha_r} \frac{u_{n_r, \alpha_r}^{\dagger}(x_r) u_{n_r, \alpha_r}(y_r)}{\pi n_r / L + g\alpha_r} \ . \tag{10}$$

5 Conclusions

As the LC gauge $A_- = 0$ does not exist on a torus we have chosen a modification due to Palumbo which preserves as many virtues and avoids as many problems of the LC gauge as

possible. The new gauge leads to a field dependent FP matrix; nevertheless it allows for an exact, non-perturbative solution of Gauss's law, a property not shared by e.g. the Coulomb gauge. What remains to be done is to solve the residual (second class) constraints and eventually obtain the finite volume LF Hamiltonian as an input for actual DLCQ calculations.

Acknowledgements

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