A-type affine Weyl group symmetry of Desargues maps and of the non-commutative Hirota–Miwa system

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Outline

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1. Motivation – P_{IV} equation
2. Desargues maps
   - Multidimensional compatibility of Desargues maps
   - The non-commutative Hirota–Miwa discrete KP system
   - Desargues maps and quadrilateral lattices
3. Affine Weyl group symmetry of Desargues maps
   - The A_N root lattice and its affine $W(A_N)$ Weyl group
   - Desargues maps of the $Q(A_N)$ root lattice
   - Action of the affine group $W(A_N)$

The symmetric form of P_IV

f_j, $j = 0, 1, 2$ – unknown functions of t, $' = \frac{d}{dt}$, α_j – parameters

$$f_0' = f_0(f_1 - f_2) + \alpha_0,$$
$$f_1' = f_1(f_2 - f_0) + \alpha_1,$$
$$f_2' = f_2(f_0 - f_1) + \alpha_2,$$

Obvious symmetries

- scaling: $t \to t/c$, $f_j \to cf_j$, $\alpha_j \to c^2\alpha_j$
- cyclic permutation: $f_j \to f_{j+1}$, $\alpha_j \to \alpha_{j+1}$, $j \in \mathbb{Z}/3\mathbb{Z}$

Obvious integral of motion

$$(f_0 + f_1 + f_2)' = \alpha_0 + \alpha_1 + \alpha_2 = k \qquad \Rightarrow \qquad f_0 + f_1 + f_2 = kt + c$$

Normalisation for $k \neq 0$

$$\alpha_0 + \alpha_1 + \alpha_2 = 1, \qquad f_0 + f_1 + f_2 = t$$

(scaling + translation in t)

Painlevé equation

Fact (V. E. Adler, 1994)

Under the given normalisation the above system is equivalent to

$$P_{IV}: \qquad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

$y = -f_1/\sqrt{2}$, $t \to \sqrt{2}t$, $\alpha = \alpha_0 - \alpha_2$, $\beta = -2\alpha_1^2$

Theorem (P. Painlevé, B. Gambier, 1900-1909)

Up to Möbius transformation there exists 50 second order ordinary differential equations whose solutions do not have movable branch oints and essential singularities. Their solutions can be expressed in terms of solutions of linear equations, elliptic functions, and one of six distinguished equations $P_I - P_{VI}$.

Other Painlevé equations

$$P_I: \quad y'' = 6y^2 + t$$

$$P_{II}: \quad y'' = 2y^3 + ty + \alpha$$

$$P_{III}: \quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_V: \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}: \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Painlevé equations in physics

- analysis of the correlation functions of two-dimensional Ising model
  (T. T. Wu, B. M. McCoy, C. A. Tracy, E. Barouch, 1976)
- analysis of the correlation functions of one-dimensional Bose gas
  (M. Jimbo, T. Miwa, Y. Môri, M. Sato, 1980)
- two-dimensional quantum gravity
  (E. Brezin, V. A. Kazakov, 1990)
- random matrices
  (C. A. Tracy, H. Widom, 1994)
- reductions of the Einstein equations
  (K. P. Tod, 1994)
- zeros of the ζ-Riemann function
  (P. J. Forrester, A. M. Odlyzko, 1996)
- ...

Acrion of the extended affine Weyl group $A_2^{(1)}$ on P_{IV}

$$\widetilde{W} = \langle r_0, r_1, r_2, \pi \rangle, \qquad r_j^2 = 1, \quad (r_j r_{j+1})^3 = 1, \quad \pi^3 = 1, \quad \pi r_j = r_{j+1}\pi$$

$$r_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \qquad r_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}, \quad \pi(\alpha_j) = \alpha_{j+1}, \quad \pi(f_j) = f_{j+1}$$

$$(a_{ij})_{i,j=0}^2 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad (u_{ij})_{i,j=0}^2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Theorem (M. Noumi, Y. Yamada, 1998)

Transformations r_0, r_1, r_2, π described above are automorphisms of the differential field $\mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ and define representation of the extended affine Weyl group Weyla typu $A_2^{(1)}$. These transformations commute with differntiation given by the normalised symmetric P_{IV} equation.

