

A Systematic Extended Perturbation Theory for Quantum Chromodynamics

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Abstract

The approximation of Euclidean QCD vertex functions Γ by a double sequence $\Gamma^{[r,p]}$ is considered, where p is a perturbative order in g^2 , and r the order of a rational approximation in the QCD scale Λ^2 , non-analytic in g^2 . Self-consistency of $\Gamma^{[r,0]}$ in the Dyson-Schwinger equations comes about by a distinctive mathematical mechanism, which limits the self-consistency problem rigorously to the seven superficially divergent vertices.

1. The Extended Approximating Sequence

In a renormalizable but not superrenormalizable field theory, the sequence of partial sums $\Gamma^{[p]\text{pert}}$ ($p = 0, 1, 2, \dots$) of the perturbative expansion in the gauge coupling g for an Euclidean proper vertex $\Gamma_N(k; g)$, $k = \{k_1 \dots k_N \mid \sum k_i = 0\}$, is known to be fundamentally incomplete. Its semi-convergence, combined with the violent non-analyticity¹ of Γ 's around $g = 0$, leaves room for a remainder term exponentially small as $g \rightarrow 0$,

$$\text{Min}_{\{p\}} \left| \Gamma(k; g) - \Gamma^{[p]\text{pert}}(k; g) \right| \propto \exp \left(-\frac{\text{const.}}{g^2} \right). \quad (1.1)$$

In asymptotically free theories, on the other hand, one knows from renormalization-group (RG) analysis that one class of terms just allowed by this bound is positively there, namely terms involving the RG-invariant mass scale (in a scheme R),

$$(\Lambda^2)_R = \nu^2 \exp \left\{ -2 \int_R^{g(\nu)} \frac{dg'}{\beta(g')} \right\} = \nu^2 \exp \left\{ -\frac{(4\pi)^2}{\beta_0 g^2(\nu)} [1 + O(g^2)] \right\}. \quad (1.2)$$

Here ν is the arbitrary renormalization scale. For QCD, it therefore seems necessary to accommodate such Λ -dependent terms, which the perturbation expansion will miss even when summed to all orders.

The present contribution briefly describes some properties of a *double approximating sequence*² for Euclidean QCD vertices,

$$\Gamma^{[r,p]}(k; g), \quad r = 0, 1, 2, \dots, \quad p = 0, 1, 2, \dots, \quad (1.3)$$

designed to account for the Λ -dependent "missing terms", while being

- perturbatively renormalizable. It preserves perturbative power counting and permits removal of divergences by the perturbative, local counterterms;
- still "perturbative" (polynomial in g^2) in its "p" direction, but on the basis of a zeroth-order $\Gamma^{[r,0]}$ differing from the Feynman-rule vertices $\Gamma^{[0]\text{pert}}$ by nontrivial Λ dependence. As compared to $\Gamma^{[p]\text{pert}}$, it is "capable of improving its own zeroth order";
- a systematic approximation to nonperturbative Λ dependence in the "r" direction;
- capable of self-reproduction, up to terms of the next higher order in p , in the Dyson-Schwinger (DS) equations of the theory via a distinctive mathematical mechanism.

Approximation in the "r" direction cannot simply consist of an expansion in powers of Λ : since Λ is a mass, unlike the dimensionless g , the Λ and k dependences are always interconnected in what is essentially a dependence on k^2/Λ^2 ($k^2 =$ set of independent Lorentz invariants formed from k). Now the dynamical equations for Γ 's (DS equations) feature loop integrations ranging over all k space, including regions where $\Lambda^2 \gg k^2$. Approximation in the "r" direction therefore must be of a *global* nature.

The only known meeting point for the dual requirements of (i) globality of approximation w.r.t. k^2/Λ^2 and (ii) preservation of perturbative power counting w.r.t. k^2 is the use of *rational approximants*. They are restricted by the boundary conditions

$$\Gamma^{[r,p]}(\Lambda = 0, g \neq 0) = \Gamma^{[p]\text{pert}}(p = 0, 1, 2 \dots) \quad (\text{"perturbative limit"}) \quad (1.4)$$

$$\Gamma^{[r,0]}(\lambda k) \rightarrow \Gamma^{[0]\text{pert}}(\lambda k), \quad \lambda \rightarrow \infty \quad (\text{"naive asymptotic freedom"}), \quad (1.5)$$

the latter condition reflecting the asymptotic freedom of QCD and the expectation that logarithmic deviations from naive asymptotic freedom arise only from resummation of $p > 0$ terms.

