The Vacuum in the Light-Cone Representation

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Abstract

The mechanism by which the physical vacuum can be different from the perturbative vacuum in the light-cone representation is described and illustrated.

In this talk I shall review the mechanism by which the physical vacuum in an interacting theory becomes a state other than the light-cone perturbative vacuum. Vacuum structure in the light-cone representation is always associated with zero modes but there are two distinctly different cases. In one case the vacuum remains the physical vacuum but some field gains a constrained zero mode due to the interaction and that zero mode generates a nonzero, and possibly symmetry breaking, V.E.V. for an operator which does not have one in free theory. Such effects have been discussed at these meetings before, especially by Robertson[1], Werner et al.[2] and Pinsky et al.[3]; I shall not discuss that type of vacuum structure in this talk. I shall discuss the case where the interacting vacuum is a different state than the perturbative vacuum. That effect must occur for theories with degenerate vacua, such as the Schwinger model, and requires the presence of unconstrained zero modes.

I shall first review the argument that the physical vacuum is the perturbative vacuum in the light-cone representation even for interacting theories, and review the mechanism by which this argument can fail. I shall then go quickly through the examples of free theory and the Schwinger model. I have spoken on these cases before and the details have been published[4,5]. I shall then apply the same methods to the case of massless QCD_2. Finally I shall speculate briefly on QCD_4.

The argument that the physical vacuum is the perturbative vacuum in the light-cone representation is as follows: the operator, P^+ has the same form in an interacting theory as it does in free theory, that is, $P^+ = P^+_{FREE}$; the physical vacuum must be an eigenstate of P^+ with eigenvalue 0; for theories which can be specified with quantization conditions on $x^+ = 0$ the only such state is the perturbative vacuum.

I want to give two arguments that $P^+ = P^+_{FREE}$ since they relate to later things. The first is simply to calculate the answer. We integrate:

$$P^+ = \frac{1}{2} \int T^{++} dx^-$$

where,

$$T^{++} = \sum_\phi \frac{\partial \phi}{\partial x_+} \frac{\partial \mathcal{L}}{\partial (\partial_+ \phi)} - g^{++} \mathcal{L}$$

Since g^{++} is zero this expression does not depend on the interaction for nonderivative coupling. I shall refer to this argument as the algebraic argument.

Another argument that $P^+ = P^+_{FREE}$, in some ways more instructive for our later work, makes use of the fact that P^+ is the generator of translations within our initial value surface, $x^+ = 0$. That is:

$$\partial_- \phi = \frac{i}{2}[P^+, \phi]$$

Since we initialize our fields to be isomorphic to free fields on the initial value surface, P^+_{FREE} will correctly generate these translations for all fields initialized on $x^+ = 0$. If we have a well posed initial value problem, and thus a complete set of fields, the only operator we can mix with P^+_{FREE} is a multiple of the identity which would have no effect on the dynamics.

The flaw in this argument is that in the presence of massless fields one cannot formulate a proper initial value problem with initial values on $x^+ = 0$. One must also specify certain zero modes—functions of x^+. Since these are true degrees of freedom they commute with the fields specified on $x^+ = 0$ and thus can mix with P^+ without contradicting the Heisenberg equations. One might ask about the algebraic argument; I shall return to that question presently.

To be definite let us consider the case of a massless Fermi field in $1 + 1$ dimensions. We can initialize the field ψ_+ on $x^+ = 0$:

$$\psi_+(0, x^-) = \frac{1}{\sqrt{2L}} \sum_{n=1}^\infty b(n)e^{-ik_-(n)x^-} + d^*(n)e^{ik_-(n)x^-}$$

The field ψ_- cannot be initialized on $x^+ = 0$ and thus furnishes the zero modes discussed above:

$$\psi_-(x^+, 0) = \frac{1}{\sqrt{2L}} \sum_{n=1}^\infty \beta(n)e^{-ik_+(n)x^+} + \delta^*(n)e^{ik_+(n)x^+}$$

Here we see that any functional of ψ_- could mix with P^+ and there would be no contradiction with the Heisenberg equation, that is, if:

$$P^+ = P^+_{FREE} + \mathcal{F}(\psi_-)$$

then still:

$$\partial_- \psi_+ = \frac{i}{2}[P^+, \psi_+]$$

While such mixing would not contradict this Heisenberg equation it would contradict the full dynamics in free theory so in free theory the ψ_- modes do not mix with P^+.