Poissona structure and Hamiltona function

Twierdzenie (M. Noumi, Y. Yamada, 1998)

The group $\widetilde{W}$ act by automorphicms of the Poisson algebra $\mathbb{C}(\alpha_0, \alpha_1, \alpha_2; f_0, f_1, f_2)$ with brackets

$$\{f_i, f_j\} = u_{ij}, \qquad \{\alpha_i, \alpha_j\} = 0, \qquad i, j = 0, 1, 2.$$

$p = f_1$, $q = f_2$, $t = f_0 + f_1 + f_2$, $\{p, q\} = 1$, $\{p, t\} = \{q, t\} = 0$.

Theorem (M. Noumi, Y. Yamada, 1998)

The symmetric P_{IV} is equivalent to the hamiltonian system with the Hamilton function

$$H = (t - q - p)pq + \alpha_2 p - \alpha_1 q + \frac{1}{3}(\alpha_1 - \alpha_2)t.$$

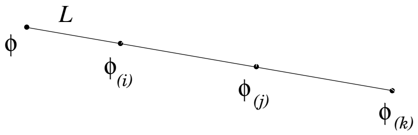
$P_{II} - \quad A_1^{(1)}$, $P_{III} - \quad C_2^{(1)}$, $P_V - \quad A_3^{(1)}$, $P_{VI} - \quad D_4^{(1)}$

Outline

Desargues maps

Definition *[AD '10]*

A map $\phi : \mathbb{Z}^N \to \mathbb{P}^M(\mathbb{D})$, $M \geq 2$, such that $\forall n \in \mathbb{Z}^N$ and all pairs of indices $1 \leq i \neq j \leq N$, the points $\phi(n)$, $\phi_{(i)}(n)$ and $\phi_{(j)}(n)$ are collinear, is called a Desargues map.



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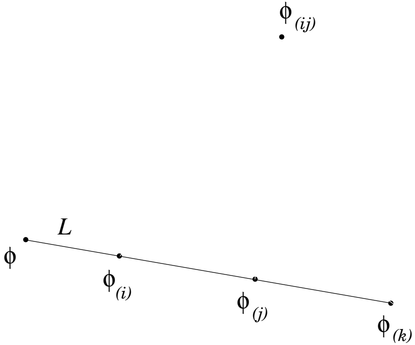

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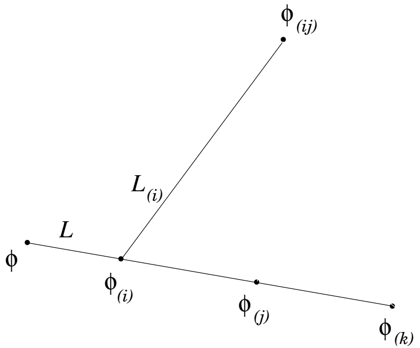

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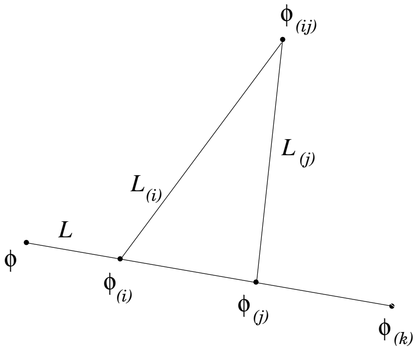

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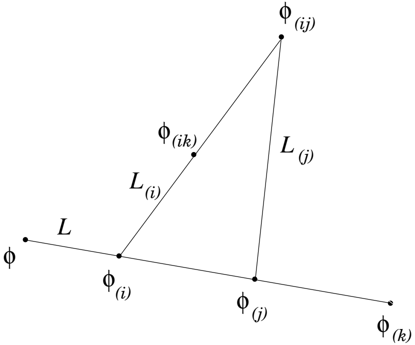

