Feynman Diagrams and Light-Cone Time Ordered Diagrams

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Abstract

We discuss some of the relations between covariant perturbation theory and Light-Cone-time-ordered perturbation theory. We avoid the ill-defined quantization procedure by establishing direct (algebraic) equivalence between terms in different perturbative expansions. Zero-modes arise from a specific class of Feynman diagrams. $(p^+)^{-1}$ ambiguities, associated with fermion spin, can be avoided by regrouping LC time-ordered diagrams.

1. Introduction

As an initial value problem null-plane field theory is ill-defined. The standard initial values on a light-like surface overdetermine the fields, *and* give a non-unique evolution. The first problem can be solved by Dirac quantization which determines the true independent initial variables. The second problem cannot be solved without restricting the initial (or propagating) fields. These problems make canonical quantization a risky undertaking. Since covariant perturbation theory is well-established, we wish to compare it with the LC time-ordered perturbation theory before extending our considerations beyond the boundaries of perturbation theory.
It turns out that in some case we have to rely on analiticity to resolve ambiguities, and we have to keep in the back of our mind which divergent LC time-ordered diagrams are generated by the same Feynman diagrams in order to cancel (non-covariant) divergences associated with the longitudinal momentum.
Details can be found in a forecoming paper.

2. Light-Front ambiguities

In a Hamiltonian formulation we distinguish kinematical variables which characterize our fields and dynamical operators (Hamiltonians) which govern the evolution of the fields in time. If we choose a light-like direction as time direction for our Hamiltonian (LC time) this picture is blurred. The following three examples illustrate this:
• If one regularizes $(p^+)^{-1}$ or $(p^-)^{-1}$ this can be done in different ways. A regularization which maintains analiticity is useful (the physical amplitude is the boundary of an analytical function

on a strip near the real axis). If the Lorentz scalar p^2 is real, p^+ and p^- are complex conjugate variables. Therefore in the regularization of the (kinematical) variable $(p^+)^{-1}$ the dynamical variable p^- appears.

$$\frac{1}{\{p^+\}} \equiv \frac{\partial \ln[p^+p^- + i\epsilon]}{\partial p^+} = \mathrm{PV}\frac{1}{p^+} - i\pi\delta(p^+)\sigma(p^-), \tag{1}$$

with $\sigma(x) = \theta(x) - \theta(-x)$. This is the well-known Mandelstam-Leibbrandt prescription. The object is singular on the light-front, as can be seen after making a Fourier transformation.

$$\mathcal{F}[\ln[p^+p^- + i\epsilon]] = \frac{i\delta^2(x_\perp)}{x^+x^- - i\epsilon} \quad \Rightarrow \quad \mathcal{F}[\frac{1}{\{p^+\}}] = \frac{x^-\delta^2(x_\perp)}{x^+x^- - i\epsilon} \tag{2}$$

The null-plane is "glowing": besides a sign-function in x^-, the dependence on x^+ is singular.
• Loop integrals in Feynman diagrams with constant p^+-momenta along the loop give rise to zero-mode contributions. These are the only diagrams where the zero-modes appear. The problem arises if, for $p^+ = 0$, the poles in p^- cross the real axis at infinity.

$$\int \mathrm{d}p^- \frac{1}{[2p^+p^- - H_1^\perp]\cdots[2p^+p^- - H_n^\perp]} = \delta(p^+)\sum_{j=1}^{n}\frac{i\pi\ln[H_j^\perp]}{\prod_{i\neq j}[H_j^\perp - H_i^\perp]} \tag{3}$$