As illustrative examples of such sequences we consider:

- (a) The Euclidean two-point vertex $\Gamma_T(k^2)$, or negative-inverse propagator, of transverse gluons, defined by the usual decomposition of the gluon propagator

$$D_{ab}^{\mu\nu}(k) = \delta_{ab} \left\{ \left[\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right] \left[-\frac{1}{\Gamma_T(k^2)} \right] + \frac{k^\mu k^\nu}{k^2} \left(\frac{\xi_0}{k^2} \right) \right\} \quad (1.6)$$

Its rational approximants of zeroth perturbative order $\Gamma_T^{[r,0]}$, with r denoting by convention the *denominator* degree, have the form

$$-\Gamma_T^{[r,0]}(k^2) = \frac{(k^2)^{r+1} + \zeta_{r,1}\Lambda^2(k^2)^r + \dots + \zeta_{r,r+1}(\Lambda^2)^{r+1}}{(k^2)^r + \eta_{r,1}\Lambda^2(k^2)^{r-1} + \dots + \eta_{r,r}(\Lambda^2)^r}, \quad (1.7)$$

where the $2r + 1$ dimensionless coefficients $(\zeta_{r,i}; \eta_{r,i})$ are real, or alternatively the partial-fraction decomposition

$$-\Gamma_T^{[r,0]}(k^2) = k^2 + u_{r,1}\Lambda^2 + \sum_{s=1}^r \frac{u_{r,2s+1}\Lambda^4}{k^2 + u_{r,2s}\Lambda^2}, \quad (1.8)$$

where the $u_{r,i}$ for $i > 1$ can be either real or pairwise complex conjugate. The boundary condition (1.5) fixes both the relative degrees and the leading relative coefficient of the numerator and denominator, and implies (1.4). In so doing, it also preserves the perturbative contribution of the propagator (1.6) to power counting in loops, which in this case therefore imposes no extra restrictions.

Among the dynamical possibilities embodied in (1.7/1.8) we mention the two one expects to dominate as *physical solutions* (most others are unphysical):

- "Particle sequence": all $r + 1$ zeroes of (1.7/1.8) (= propagator poles) real and at negative Euclidean (timelike Minkowskian) k^2 , with m of them well separated, and the remaining $r + 1 - m$ interspersed with real poles (= propagator zeroes), in the manner known in the context of rational approximants as approximating a branch cut. This would describe a gauge field creating m stable gluon particles at different masses (Schwinger mechanism).
- "Quasiparticle Sequence": r odd, with all $r + 1$ zeroes of (1.7/1.8) (= propagator poles) and all but one of the poles (= propagator zeroes) coming in complex-conjugate pairs. In the simplest case, one would expect one well-separated pair of complex propagator poles, the remaining ones settling (together with the propagator zeroes) into a two-complex-cuts pattern. The conceptual problems apparently arising from the complex propagator structure are not unsurmountable, provided the solution is used consistently. This subsequence is then of considerable interest for QCD, since its propagators describe gluons as the *short-lived elementary excitations* with lifetime $\sim \Lambda^{-1}$ that are "seen" at the origin of gluon jets.

(b) The Euclidean three-gluon vertex. Here it will be stated without elaboration that for the vertices with more than one invariant momentum variable, DS self-consistency will require the use of $p = 0$ approximants with a factorizing-denominator structure, so that e.g. the 3-gluon vertex Γ_{3g} will admit partial-fraction decompositions of the form

$$\Gamma_{3g}^{[r,0]} = B_0(k_1^2, k_2^2, k_3^2) + \sum_{s=1}^r B_s(k_1^2, k_2^2) \left(\frac{\Lambda^2}{k_3^2 + u'_{r,2s}\Lambda^2} \right). \quad (1.9)$$