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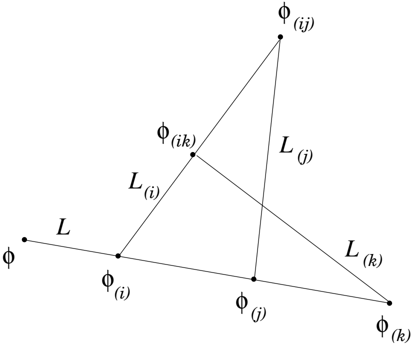

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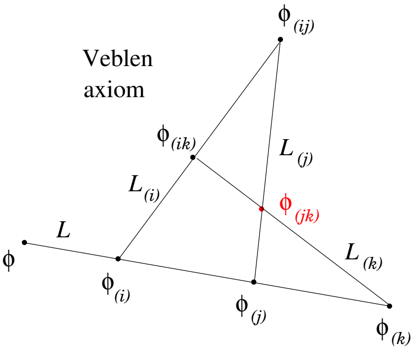
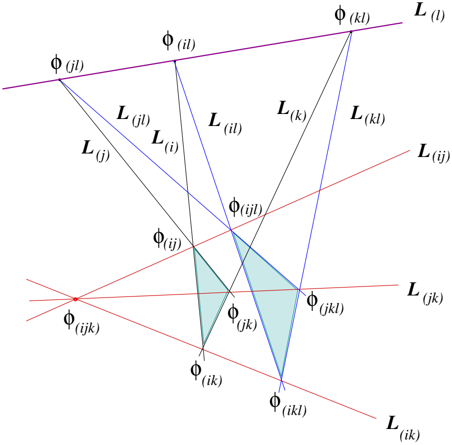
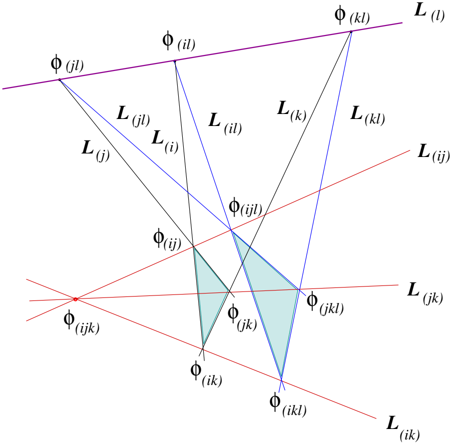


4D compatibility of Desargues maps



Where is the S_5 symmetry group of the Desargues configuration?

4D compatibility of Desargues maps



Where is the S_5 symmetry group of the Desargues configuration?

Non-commutative Hirota–Miwa system

In homogeneous coordinates $\phi : \mathbb{Z}^N \to \mathbb{D}_*^{M+1}$

$$\phi + \phi_{(i)} A_{ij} + \phi_{(j)} A_{ji} = 0, \qquad i \neq j,$$

where $A_{ij} : \mathbb{Z}^K \to \mathbb{D}_*$.

The compatibility condition of the above linear system reads

$$A_{ij}^{-1} A_{ik} + A_{kj}^{-1} A_{ki} = 1,$$

$$A_{ik(j)} A_{jk} = A_{jk(i)} A_{ik},$$

where i, j, k are distinct

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The Hirota system gauge

One can find homogeneous coordinates such that $A_{ij} = -A_{ji}$

$$\phi_{(i)} - \phi_{(j)} = \phi U_{ij}, \qquad 1 \leq i \neq j \leq N,$$

where the functions $U_{ij} = A_{ji}^{-1}$ satisfy the non-commutative Hirota–Miwa system

$$U_{ij} + U_{ji} = 0, \qquad U_{ij} + U_{jk} + U_{ki} = 0$$

$$U_{ij} U_{ik(j)} = U_{ik} U_{ij(k)}, \qquad i, j, k \text{ distinct}$$

[Nijhoff '85, Nimmo '07]

The second set of equations implies existence of potentials $\rho_i : \mathbb{Z}^K \to \mathbb{D}_*$, $i = 1, \dots, K$, such that

