Gauge Invariant Variables for SU(2) Yang-Mills Theory

Cécile Martin
Division de Physique Théorique, Institut de Physique Nucléaire
F-91406, Orsay Cedex, France.

Abstract

We describe a nonperturbative calculation of the spectrum of SU(2) Yang-Mills theory based on a Hamiltonian formulation. Our approach exploits gauge invariant variables similar to those used in nuclear physics to describe collective motion in nuclei.

1. Introduction

Our understanding of the low energy behaviour of QCD depends crucially on the development of nonperturbative methods. The variational approach in the Schrödinger picture has been successfull in scalar field theory. However, it remains difficult to apply for a nonabelian gauge theory because of the requirement to satisfy the Gauss law constraint. The necessity to maintain gauge invariance is an essential ingredient in practical calculations. In order to study the configurations which contribute significantly to the nonperturbative ground state, we have to take into account the gauge invariant functional measure [1,2]. The nontrivial gauge invariant volume element will induce a centrifugal effect and the boundary conditions on the wave functional will differ drastically from those in perturbative calculations .

In this presentation, I will investigate the properties of the ground state in SU(2) gauge theory and give predictions for the lowest glueball states [3]. The polar representation for the SU(2) vector potential [4] allows to separate explicitly the gauge degrees of freedom. More appropriate gauge invariant variables $\rho(x), \beta(x), \gamma(x)$ are introduced. These variables describe the field configuration in an intrinsic frame and they can be interpreted as "density" and "deformation" variables. They are analogous to the collective variables introduced by Bohr and Mottelson to described the dynamics of deformed nuclei [5]. In these gauge invariant variables, the hamiltonian becomes non local. However, a derivative expansion (or strong coupling expansion) valid in the nonperturbative domain and for slowly-varying fields allows one to write expliciltly the first few terms of an effective hamiltonian. One can thus obtain approximate solutions for the dynamics which are gauge invariant.

2. The polar representation

In the hamiltonian formalism, we choose the temporal gauge $A_0^a = 0$. The hamiltonian reads :

$$H = \frac{1}{2} \int d^3x \, tr \left(E^2 + B^2 \right) . \tag{1}$$

For the SU(2) color group, the vector potential A_{ia} (where i is a space index and a is a color index) is a 3×3 matrix. The polar representation is given by [4] :

$$A_{ia} = f_{in} \lambda_n h_{na} - \frac{1}{2g} h_{kb} \partial_i h_{kc} \epsilon_{abc} , \tag{2}$$

where $\lambda_n, n = 1, 2, 3$ are three numbers, $\hat{f}(\theta_i)$ and $\hat{h}(\phi_a)$ are orthogonal matrices parametrized by two sets of three Euler angles θ_i and ϕ_a and g is the bare coupling constant. The matrix $\hat{f}$ describes a rotation in ordinary space and the matrix $\hat{h}$ a rotation in color space. In general, λ_n, θ_i and ϕ_a are space-dependent. Under a local gauge transformation, the matrix $\hat{h}$ is simply rotated while $\hat{f}$ and λ_n remain unchanged. Therefore, among the nine variables A_{ia}, six gauge invariant variables λ_n and θ_i are explicitly separated from the three gauge degrees of freedom ϕ_a. The functional integration measure becomes

$$\prod_{i,a} DA_i^a(x) = \prod_{n>m} \left| \lambda_n^2(x) - \lambda_m^2(x) \right| \prod_p D\lambda_p(x) \, d\hat{f} \, d\hat{h} . \tag{3}$$

In terms of the new variables $\lambda_n, \theta_i, \phi_a$, the Gauss law operator becomes a local operator. The wave functional for a color singlet state depends only on the six gauge invariant variables : $\Psi(\lambda_n(x), \theta_i(x))$.

In terms of the new gauge invariant variables, the hamiltonian becomes non local. However, we can use a derivative expansion to write explicitly the first few terms. In the following, we will consider only the lowest order in $1/g^2$, which corresponds also to the constant field approximation.