2. The Self-Consistency Mechanism

The characteristic new problem of this extended approximation is the self-consistency of $\Gamma^{[r,0]}$ in the DS equations. The latter take the general form

$$\Gamma_N = \Gamma_N^{[0]\text{pert}} + \left(\frac{g_0}{4\pi}\right)^2 \Phi_N[\Gamma_2, \dots, \Gamma_N, \Gamma_{N+1}, \Gamma_{N+2}], \quad (2.1)$$

with the dressing functionals Φ_N consisting of loop integrals (generally divergent) over combinations of Γ 's, and being preceded by at least two powers of the bare gauge coupling. Self-consistency here means that

- when inserted collectively into the Φ_N , and after renormalization, the $\Gamma^{[r,0]}$ should reproduce themselves *up to corrections of order $p = 1$* , which are also generated in the process;
- for a rational approximant with $n(r)$ parameters, matching between the input function $\Gamma^{[r,0]}$ and the output of Φ_N can be achieved with respect to $n(r)$ "comparison data" (function values, derivatives, residues ...). In all other data, a matching error remains that can be improved only by going to higher r .

The problem with (2.1) is that the dressing terms must supply the nonperturbative parts $\Gamma^{[r,0]} - \Gamma^{[0]\text{pert}}$ with no $g^2(\nu)$ prefactor, in spite of the fact that they always come with at least one g_0^2 prefactor. The mathematical mechanism making this possible is, in my opinion, nontrivial. To explain it again on the example of the self-energy function $\Gamma_T(k^2) = -1/D_T(k^2)$, consider the corresponding DS equation at one loop,

$$\Gamma_T(k^2) = -k^2 + \left(\frac{g_0}{4\pi}\right)^2 \sum_{M=A,B,C,D} \Phi_T^{M(l=1)} [\Gamma^{[r,0]}]. \quad (2.2)$$

Here $\Phi_T^{A\dots D}$ stand for the well-known four terms: 2-gluons loop, ghost-antighost loop, gluon tadpole, and quark-antiquark loop, respectively. While Φ_T^C is a constant, the other three involve 3-point vertices $\Gamma_{3g}^{[r,0]}$, $\Gamma_{cg\bar{c}}^{[r,0]}$, $\Gamma_{qg\bar{q}}^{[r,0]}$, which all have partial-fraction decompositions w.r.t. k^2 of the general form (1.9). The structure of (2.2) therefore is

$$\left(\frac{g_0}{4\pi}\right)^2 \Phi_T^{(l=1)} = \left(\frac{g_0}{4\pi}\right)^2 \left\{ I_0^{(r)}(k^2) + \sum_{s=1}^r I_s^{(r)}(k^2) \left(\frac{\Lambda^2}{k^2 + u'_{r,2s}\Lambda^2} \right) \right\}, \quad (2.3)$$

with quadratically divergent integrals $I_0, I_1 \dots I_r$. The natural "comparison data", to be matched to the input function (1.8), are the positions and residues of the r poles in the k^2 plane, and the value of the smooth remainder function, denoted $J_0^{(r)}(k^2)$, at some still arbitrary point $k^2 = -u_0\Lambda^2$:

$$u_{r,2s} = u'_{r,2s} \quad (s = 1 \dots r); \quad (2.4)$$

$$\begin{aligned}
u_{r,2s+1}\Lambda^2 &= \left(\frac{g_0}{4\pi}\right)^2 I_s^{(r)}(-u_{r,2s}\Lambda^2) \quad (s = 1 \dots r); \\
u_{r,1}\Lambda^2 &= \left(\frac{g_0}{4\pi}\right)^2 J_0^r(-u_0\Lambda^2).
\end{aligned}
\tag{2.5}$$

Eq. (2.4) expresses the explicit self-reproduction of *momentum* structure occurring in this scheme: the 3-point approximants, by virtue of their factorizing denominators, "hand down" their rational structure to a 2-point vertex. (One noteworthy consequence of this is that all poles of the 3-point approximants reappear as *zeroes* of the corresponding propagators.) The r. h. sides of (2.5), dimensionally regularized in $D = 4 - 2\epsilon$, come out as