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The Hirota–Miwa discrete KP system

When $\mathbb{D} = \mathbb{F}$ is comutative then the functions ρ_i can be parametrized in terms of a single potential $\tau : \mathbb{Z}^K \to \mathbb{F}$

$$\rho_i = (-1)^{\sum_{k<i} n_k} \frac{\tau_{(i)}}{\tau}$$

The linear problem [Date-Jimbo-Miwa '82]

$$\phi_{(i)} - \phi_{(j)} = \frac{\tau \tau_{(ij)}}{\tau_{(i)} \tau_{(j)}} \phi, \qquad 1 \leq i < j \leq K$$

The nonlinear system [Hirota '81], [Miwa '82]

$$\tau_{(i)}\tau_{(jk)} - \tau_{(j)}\tau_{(ik)} + \tau_{(k)}\tau_{(ij)} = 0, \qquad 1 \leq i < j < k \leq K$$

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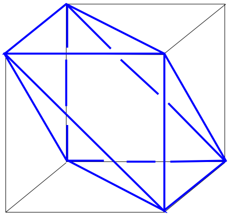
The Hirota–Miwa system in combinatorics

3D Hirota–Miwa (equation) is the same as the so called octahedron recurrence in combinatorics



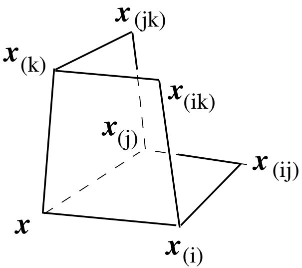
[Bobenko–Suris, Schief '09]

On the level of the τ-function the 4D compatibility of the Hirota–Miwa equation involves the graph of the Desargues configuration

Quadrilateral lattices

Definition *[AD-Santini '97]*

A quadrilateral lattice is a map $x : \mathbb{Z}^K \to \mathbb{P}^M(\mathbb{D})$, $3 \leq K \leq M$, whose all elementary quadrilaterals are planar.



Three planes in $\mathbb{P}^3$ intersect generically at one point.

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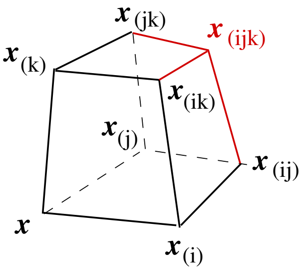

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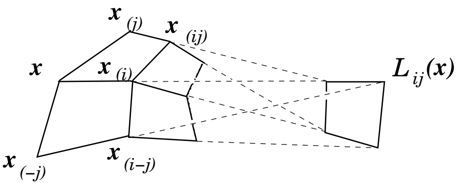
Laplace transformations of quadrilateral lattices



[Sauer '37], [AD '97]

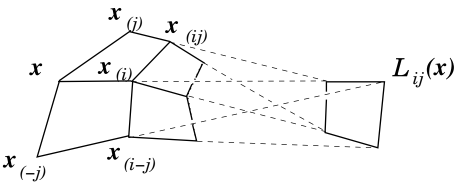
$$\mathcal{L}_{ij} \circ \mathcal{L}_{ji} = \mathrm{id}\,, \qquad \mathcal{L}_{jk} \circ \mathcal{L}_{ij} = \mathcal{L}_{ik}, \qquad \mathcal{L}_{ki} \circ \mathcal{L}_{ij} = \mathcal{L}_{kj}.$$

Laplace transformations of generic K-dimensional quadrilateral lattices are parametrized by points of the root lattice $Q(A_{K-1})$

$$Q(A_{K-1}) = \{(\ell_1, \dots, \ell_K) \in \mathbb{Z}^K | \ell_1 + \dots + \ell_K = 0\}$$
$$\mathcal{L}_{ij} : \ell_i \mapsto \ell_i + 1, \quad \ell_j \mapsto \ell_j - 1$$

[AD-Santini-Mañas '00]

Laplace transformations of quadrilateral lattices



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[AD-Santini-Mañas '00]