3. The gauge invariant collective coordinates $\rho(x), \beta(x), \gamma(x)$

We introduce now three gauge invariant variables ρ, β, γ defined as :

$$\rho^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 , \tag{4}$$

$$\lambda_2^2 - \lambda_1^2 = \frac{2}{\sqrt{3}} \rho^2 \beta \sin\gamma , \tag{5}$$

$$2\lambda_3^2 - \left(\lambda_1^2 + \lambda_2^2 \right) = 2\rho^2 \beta \cos\gamma , \tag{6}$$

where $0 < \beta < 1$ and, from symmetry properties, we can restrict the angle γ between 0 and $\pi/3$. The range of variation of β and γ is also restricted to $\cos\left(\gamma + \frac{2\pi}{3}\right) \geq -\frac{1}{2\beta}$ to make the three λ_n^2 positive. The variable ρ has the dimension of $1/L$.

The vibrational part of the wave function (i. e. the part of the wave function which is independent of the three Euler angles θ_i) is a function of the three gauge invariant collective coordinates ρ, β, γ : $\Psi(\rho, \beta, \gamma)$. In the β, γ plane, the axis $\beta = 0$ corresponds to a "spherical" field configuration : $\lambda_1 = \lambda_2 = \lambda_3$. The axis $\gamma = 0$ corresponds to an "axial symmetric" field configuration. An arbitrary point in the β, γ plane corresponds to a "triaxial" field configuration. We will show that ρ, β and γ are convenient coordinates to perform practical calculations. Furthermore, they give a physical insight in the structure of the vacuum state and the lowest excited states.

The expression for the volume element of the vibrational coordinates is given by :

$$d\tau = \frac{1}{9}\,\rho^8\,\beta^4 \frac{|\sin 3\gamma|}{\left(\frac{1}{3} - \beta^2 + \frac{2}{3}\beta^3 \cos 3\gamma\right)^{1/2}}\, d\rho\, d\beta\, d\gamma\ . \tag{7}$$

2. Strong coupling expansion for the 0^+ state

In the constant field approximation (or in the strong coupling approximation), we can write explicitly the SU(2) hamiltonian in terms of the gauge invariant variables ρ, β, γ and the Euler angles θ_i :

$$H = T_{vib} + V + T_{rot}\ . \tag{8}$$

The vibrational energy T_{vib} is given by

$$T_{vib} = -\frac{1}{2L^3}\left\{\frac{1}{\rho^8}\,\frac{\partial}{\partial\rho}\left(\rho^8 \frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\,F(\beta,\gamma)\right\}\ , \tag{9}$$

where $F(\beta, \gamma)$ is a differential operator in the β, γ variables. We have introduced a length L, the total volume being L^3. T_{rot} is the rotational kinetic energy. The potential energy $B^2/2$ is equal to :

$$V = L^3\,\frac{g^2}{6}\rho^4\left(1 - \beta^2\right)\ . \tag{10}$$

In order to work out approximate solutions of the collective Schrödinger equation, we will perform an average with respect to the deformation variables β, γ, taking into account the nontrivial gauge invariant volume element (7). This corresponds to the assumption that the β and γ vibrations are stiffer than the ρ vibration. We will subsequently use a semiclassical approximation to describe the ρ vibration : we look for the minimum of the $\rho-$dependent effective potential and calculate small vibrations around this point.

In the following we will consider only the zero angular momentum states of positive parity. In reference [3], we describe also the lowest 0^- and 2^+ states.

For the 0^+ states, we introduce the following transformation : $\tilde{\Psi}(\rho, \beta, \gamma) = \rho^4\,\Psi(\rho, \beta, \gamma)$. The advantage of this transformation is that the wave function $\tilde{\Psi}$ is now normalized according

to a measure flat in the ρ direction. The new kinetic energy in the ρ variable is equal to $\tilde{T}_\rho = -\frac{1}{2L^3}\frac{\partial^2}{\partial\rho^2}$ which makes it adequate to perform a semiclassical calculation of the ground state energy. The average over the variables β, γ with respect to the measure (**??**) gives $\langle 1-\beta^2\rangle_{0^+} = \frac{6}{11}$ and $\frac{<\beta^3\cos 3\gamma>}{<\beta^3>} \ll 1$, which indicates a large asymmetry. We end up with a $\rho-$dependent effective potential :

$$\langle V_{0^+}\rangle = g^2L^3\rho^4\frac{1}{11} + \frac{6}{L^3\rho^2}\ . \tag{11}$$

