Similarity Renormalization Scheme for Hamiltonians

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Abstract

Elements of the similarity renormalization scheme are illustrated by a second order calculation of the effective light-front QCD Hamiltonian matrix elements between one-gluon states.

1. Introduction

Our discussions indicate a need for a presentation of the similarity renormalization scheme for Hamiltonians. I have chosen an example of an application to gluons in light-front QCD. I describe all steps from a Lagrangian to the gluon mass counterterm in the Hamiltonian. The similarity renormalization scheme is defined in Ref. 1.

2. Formulation of the problem

Our construction of a light-front quantum Hamiltonian for quarks and gluons draws on the classical Lagrangian of chromodynamics. The Lagrangian depends on the space-time coordinates through the fields ψ and A only. Therefore, it defines a conserved energy-momentum tensor, $T^{\mu\nu}$. The T^{+-} component defines a light-front energy density that produces a candidate for the light-front Hamiltonian by integration over the light front space-time hyperplane. Using equations of motion for the fields and partial integrations (neglecting terms at infinity) one can blindly express the Hamiltonian in terms of fields ψ_+ and $A^\perp$ and one or two inverse powers of the operator $i\partial^+$ acting on some products of these fields. The fields are then expanded into creation and annihilation operators to obtain the quantum theory. For example, for $x^+ = 0$,

$$A^\perp(x) = \sum_{\sigma c} \int [dk] \ \exp\left[-E_\nu(k)/\Lambda\right] \ \left[a_{k\sigma c}\, \varepsilon^\perp_\sigma\, T^c\, e^{-ikx} + a^\dagger_{k\sigma c}\, \varepsilon^{\perp *}_\sigma\, T^c\, e^{ikx}\right].$$

$[dk] = dk^+ d^2k^\perp (2k^+)^{-1}(2\pi)^{-3}$. The Boltzmann factor cuts off large energies, $2E_\nu = k^+ + k^-(\nu)$, $k^-(\nu) = (k^{\perp 2}+\nu^2)/k^+$, when they are much larger than the cutoff parameter Λ. A similar expansion is written for ψ_+ with, possibly, a different parameter ν. For inverse powers of $i\partial^+$, the rule is $(i\partial^+)^{-n} \to (i\partial^+)^{-n} \exp\left\{-\left[|i\partial^+| + (-\partial^{\perp 2}+\nu^2)/|i\partial^+|\right]/2\Lambda\right\}$ for seagulls associated with gluon fields, with a similar rule for seagulls associated with fermions. The parameters ν may be equal to bare masses in the free part of the Hamiltonian, they may depend on Λ but, generally, they need to be different from zero in order to cut off divergences for small k^+. The resulting Hamiltonian is finite. We normal order it and we throw away all terms resulting from commuting creation and annihilation

operators until only normal-ordered terms are left. This procedure may look naive. But, if we apply it to electrodynamics and then calculate some tree diagrams for scattering processes using the Hamiltonian, we reproduce Feynman results in the limit $\Lambda \to \infty$. Similar things happen in QCD. This is a wonderful Hamiltonian, $H_{\Lambda QCD}$. The problem is that when we try to solve for its eigenvalues using various methods, any method you want, our answers depend on Λ and we need to think about it. In perturbation theory, the problem appears when we calculate a loop. In QCD, we need a strategy to define the Hamiltonian in a way that will allow for solving the relativistic hadronic bound state problem, not only order by order loop diagram problems.

The key simplifying property of the Boltzmann regulating factor that we will use in the example is that $\exp(-E_1/\Lambda) \cdot \exp(-E_2/\Lambda) = \exp[-(E_1 + E_2)/\Lambda]$, and the sum of free light-front energies of two particles can be written in terms of the free invariant mass of both particles together and their total $+$ and $\perp$ momenta. Therefore, the Boltzmann regulating factor in the field expansions will imply the invariant mass regularization in second order corrections to the gluon matrix elements in the effective Hamiltonian. We will not need to consider more than two particles in the example. It is important that exp is an analytic function, for simplicity of the counterterms.