Desargues maps and quadrilateral lattices

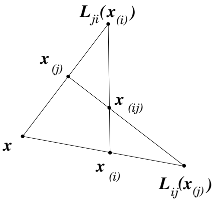
Proposition *[AD '10]*

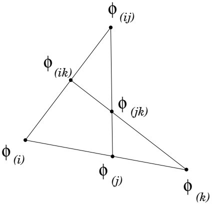
The theory of K-dimensional quadrilateral lattices and their Laplace transformations is the same as the theory of $2K-1$ dimensional Desargues maps

the change of variables: $n \in \mathbb{Z}^{2K-1}$ and $(m,\ell) \in \mathbb{Z}^K \times Q(A_{K-1})$

$$n_{2i-1} = m_i, \qquad n_{2i} = -m_i - \ell_i, \qquad i = 1, \dots, K,$$

where $n_{2K} = -n_1 - n_2 - \cdots - n_{2K-1}$





Outline

The A_N root lattice

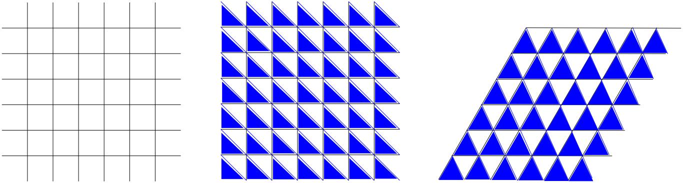
$Q(A_N)$ is the lattice generated by vectors along the edges of regular N-simplex. If we take the vertices of the simplex to be the vectors of the canonical basis in $\mathbb{R}^{N+1}$

$$\boldsymbol{e}_i = (0, \ldots, \overset{i}{1}, \ldots, 0), \qquad 1 \leq i \leq N+1,$$

then the generators are

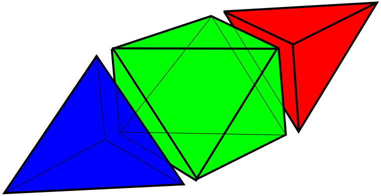
$$\varepsilon^i_j = \boldsymbol{e}_i - \boldsymbol{e}_j, \qquad 1 \leq i \neq j \leq N+1.$$

$$Q(A_N) = \{(n_1, \ldots, n_{N+1}) \in \mathbb{Z}^{N+1} | n_1 + \cdots + n_{N+1} = 0\}$$



Tiles (Delunay polytopes) of the A_N root lattice

Holes - points locally maximally distant from the lattice
Delaunay polytope - convex hull of the lattice points closest to the hole



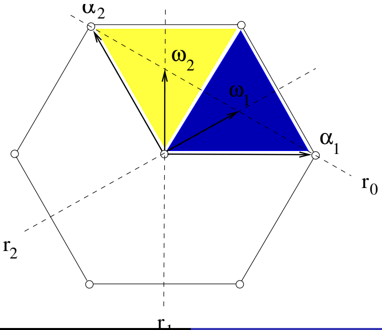
The Delaunay polytopes of $Q(A_N)$ are called regular "hypersimplices" $P(k, N)$, $k = 1, 2, \ldots N$
$P(1, N)$ - regular N-simplex
$P(k, N)$ - truncation of order $k - 1$ of the regular N-simplex

The A_N Weyl group

The *Weyl group* $W_0(A_N)$ is generated by the reflections r_i, $1 \leq i \leq N$,

$$r_i : \boldsymbol{v} \mapsto \boldsymbol{v} - \frac{2(\boldsymbol{v}|\boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i|\boldsymbol{\alpha}_i)}\boldsymbol{\alpha}_i$$

$\boldsymbol{\alpha}_i = \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ - simple root vectors
$W_0(A_N) \equiv S_{N+1}$, $r_i \equiv (i, i+1)$



The A_N affine Weyl group

The *affine Weyl group* $W(A_N)$ is generated by the reflections r_i, $1 \leq i \leq N$, and by the affine reflection r_0

$$r_0 : \boldsymbol{v} \mapsto \boldsymbol{v} - \left(1 - \frac{2(\boldsymbol{v}|\tilde{\alpha})}{(\tilde{\alpha}|\tilde{\alpha})}\right)\tilde{\alpha}$$