In the Schwinger model they do. We shall work in the gauge $\partial_- A^+ = 0$ and find that the equations of motion are:

$$\frac{\partial^2 A^-}{\partial x^{-2}} = -\frac{1}{2} J'^+$$

$$-\frac{\partial^2 A^+}{\partial x^{+2}} + \frac{\partial^2 A^-}{\partial x^+ \partial x^-} = \frac{1}{2} J'^-$$

The prime on the J's reflects the need to subtract an overall zero mode from J^0 before coupling the current to the Maxwell field. We define gauge invariant products of Fermi fields as:

$$:\psi_+^*(x)\psi_+(x): \equiv \lim_{\epsilon^- \to 0}\left(e^{-ie\int_x^{x+\epsilon^-} A_-^{(-)} dx^-} \psi_+^*(x+\epsilon^-)\psi_+(x) e^{-ie\int_x^{x+\epsilon^-} A_-^{(+)} dx^-} - \text{V.E.V.}\right)$$

$$:\psi_-^*(x)\psi_-(x): \equiv \lim_{\epsilon^+ \to 0}\left(e^{-ie\int_x^{x+\epsilon^+} A_+^{(-)} dx^+} \psi_-^*(x+\epsilon^+)\psi_-(x) e^{-ie\int_x^{x+\epsilon^+} A_+^{(+)} dx^+} - \text{V.E.V.}\right)$$

From which we calculate the zero modes of the currents to be:

$$J^{+\prime}(0) = \frac{1}{2L}Q_+ - \frac{e^2}{2\pi}A^+ - \frac{1}{2}\frac{1}{2L}Q_+ + \frac{1}{2}\frac{e^2}{2\pi}A^+ - \frac{1}{2L}Q_- + \frac{1}{2}\frac{e^2}{2\pi}A^-(0)$$

$$J^{-\prime}(0) = \frac{1}{2L}Q_- - \frac{e^2}{2\pi}A^-(0) - \frac{1}{2}\frac{1}{2L}Q_- + \frac{1}{2}\frac{e^2}{2\pi}A^-(0) - \frac{1}{2L}Q_+ + \frac{1}{2}\frac{e^2}{2\pi}A^+ = -J^{+\prime}(0)$$

Which in turn allows us to solve for the zero modes of the gauge fields:

$$A^+ = -\frac{1}{Lm^2}Q_-$$

$$A^-(0) = -\frac{1}{Lm^2}Q_+$$

To find P^+ we must integrate the density:

$$T^{++} = 2i\lim_{\epsilon^- \to 0}\left(e^{-ie\int_x^{x+\epsilon^-} A_-^{(-)} dx^-} \psi_+^*(x+\epsilon^-)\partial_-\psi_+(x) e^{-ie\int_x^{x+\epsilon^-} A_-^{(+)} dx^-} - \text{C.C.} - \text{V.E.V.}\right)$$

It is here that we see that the algebraic argument that P^+ is trivial has failed. The effect is precisely like an anomaly: we have a singular operator product and we find that not all properties of the classical product can be maintained in the quantum theory; here we must give up gauge invariance or the purely kinematical nature of P^+. For P^+ we get:

$$P^+ = \frac{1}{2}\int_{-L}^{L} :2i\left(\psi_+^*\partial_-\psi_+ - \partial_-\psi_+^*\psi_+\right): dx^- = P_{FREE}^+ - \frac{1}{4Lm^2}Q_-^2$$

if we define a set of special states, $|M, N\rangle$, as:

$$
\begin{aligned}
|M, N\rangle &= \delta^*\big(M\big)\dots\delta^*\big(1\big)d^*\big(N\big)\dots d^*\big(1\big)|0\rangle &\quad (M > 0, N > 0)\\
|M, N\rangle &= \beta^*\big(M\big)\dots\beta^*\big(1\big)d^*\big(N\big)\dots d^*\big(1\big)|0\rangle &\quad (M < 0, N > 0)\\
|M, N\rangle &= \delta^*\big(M\big)\dots\delta^*\big(1\big)b^*\big(N\big)\dots b^*\big(1\big)|0\rangle &\quad (M > 0, N < 0)\\
|M, N\rangle &= \beta^*\big(M\big)\dots\beta^*\big(1\big)b^*\big(N\big)\dots b^*\big(1\big)|0\rangle &\quad (M < 0, N < 0)
\end{aligned}
$$

we find that:

$$
P^+|M, N\rangle = 0
$$

For $M = -N$ these states are in the physical subspace and form the degenerate ground states of the Schwinger model. To form a θ-state we take:

$$
|\theta\rangle = \sum e^{iM\theta}|M, -M\rangle
$$

The point is not just that degenerate ground states can be accommodated within the light-cone representation but that there is a limited number of ways that that can occur. That fact leads, in the case of the Schwinger model, to the fact that the ground states are much simpler in the light-cone representation than they are in the equal-time representation—simpler to express and simpler to find. It also suggests that the light-cone representation may be useful in the analysis of vacuum structure for more complicated theories; the number of operators which can mix with P^+ grows rather slowly with the dimension of space-time and the vacuum structure is controlled substantially by P^+.