3. Similarity

We introduce a giant cutoff $\Omega \gg \Lambda$ and we totally discard all Fock states that have total free energies larger than Ω. This introduces no error for finite Λ and we can always keep Ω arbitrarily larger than Λ. In the Fock space of states with energies smaller than Ω, our Hamiltonian is now a giant matrix with matrix elements approaching quickly zero for energies larger than Λ. The question is why results of diagonalization of this matrix depend on Λ. The answer is that we have suppressed states with energies much bigger than Λ. Including effects of coupling to those states would produce additional terms in the Hamiltonian with the cutoff Λ if it were to be equivalent to the initial Hamiltonian with no such cutoff. Without including those terms, the spectrum and most other observables do not have a limit when $\Lambda \to \infty$. The similarity renormalization scheme tells us how to find the missing terms. Then, our results will not depend on the arbitrary Λ any more, no matter how large it is (although always infinitely smaller than Ω), by construction. The similarity output is, (1) the missing terms (counterterms) in the initial Hamiltonian with very large Λ and, (2) another equivalent Hamiltonian with a finite and arbitrarily chosen cutoff, λ, that can be used for calculations. I shall consider only perturbative analysis; for examples of nonperturbative iterative procedures see Ref. 1. The counterterms are important because we cannot restore symmetries violated by the cutoff without including counterterms. One of the most important examples is the special relativity symmetry that is initially included in the Lagrangian through the abstract notions of local objects and events. In Dirac's program from Ref. 2 the relevance of renormalization to the problem of constructing relativistic quantum theory is not mentioned.

The QCD Hamiltonian is equal $H_{QCD} = H_{\Lambda QCD} + X_\Lambda$ in the limit $\Lambda \to \infty$. X_Λ does not vanish in the limit. We find X_Λ by making the following steps. H_{QCD} is transformed by a similarity transformation, S, to $H'_{QCD}(\lambda)$. λ defines a width of $H'_{QCD}(\lambda)$, for example, as a width of the band-diagonal Hamiltonian matrix in the Fock basis. The width says how large free energy jumps can be produced by the Hamiltonian. The small width λ excludes a possibility that the cutoff Λ may appear in perturbative calculations to orders smaller than $\sim \Lambda/\lambda$ if the matrix elements of $H'_{QCD}(\lambda)$ are independent of Λ. Therefore, the only potential source of Λ-dependence in the spectrum of $H'_{QCD}(\lambda)$ in perturbation theory is the explicit Λ-dependence of its matrix elements in the narrow band. We find X_Λ by demanding that this explicit Λ-dependence of the matrix elements disappears for $\Lambda \to \infty$. The resulting effective Hamiltonian, $H'_{QCD}(\lambda)$, has the same cutoff-independent spectrum as H_{QCD} because S is unitary. The free part of $H'_{QCD}(\lambda)$ is the same as in H_{QCD}. Now, $S = 1 + T = 1 + a + h$, where $h = (a^2 + h^2)/2$ and $a^\dagger = -a$ and $h^\dagger = h$. This notation is used to explain the method of defining S by a recursion. a and h must vanish when H_I vanishes, because H_0 is already diagonal. Note that h is of higher order in H_I than a. We find S by expanding a and h in power serieses in the interactions so that the lowest order terms in a induce higher order terms in a and h. Algebraic manipulations on the relation $H' = S^\dagger H S$ produce the following equation with the lowest-order unknown terms on its left-hand side,

$$H'_I + [a, H_0] = H_I + \{H_0, h\} + T^\dagger H_I + H_I T + T^\dagger H T = Q.$$