$\tilde{\boldsymbol{\alpha}} = -\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_1 + \cdots + \boldsymbol{\alpha}_N = \boldsymbol{e}_1 - \boldsymbol{e}_{N+1}$ - the highest root vector

$$W(A_N) = Q(A_N) \rtimes W_0(A_N)$$

Theorem (Coxeter)

The affine Weyl group acts on the Delaunay tiling by permuting tiles within each class $P(k, N)$.

Affine Weyl group symmetry of the Desargues maps

Proposition

Under the identification $\mathbb{Z}^N = \sum_{i=1}^{N} \mathbb{Z}\varepsilon_i^{N+1} = Q(A_N)$ the Desargues maps are maps $\phi : Q(A_N) \to \mathbb{P}^M$ such that the vertices of any N-simplex $P(1, N)$ are mapped into collinear points.

Theorem

If $\phi : \sum_{i=1}^{N} \mathbb{Z}\varepsilon_i^{N+1} \to \mathbb{P}^M$ is a Desargues map then also for any element g of the affine Weyl group $W(A_N)$ the map $\phi \circ g$ is a Desargues map.

Images of $P(K-1,K)$ under the Desargues maps

$P(K-1,K)$ has exactly $\binom{K+1}{2}$ vertices and $\binom{K+1}{3}$ 2-facets $P(1,2)$. Under Desargues map it gives a configuration of $\binom{K+1}{2}$ points and $\binom{K+1}{3}$ lines such that:

(i) every line is incident with exactly three points;
(ii) every point is incident with exactly $K-1$ lines

$K=3$ – the Veblen configuration – definition *[Schief '09]* of the Laplace–Darboux maps of $FCC = Q(A_3)$ lattice in $\mathbb{R}^3$

$K=4$ – the Desargues configuration

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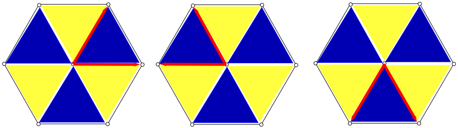
The linear problem of the Hirota-Miwa system on the root lattice

By the identification $\mathbb{Z}^N = \sum_{i=1}^{N} \mathbb{Z}\varepsilon_i^{N+1} = Q(A_N)$
we have the linear problem

$$\phi^{N+1}(n+\varepsilon_i^{N+1}) - \phi^{N+1}(n+\varepsilon_j^{N+1}) = \phi^{N+1}(n) U_{ij}^{N+1}(n), \quad 1 \leq i \neq j \leq N$$

$$U_{ij}^{N+1}(n) = \left[\rho_i^{N+1}(n)\right]^{-1} \rho_i^{N+1}(n+\varepsilon_j^{N+1})$$

Observation: There are $N+1$ equivalent choices of $\mathbb{Z}^N$ coordnates in $Q(A_N)$ (with fixed origin) respecting geometrically the Desargues map condition!



The "rotated" linear problems

Theorem

The functions $\phi^i : \mathbb{Z}^N = \sum_{j=1}^N \mathbb{Z}\varepsilon^i_j \to \mathbb{D}_*^{M+1}$ *given by*

$$\phi^i(n) = (-1)^{(n|\varepsilon_i^{N+1})} \phi^{N+1}(n) \left[\rho_i^{N+1}(n)\right]^{-1}$$

satisfy the linear system

$$\phi^i(n+\varepsilon^i_j) - \phi^i(n+\varepsilon^i_k) = \phi^i(n) U^i_{jk}(n), \qquad i,j,k \quad \textit{distinct},$$

where

$$U^i_{jk}(n) = \left[\rho^i_j(n)\right]^{-1} \rho^i_j(n+\varepsilon^i_k),$$

$$\rho^i_j(n) = \begin{cases} \rho_j^{N+1}(n) \left[\rho_i^{N+1}(n)\right]^{-1}, & j \neq N+1, \\ \left[\rho_i^{N+1}(n)\right]^{-1}, & j = N+1. \end{cases}$$