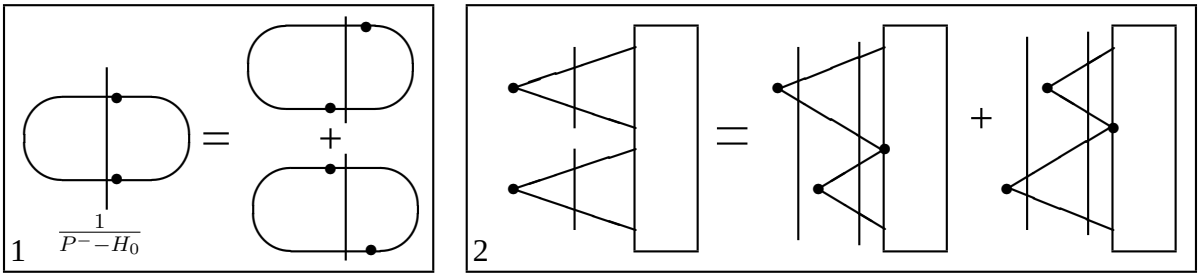
One can calculate this expression analytically for $p^+ > 0$ and $p^+ < 0$ separately and take the limit to real p^+ only after integration. Tadpole-diagrams are a special case of this type of diagrams. Tadpoles are removed from perturbation theory by normal ordering of the Hamiltonian, zero-modes are in this way related with normal ordening. A likely interpretation of these delta-like contributions is that the infinite density of the phase space $(p^+)^{-1}$ near $p^+ = 0$ competes with the infinite off-shell energy $(p_\perp^2 + m^2)/(2p^+)$. The amplitude calculated in this way is the same as for a covariant calculation using a Wick-rotation.
• In Hamiltonian perturbation theory as well as in Feynman diagrams we have for fermions the so-called instantaneous interaction: $\gamma^+/(2p^+)$. The interpretation of this term is confused, but it cannot be interpreted as a constraint although it contains kinematical variables only. If one assumes $\gamma^+/(2p^+)$ as a kinematical operator one can restrict it to the Hilbert-space of physical states. There it acts as a local operator with the strength $1/(2m)$.

$$\frac{\gamma^+}{2p^+} = \sum_s\sum_{s'}\int \mathrm{d}^3p\int \mathrm{d}^3p' |\psi_s(p)\rangle\langle\psi_s(p)|\frac{\gamma^+}{2p^+}|\psi_{s'}(p')\rangle\langle\psi_{s'}(p')| = \frac{1}{2m} \tag{4}$$

Inserting $1/(2m)$ at the places where $\gamma^+/(2p^+)$ appear in the perturbative expansion leads to amplitudes that differ from the original expressions.

3. Equivalence

A number of algebraic relations allow one to reduce any Feynman diagram to time-ordered diagrams. The successive application of these formulae reduces any Feynman diagram step by step to a number of LC time-ordered diagrams.
The application itself is governed by the topology of the generating Feynman diagram and the longitudinal p^+-momenta in the internal lines of the diagram. As the result of causality a line should always have positive p^--momentum if it goes forward in time. Positive energy is related with positive p^+-momentum of a free state. Completeness tells us that all states are superpositions of free states, thus this property holds for an arbitrary state.
The direction of each line is fixed in LC time. Only the relatively unconnected vertices: space-like separated and not connected by a line with has a specific time direction, have to be ordered with respect to each other. This gives rise to different LC time-ordered diagrams, although less in number than the time-ordered diagrams obtained in "old-fashioned perturbation theory".
The idea that governs the application of the reduction algorithm is an equal-LC-time surface that crosses a Feynman diagram. Vertices in the Feynman diagram cross this surface at certain times, we call the crossing an event. Each event terminates an intermediate state, so we extract the energy denominator of this state, from the Feynman diagram. Events can occur in different orders, so each of the possible orders appear at each step of the reduction algorithm, and we end up with a sum of terms, each representing a specific ordering. The reduction gives us a sum of products of energy denominators.
If a loop crosses the equal time surface it does so an even number of times; if it goes into the surface is must come out as well. Now the algorithm has two steps:



• 1 A vertex, on each of the lines going into and out of the surface, can cross the surface, giving rise to one energy denominator and two different time-orderings.
• 2 Two sets of lines going into and out of of the surface might connect and become just one set of lines in and out of the surface. Then the two reduced parts of LC time-ordered diagrams contribute to the sum of all relative orderings of both the LC time-ordered parts.