We define $H'_I = u_\lambda[Q]$ and $[a, H_0] = (1 - u_\lambda)[Q]$. This is only one out of many possible ways of splitting the right-hand side into two parts on the left-hand side in order to define H' and S. The operation u_λ makes H' have the width λ. This is explained by an example of the action of u_λ on a generic Hamiltonian term containing products of creation and annihilation operators:

$$u_\lambda \left[\prod_{i \in I} a_i^\dagger \prod_{j \in J} a_j \right] = u(I, J, \lambda) \prod_{i \in I} a_i^\dagger \prod_{j \in J} a_j.$$

The function $u(I, J, \lambda)$ can be chosen, for our example, in the form $u(I, J, \lambda) = \theta[\lambda^2 - (E_I - E_J)^2]$, where $E_K = \sum_{k \in K} E_k$ and E_k's denote free energies of particles created by the correspodning $a_k^\dagger$'s. It is clear that matrix elements of H' equal zero for states that differ in energies by more than λ. At the same time, the matrix elements of a vanish near the diagonal and one can solve for a knowing the commutator because matrix elements of a equal 0 when $\theta[\lambda^2 - (E_I - E_J)^2] = 1$. Thus, there is no small-denominator problems in our perturbative example despite the presence of degenerate states. In general, one needs smooth, analytic functions $u(I, J, \lambda)$ in order to avoid complicated theoretical analysis. Physically, the diagonal proximity condition means that the effective Hamiltonian describes interactions of soft, composite constituent quarks and gluons that cannot

suddenly change their momenta by arbitrary amounts in local interactions. This point and the related theory will be described elsewhere.

Expanding in power serieses in the coupling constant, $H'_{I\,1} = u_\lambda[H_{I\,1}]$, $a_1 = \underline{(1-u_\lambda)[H_{I\,1}]}$ (the line denotes the energy denominator), one obtains (λ is dropped for brevity)

$$H'_{QCD2} = u\left\{H_{QCD2} - \left[\underline{(1-u)[H_{QCD1}]}, u[H_{QCD1}]\right] + \frac{1}{2}\left[(1-u)[H_{QCD1}], \underline{(1-u)[H_{QCD1}]}\right]\right\}.$$

4. Calculation of $X_{\Lambda 2}$

The last equation above defines the effective light-front Hamiltonian for QCD in second order perturbation theory. We shall describe its matrix elements in one gluon states, $|g\rangle = |p^+\, p^\perp\, \varepsilon_\sigma\, c\rangle = a^\dagger_{p\sigma c}|0\rangle$. The Hamiltonian conserves momenta p^+ and $p^\perp$ and, therefore, the left and right free energies of the one-gluon states are equal. The second term does not contribute because of the $u(1-u)$-factor in the commutator. It would contribute a Λ-independent term for a smooth u. The gluon part of the second-order counterterm $X_{\Lambda 2}$ in H_{QCD2}, which we denote by $X_{g\Lambda 2}$, is determined from the condition that

$$\lim_{\Lambda\to\infty} \langle g'|\left\{X_{g\Lambda 2} + H_{QCD1}(p^-_g - H_0)^{-1}\theta[(p^-_g - H_0)^2 - \lambda^2]H_{QCD1}\right\}|g\rangle$$

exists. The above expression results from a number of simplifications that occur due to the same left and right free energies in the matrix element and properties of the θ-function. Note that the intermediate states with free energies that are close to p^-_g do not contribute and the main contribution comes from states with free energies that are bigger than $p^-_g + \lambda$. The Λ-dependence comes from high energy states. The effective gluon Hamiltonian term that has the same matrix elements in the one gluon states is (the outer $u = 1$ in this case and the same ν is used for quarks and gluons)

$$H'_{g2} = \sum_{\sigma c}\int [dp]\, \exp\left[-2E_\nu(p)/\Lambda\right] a^\dagger_{p\sigma c}\, a_{p\sigma c}[x_{g2}(p,\Lambda,\lambda) + f_{g2}(p,\Lambda,\lambda)],$$