Geometric description of symmetries of the $\mathbb{Z}^N$-Desargues maps and of the non-comutative Hirota–Miwa system

- translations of the origin
- "rotations" of the $\mathbb{Z}^N$ basis in $Q(A_N)$
- permutation of indices with fixed basis

The affine Weyl group action on the edge potentials

$E(A_N) = \{[n, n+\varepsilon^i_j] | n \in Q(A_N)\}$ - oriented edges of $Q(A_N)$

$\rho : E(A_N) \to \mathbb{D}$ - the oriented edge potentials

$$\rho([n, n+\varepsilon^i_j]) = \rho^i_j(n)$$

Define the action of the affine Weyl group on the functions ρ^i_j through its action on the oriented edges of the root lattice

$$[g(\rho)]([n, n+\varepsilon^i_j]) = \rho(g^{-1}[n, n+\varepsilon^i_j]), \qquad g \in W(A_N)$$

The simple root potentials

$\rho^i(n) = \rho^i_{i+1}(n)$ - the simple root potentials

$$\rho^i_j = \rho^{j-1} \dots \rho^i, \qquad \rho^j_i = (\rho^i_j)^{-1}, \qquad i < j$$

Define also the α_0 potential

$$\rho^0 = (\rho^N \rho^{N-1} \dots \rho^1)^{-1}$$

Theorem

The action of the generators r_i, $i = 0, \ldots, N$, of the affine Weyl group on the functions ρ^j, $j = 0, \ldots, N$, is given by

$$[r_i(\rho^j)](n) = [(\rho^i)^{-a^U_{ji}} \rho^j (\rho^i)^{-a^L_{ji}}](r_i(n)),$$

where a^U_{ji} and a^L_{ji} are the "upper" and the "lower" parts of the Cartan matrix of the affine Weyl group $W(A_N)$

$$a^L_{ij} = \begin{bmatrix} 1 & 0 & & & -1 \\ -1 & 1 & 0 & & \\ & -1 & 1 & \ddots & \\ & & \ddots & \ddots & 0 \\ 0 & & & -1 & 1 \end{bmatrix}, \quad a^U_{ij} = \begin{bmatrix} 1 & -1 & & & 0 \\ 0 & 1 & -1 & & \\ & 0 & 1 & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & 0 & 1 \end{bmatrix}$$

The non-commutative τ-functions

$$\rho^i_j(n)\rho^k_i(n) = \rho^k_j(n), \qquad \text{where} \quad \rho^i_i = 1.$$

implies existence of $N+1$ τ-functions $\tau_i : Q(A_N) \to \mathbb{D}$

$$\rho^i_j(n) = \tau_j(n)\,[\tau_i(n)]^{-1}\,.$$

The action of the affine Weyl group on the edge potentials ρ^i_j follows from the action on the τ-functions by transpositions

$$r_i(\tau_j) = \begin{cases} \tau_{i+1} & j = i \\ \tau_i & j = i+1 \\ \tau_j & j \neq i, i+1 \end{cases}, \qquad \text{with} \quad n \mapsto r_i(n)$$

indices are considered modulo $N+1$ within their range.

For $\mathbb{D}$ commutative, i.e. a field, one can find simple expressions for all τ-functions in terms of one of them, e.g.

$$\tau_i(n) = (-1)^{\sum_{\ell<i} n_\ell^{N+1}} \tau_{N+1}(n + \varepsilon_i^{N+1}), \qquad i \neq N+1$$

$n = \sum_{\ell=1}^{N} n_\ell^{N+1} \varepsilon_\ell^{K+1}$

Papers

- A. Doliwa, *Desargues maps and the Hirota–Miwa equation*, Proc. R. Soc. A **466** (2010) 1117–1200.
- A. Doliwa, *The affine Weyl group symmetry of Desargues maps and of the non-commutative Hirota–Miwa system*, `arXive` soon