Let us now apply these same considerations to QCD_2 with color group $SU(2)$ and quarks in the fundamental representation. We initialize the fields as before:

$$
\psi^i_+(0, x^-) = \frac{1}{\sqrt{2L}}\sum_{n=1}^{\infty} b^i(n)e^{-ik_-(n)x^-} + d^{i*}(n)e^{ik_-(n)x^-}
$$

$$
\psi^i_-(x^+, 0) = \frac{1}{\sqrt{2L}}\sum_{n=1}^{\infty} \beta^i(n)e^{-ik_+(n)x^+} + \delta^{i*}(n)e^{ik_+(n)x^+}
$$

where the i's are color indices. Gauge invariant currents are given by, for instance:

$$
{:}\psi^*_+(x)T^a\psi_+(x){:} \equiv \lim_{\epsilon^- \to 0}\left(e^{-ie\int_x^{x+\epsilon^-} A_-^{b(-)}T^b_{jk}dx^-}\psi^{i*}_+(x + \epsilon^-)T^a_{ij}\psi^k_+(x)e^{-ie\int_x^{x+\epsilon^-} A_-^{c(+)}T^c_{jk}dx^-} - \text{V.E.V.}\right)
$$

We find for the currents:

$$
J^{a+}(0, x^-) = \frac{g}{2L}\sum_{n=1}^{\infty}\left(C^{a*}(n)e^{ik_-(n)x^-} - C^a(n)e^{-ik_-(n)x^-}\right) + \frac{g}{2L}Q^a_+ - \frac{g^2}{4\pi}A^{a+}
$$

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$$J^{a-}(x^+,0) = \frac{g}{2L}\sum_{n=1}^{\infty}\left(D^{a*}(n)e^{ik_+(n)x^+} - D^a(n)e^{-ik_+(n)x^+}\right) + \frac{g}{2L}Q^a_- - \frac{g^2}{4\pi}A^{a-}$$

where the C's and D's are the fusion operators and the Q's are the charges. A fully symmetrized equation of motion for the gauge field is:

$$
\begin{aligned}
-\partial_-^2 A^{1-} \quad &+\tfrac{g}{2}\Big(-A^{2+}\partial_- A^{3-} - \partial_- A^{3-}A^{2+} + A^{3+}\partial_- A^{2-} + \partial_- A^{2-}A^{3+} \\
&+\tfrac{1}{2}A^{2+}\partial_+ A^{3+} + \tfrac{1}{2}\partial_+ A^{3+}A^{2+} - \tfrac{1}{2}A^{3+}\partial_+ A^{2+} - \tfrac{1}{2}\partial_+ A^{2+}A^{3+}\Big) \\
&+\tfrac{g^2}{16}\Big(A^{1-}A^{2+}A^{2+} + A^{2+}A^{2+}A^{1-} + 2A^{2+}A^{1-}A^{2+} \\
&-A^{2-}A^{1+}A^{2+} - A^{2+}A^{1+}A^{2-} - A^{1+}A^{2-}A^{2+} - A^{2+}A^{2-}A^{1+} \\
&-A^{3-}A^{1+}A^{3+} - A^{3+}A^{1+}A^{3-} - A^{1+}A^{3-}A^{3+} - A^{3+}A^{3-}A^{1+} \\
&+A^{1-}A^{3+}A^{3+} + A^{3+}A^{3+}A^{1-} + 2A^{3+}A^{1-}A^{3+}\Big) = \tfrac{1}{2}J'^{1+}
\end{aligned}
$$

and similarly for the other Maxwell equation. The zero modes of these equations are:

$$
\begin{aligned}
\tfrac{g}{4}(A^{2+}\partial_+ A^{3+} &+ \partial_+ A^{3+}A^{2+} - A^{3+}\partial_+ A^{2+} - \partial_+ A^{2+}A^{3+}) \\
&+\tfrac{g^2}{16}(A^{1-}A^{2+}A^{2+} + A^{2+}A^{2+}A^{1-} + 2A^{2+}A^{1-}A^{2+} \\
&-A^{2-}A^{1+}A^{2+} - A^{2+}A^{1+}A^{2-} - A^{1+}A^{2-}A^{2+} - A^{2+}A^{2-}A^{1+} \\
&-A^{3-}A^{1+}A^{3+} - A^{3+}A^{1+}A^{3-} - A^{1+}A^{3-}A^{3+} - A^{3+}A^{3-}A^{1+} \\
&+A^{1-}A^{3+}A^{3+} + A^{3+}A^{3+}A^{1-} + 2A^{3+}A^{1-}A^{3+}) \\
&= -\tfrac{1}{2}\left(-\tfrac{g}{2L}Q^a_- - \tfrac{g^2}{4\pi}A^{a+}\right)
\end{aligned}
$$

$$
\begin{aligned}
-\partial_+^2 A^{1+} &+\tfrac{g}{2}\Big(-A^{2-}\partial_+ A^{3+} - \partial_+ A^{3+}A^{2-} + A^{3-}\partial_+ A^{2+} + \partial_+ A^{2+}A^{3-}\Big) \\
&+\tfrac{g^2}{16}\Big(A^{1+}A^{2-}A^{2-} + A^{2-}A^{2-}A^{1+} + 2A^{2-}A^{1+}A^{2-} \\
&-A^{2+}A^{1-}A^{2-} - A^{2-}A^{1-}A^{2+} - A^{1-}A^{2+}A^{2-} - A^{2-}A^{2+}A^{1-} \\
&-A^{3+}A^{1-}A^{3-} - A^{3-}A^{1-}A^{3+} - A^{1-}A^{3+}A^{3-} - A^{3-}A^{3+}A^{1-} \\
&+A^{1+}A^{3-}A^{3-} + A^{3-}A^{3-}A^{1+} + 2A^{3-}A^{1+}A^{3-}\Big) = -\tfrac{1}{2}\left(-\tfrac{g}{2L}Q^a_+ - \tfrac{g^2}{4\pi}A^{a-}\right)
\end{aligned}
$$

I do not think these equations can be implemented at the operator level. A construction which works is as follows: Set

$$A^{a+} = -\frac{2\pi}{Lg}Q^a_- \quad ; \quad A^{a-}(0) = -\frac{2\pi}{Lg}Q^a_+$$

$$P^+ = P^+_{FREE} - \frac{\pi}{2L}(Q^a_- Q^a_-)$$

$$P^- = P^-_{FREE} - \frac{\pi}{2L}(Q^a_+ Q^a_+) + \frac{g^2}{2}\sum\frac{1}{k_-^2(n)}C^{a*}(n)C^a(n)$$

Define the physical subspace by:

$$D^a(n)|\mathcal{P}\rangle = 0 \qquad a = 1, 2, 3$$

$$(Q^a_- Q^a_-)|\mathcal{P}\rangle = (Q^a_+ Q^a_+)|\mathcal{P}\rangle = 0$$

The unexpected thing is that these last two restrictions hold separately. It is that fact that holds the ground state of the system in the perturbative vacuum. If the restriction were the naively expected:

$$(Q^a_- Q^a_- + Q^a_+ Q^a_+)|\mathcal{P}\rangle = 0$$

the perturbative vacuum would decay into some combination of $|1\rangle = (d^{1*}(1)\beta^{1*}(1) + d^{2*}(1)\beta^{2*}(1))|0\rangle$ and $|2\rangle = (\delta^{1*}(1)b^{1*}(1) + \delta^{2*}(1)b^{2*}(1))|0\rangle$. The physical effect is that long range interactions (the zero modes) stabilize the the perturbative light-cone vacuum and prevent the occurrence of degenerate ground states. Eric Zhitnitsky has told me that Andi Smilga has reached a similar conclusion on the basis of lattice calculations.

Issues in $1 + 1$ dimensions which would be interesting to examine include the problem of adjoint matter, which semiclassical arguments suggest should be different from fundamental matter, and whether or not twisted boundary conditions make a difference[6]. In four dimensions one expects the operators which can mix with P^+ to be formed from sixteen gluon and six quark fields, in each case a function of the single variable x^+. The problem is to calculate the mixing. As is seen from the discussion above, operator mixing induced by renormalization plays a central role in the analysis. In higher dimensions that problem is more important and more difficult. Indeed that problem is central not only to the zero mode problem but the whole field of light-cone techniques—as the Ohio State group keeps reminding us. While the problem is difficult and I do not know how to solve it yet, I believe the light-cone representation may prove to be a valuable tool in the analysis of vacuum structure.

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References

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