where

$$f_{g2}(p,\Lambda,\lambda) = -\frac{\alpha}{4\pi p^+}\, \exp\left\{-(p^+ + p^-)/\Lambda\right\} \int_0^1 dx \int_{\lambda p^+}^\infty dz \exp\left\{-\{z + \frac{\nu^2}{x(1-x)}\}/\Lambda p^+\right\}\cdot$$

$$\cdot\left\{N_f[1 - 2x(1-x)] + N_c[(1-x)/x + x(1-x)/2]\right\}.$$

The bare masses from the initial Hamiltonian are put equal zero for simplicity. A smooth u would produce a different result. For example, $u(I,J,\lambda) = \lambda^2[(E_I - E_J)^2 + \lambda^2]^{-1}$ leads to the factor $z^4[\lambda^2 p^{+2} + z^2]^{-2}$ instead of $\theta(z - \lambda p^+)$ above. The first term in the bracket, $\sim N_f$, is the quark-loop contribution to the gluon self-energy and the second term, $\sim N_c$,

is the gluon-loop contribution. The gluon part is singular for $\nu = 0$. In the limit $\nu \to 0$ we have

$$f_{g2}(p,\Lambda,\lambda) = -\frac{\alpha}{4\pi}\left[\Lambda - \lambda - p^+ - \frac{p^{\perp 2}}{p^+}\right]\left[\frac{2}{3}N_f + N_c\left(\log\frac{\Lambda p^+}{\nu^2} - \gamma - \frac{11}{12}\right)\right].$$

The existence of $\lim_{\Lambda\to\infty} H'_{g2}$ requires that

$$x_{g2}(p,\Lambda,\lambda) = \frac{\alpha}{4\pi}\left\{\Lambda\left[\frac{2}{3}N_f + N_c\left(\log\frac{\Lambda p^+}{\nu^2} - \gamma - \frac{11}{12}\right)\right] - \left[\lambda + p^+ + \frac{p^{\perp 2}}{p^+}\right]\log\frac{\Lambda}{\nu} + c_{g2}\right\}.$$

$c_{g2} = c_{g2}(p^+, p^{\perp 2}, \lambda, \nu)$ is a finite (i.e. Λ-indpendent) and unknown term. However, the finite term should exactly cancel terms that depend on the ratio p^+/ν and violate Lorentz symmetry, and other terms that scale under Lorentz boosts like a constant or like p^+ (unless some other cancellation mechanism for those terms is present). Note that the most divergent parts of the counterterm are of the form of a chemical-potential operator $\sim \Lambda\hat{N}_g$, not a mass operator. A gluon mass term arises but it is proportional to ν^2 and it is not displayed in the limiting expression above. Thus,

$$H'_{g2} = \sum_{\sigma c}\int [dp]\, \exp\left[-2E_\nu(p)/\Lambda\right] a^\dagger_{p\sigma c}\, a_{p\sigma c}\frac{\alpha}{4\pi}\left[d_1\frac{p^{\perp 2}}{p^+} + d_2\frac{\nu^2}{p^+}\right].$$

The constants d_1 and d_2 must be determined from physical conditions, such as current conservation, and "coupling coherence" may constrain their values and dependence on λ.[3]

We wish to add a few comments. The Boltzmann regularization factor leads to more complicated structures for overlapping divergences. For example, we could not exclude appearance of the term $\partial^i A^i \partial^j A^j$. The similarity transformation can be performed in a variety of ways. For example, the unitarity condition can be satisfied by writing $S = e^T$, where T is antihermitean. Splitting of Q can be arranged in more convenient ways for specific calculations. An exciting calculation of $q\bar{q}$ matrix elements of H'_{QCD2} using coupling coherence to obtain confinement potential has been done by Robert Perry.[3] Finally, but certainly not the least importantly, the similarity can be defined in terms of operators without using Fock space restrictions of the Tamm-Dancoff type.

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