The explicit formulae are lengthy, so we lack space here to write them down. But we can summarize them in the following way:

1. Energy integration; for each physical sector, depending on p^+, we get a determinantal form for the Feynman diagram:

$$\int \mathrm{d}p^-_{loop} FD = \sum_{\text{physical sectors}} W(H_1, \cdots, H_n) \tag{5}$$

where the H's are the one particle Hamiltonians.

2. Basic step; (step 1 in the figure).

$$W(H_1, H_2, H_3, \cdots, H_n) = \frac{1}{P^- - H_0}(W(H_2, H_3, \cdots, H_n) + W(H_1, H_3, \cdots, H_n)) \quad (6)$$

Either the event that ends line 1, with on-shell energy H_1, happened or the one that ended the line 2. It yields the energy denominator of the intermediate state of H_1 and H_2.

3. Simultaneous combinations; the product of two ordered parts is the sum of all relative orderings (step 2 in the figure):

$$\left(V\frac{1}{P^- - H_1}V\cdots V\frac{1}{P^- - H_n}V\right)\left(V\frac{1}{P^- - H'_1}V\cdots V\frac{1}{P^- - H'_m}V\right) = \quad (7)$$

$$\sum_{\text{all relative orderings}} V\frac{1}{P^- - H''_1}V\frac{1}{P^- - H''_2}V\cdots V\frac{1}{P^- - H''_{n+m}}V$$

The H'''s are the composite Hamiltonians containing the H's and the H''s.

4. Multiple loops are reduced loop by loop, the lines are now composite lines. The Hamiltonians in de W-functions are the sum of one particle Hamiltonians.

4. Fermions

In the case of fermions the Feynman diagram has energy dependence in the numerator of the integral. Before doing the energy integration this dependence must be removed, since the numerator must contain only operators on the Hilbert space, and thus can only depend on kinematical variables. Technically we are making a Laurent expansion, in the energy, of the propagator:

$$\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} = \frac{1}{2p^+}\frac{u(\vec{p}) \otimes \bar{u}(\vec{p})}{P^- - H_0(\vec{p})} + \frac{\gamma^+}{2p^+} \quad (8)$$

H_0 is the on-shell energy: $(p_\perp^2 + m^2)/(2p^+)$. The second term must be included in the interaction (instantaneous). This leads to two problems: the energy integration is divergent for many Feynman diagrams, secondly; there seems to appear additional singularities of the form: $\frac{1}{p^+}$. We can deal with both in a consistent way:

• 1 We can make the contour integration symmetric with respect to the origin, then one can subtract the contributions from the semicircles in the contour integration, this leads to the following regularization of divergent integrals:

$$\int \mathrm{d}p^- = 0 \quad \wedge \quad \int \mathrm{d}p^- \frac{1}{p^- - \alpha} = \pi i \sigma(\alpha) \quad (9)$$

The regularization preserves the linearity of the integral and is independent of the shift in integration variable (which is to be expected since a "pole" at infinity is subtracted).
This regularization is consistent with the causal formulation.
• 2 Among the LC time-ordered diagrams there are a number of p^+-divergent ones. They contain "one-state-particles", these are intermediate states which start at the creation of a fermion which is annihilated at the end of the same intermediate state. The creation point can approach the annihilation point, this gives rise to a divergence. But this divergence is cancelled against the same LC time ordered diagram but with an instantaneous interaction instead of this state. The only remaining ambiguities are related to the creation or annihilation of a fermion pair where both lines are instantaneous interactions. But the associated $\frac{1}{p^+}$ singularity lies always inside a specific physical domain, and does not lead to a divergence. (Principal value regularization appears natural since $\frac{1}{p^+}$ can be interpreted as phase space, and an $i\delta(p^+)$ term would not make sense in that respect.)

It is hard to check whether these methods for fermions lead to the same answers as the covariant calculation since both contain at low order overall divergences.
A consistent method should also deal with divergent diagrams.

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