$$\begin{aligned}
I_s^{(r)}(-u_{r,2s}\Lambda^2) &= \left(\frac{\Lambda_\epsilon^2}{\nu_0^2}\right)^{-\epsilon} \left\{ \frac{A_s^{(l=1)}}{\epsilon} + (\text{finite as } \epsilon \rightarrow 0) + 0(\epsilon) \right\}, \\
I_0^{(r)}(-u_0\Lambda^2) &= \left(\frac{\Lambda_\epsilon^2}{\nu_0^2}\right)^{-\epsilon} \left\{ \frac{A_0^{(l=1)}}{\epsilon} + (\text{finite as } \epsilon \rightarrow 0) + 0(\epsilon) \right\},
\end{aligned}
\tag{2.6}$$

with Λ_ϵ the generalization of the scale (1.2) to $\epsilon \neq 0$. The A_s, A_0 depend on the nonperturbative constants of all the 2-point and 3-point vertices entering the $l = 1$ functional. Upon postulating that perturbative coupling-constant renormalization remain valid,

$$g_0^2 \nu_0^{2\epsilon} = Z_\alpha(g^2(\nu), \epsilon) g^2(\nu) \nu^{2\epsilon}, \tag{2.7}$$

one finds in (2.5) the divergent terms of (2.6) associated with a factor

$$\Pi(\epsilon, g^2(\nu)) = \left\{ \left[\frac{g(\nu)}{4\pi} \right]^2 Z_\alpha(g^2(\nu), \epsilon) \right\} \left\{ \left(\frac{\Lambda_\epsilon^2}{\nu^2} \right)^{-\epsilon} \frac{1}{\epsilon} \right\}. \tag{2.8}$$

The point now is that by using the exact integral representations of both Z_α and Λ_ϵ^2 in terms of the RG beta function, one immediately sees that Π is *in fact independent of $g^2(\nu)$* and finite at $\epsilon = 0$, and a few lines of algebra give its behavior near $\epsilon = 0$ as

$$\Pi(\epsilon) = \frac{1}{\beta_0} [1 + 0(\epsilon, \epsilon \ln \epsilon)]. \tag{2.10}$$

This is a compact statement of the "eating" mechanism. Its properties, partly discussed already in³, are:

- It makes the $p = 0$ nonperturbative terms of the input (1.8) reproduce themselves *analytically* upon imposition of the matching conditions.

$$u_{r,s} = \frac{1}{\beta_0} A_s^{(l=1)} \quad (s = 1 \dots 2r + 1), \tag{2.10}$$

thus enabling the sequence to establish self-consistently its own zeroth order.

- It establishes these terms without divergences as $\epsilon \rightarrow 0$, so that *no nonlocal counterterms* are needed (as necessary for perturbative renormalizability), and even without finite $O(g^2)$ corrections.
- It is closely tied to the divergence structure of the theory, as shown by the presence of the terms A_s in (2.10). It is not hard to demonstrate that, as a result, the *self-consistency problem of the $\Gamma^{[r,0]}$ gets rigorously* (i.e. without decoupling approximations) *restricted to the small finite set of DS equations for the superficially divergent vertices* of QCD.
- It leads to a *decoupling of the perturbative (p) and loop (l) orders*, with the l -loop terms arising in the iterative evaluation of the DS functional contributing to all terms of orders $p = 0, 1, 2, \dots, l$ of the sequence. In particular, the r. h. sides of the self-consistency equations (2.10), at l loops, become l -th order polynomials in $1/\beta_0$.

To complete the argument, one notes that the remainder of (2.3), the logarithmically divergent function $(g_0/4\pi)^2[J_0^{(r)}(k^2) - J_0^{(r)}(-u_0\Lambda^2)]$, can be renormalized by the *perturbative, local* one-loop counterterm if and only if $u_0 = 0$, so that perturbative renormalizability fixes this last undetermined parameter. In the framework of the "particle" subsequence, this implies in particular that a gauge-boson mass term, $u_{r,1}\Lambda^2$ in eq. (1.8), can in principle be established self-consistently *while using only the massless perturbative counterterm*.

To establish the full system of self-consistency equations, even at the lowest nontrivial ($r = 1$ and $l = 1$) level, by isolating the divergent parts of all relevant nonperturbative DS loops for the seven vertices, represents a substantial research program. However, the partial results (for $r = 1$ and $l = 1$) described in the second of Refs. 3 for the pure gluon theory, and analogous unpublished results of Koenning for the two basic fermion vertices, while based on crude truncations of that system, are encouraging, and would make the program seem worthwhile.

References

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