According to our semiclassical strategy we first look for the minimum of $\langle V_{0^+}\rangle$. It corresponds to

$$L^2\bar{\rho}^2_{0+} = g^{-2/3}\ (33)^{1/3} = g^{-2/3}\ 3.2\ , \tag{12}$$

which leads to a magnetic energy $L^4B^2 = g^{2/3}\ 1.87$. In our approach, the vacuum is thus described as a strongly deformed minimum with $\beta \simeq 1/\sqrt{2}$ and a nonvanishing value of $\bar{\rho}^2$ or B^2. Its wave function is of the form $\rho^{-4}\ \exp(-L^3\,\omega_\rho(\rho-\bar{\rho})^2)$. This minimum is not optimally described by a perturbative approach which starts from a state located arround $B^2 = 0$.

The frequency of the ρ vibration around the minimum (12) is equal to : $\omega_\rho = g^{2/3}\ 2.64$. The 0^+ sector is described in this harmonic approximation as ρ-vibrational states. In the strong coupling approximation, the energies are proportional to $g^{2/3}$. For the ground state energy, we obtain the value :

$$E_0 = \frac{g^{2/3}}{L}\ 4.13\ . \tag{13}$$

For the lowest 0^+ excited state, we find : $E(0^+) = 6.77$ (all the masses are in unit of $g^{2/3}/L$). The difference between the ground state energy and the first excited 0^+ energy is the mass of the lightest glueball : $M(0^+) = 2.64$.

2. Discussion and Perspectives

The great advantage of our approach is that it provides simple and transparent gauge invariant wave functions describing the ground state and the lowest excited states in SU(2) Yang Mills theory. The gauge invariant variables we have introduced allow an attractive interpretation of the spectrum in terms of vibrational and rotational states.

The results $\bar{\rho}^2 \neq 0$ and $B^2 \neq 0$ at the minimum suggest that a perturbative approach arround $\bar{\rho} = 0$ is not optimal. We have found a strongly deformed ground state with an energy (in units of $g^{2/3}/L$) : $E_0 = 4.13$. For the lowest 0^+ state, our result is (in units of $g^{2/3}/L$) : $E(0^+) = 6.77$. For the lowest 0^- and 2^+ states, we obtain [3] : $E(0^-) = 9.52, E(2^+) = 8.13$, which leads to the following dimensionless ratios : $M(0^-)/M(0^+) = 2.04$ and $M(2^+)/M(0^+) = 1.51$.

These results are to be compared to those of the authors of [6,7,8] who have performed a perturbative calculation for the $SU(2)$ gauge theory in a finite volume. Our Hamiltonian in the constant field limit agrees with Lüsher effective hamiltonian in lowest order [6]. However, whereas the authors of references [6,7,8] perform at this stage an exact diagonalization using a

large polynomial basis adapted to describe the potential energy, we include in contrast the dynamical effects arising from the coordinate dependance of the kinetic energy. This dependance is a consequence of the fact that our method takes into account the nontrivial gauge invariant measure. This is a nonperturbative effect. The results of [7] for the ground state energy and the energies of the lowest 0^+,0^- and 2^+ excited states are the following (in unit of $g^{2/3}/L$) : $E_0 = 4.116, E(0^+) = 6.386, E(0^-) = 8.786, E(2^+) = 6.01$, which leads to the following ratios : $M(0^-)/M(0^+) = 2.057$ and $M(2^+)/M(0^+) = 0.834$. We therefore obtain a excellent agreement for the ground state energy. Our results are also very good for the lowest 0^+ and 0^- excited states. For the 2^+ state, our result should be improved by calculating more accurately the average over the $\beta - \gamma$ part of our wave function.

Let us finish with some remarks about the derivative expansion. This is an expansion in powers of $1/g^2$. It is expected to be valid when the derivatives of gauge invariant quantities are small compared to some scale, which is set by B^2, for instance $\left(\vec{\nabla}\rho\right)^2/B^2 \ll 1$. In lowest order in the derivative expansion, the wavefunctional reduces to a function of gauge invariant quantities constant in space. The next step is to investigate the coupling due to gradient terms and its effect on the vacuum state properties and the glueball spectrum.

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