Necessary conditions for partial integrability of Hamiltonian systems with homogeneous potential

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The planar three body problem

- Masses m_1, m_2 and m_3; inertial coordinates $\boldsymbol{r}_1 = (x_1, x_2)$, $\boldsymbol{r}_2 = (x_3, x_4)$ and $\boldsymbol{r}_3 = (x_5, x_6)$; the respective momenta $\boldsymbol{y}_1 = (y_1, y_2)$, $\boldsymbol{y}_2 = (y_3, y_4)$ and $\boldsymbol{y}_3 = (y_5, y_6)$;
- Hamiltonian

$$K = \frac{1}{2m_1}(y_1^2 + y_2^2) + \frac{1}{2m_2}(y_3^2 + y_4^2) + \frac{1}{2m_3}(y_5^2 + y_6^2) + U(\boldsymbol{x}),$$

where

$$U(\boldsymbol{x}) = -\frac{m_1 m_2}{r_{12}} - \frac{m_2 m_3}{r_{23}} - \frac{m_3 m_1}{r_{31}}$$

and

$$r_{12} := \sqrt{(x_1 - x_3)^2 + (x_2 - x_4)^2},$$
$$r_{23} := \sqrt{(x_3 - x_5)^2 + (x_4 - x_6)^2},$$
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Three body problem. First integrals

- Total linear momentum:

$$Y_1 := y_1 + y_3 + y_5, \qquad Y_2 := y_2 + y_4 + y_6.$$

- Total angular momentum:

$$C := x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5.$$

- Poisson brackets:

$$\{C, Y_1\} = Y_2, \quad \text{and} \quad \{Y_2, C\} = Y_1.$$

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Three body problem. Integrability.

Ziglin, S. L.: 2000, ‘On involutive integrals of groups of linear symplectic transformations and natural mechanical systems with homogeneous potential’. *Funktsional. Anal. i Prilozhen.* **34**(3), 26–36, 96.

Theorem

In the positive octant of $\mathbb{R}^3$ with coordinates $m=(m_1,m_2,m_3)$ there exists a neighbourhood U of the bisector planes $m_i=m_j$, $i\neq j$ with deleted lanes $m_k/m_j=11/12, 1/4, 1/24$, such that for $m\in U$, the system is not integrable in the Liouville sense.

Three body problem. Reduction I.

$$(q_1, q_2) := \boldsymbol{r}_1 - \boldsymbol{r}_3, \qquad (q_3, q_4) := \boldsymbol{r}_2 - \boldsymbol{r}_3,$$

$$(q_5, q_6) := \frac{1}{m}\sum_{i=1}^{3} m_i \boldsymbol{r}_i.$$

$$H(\boldsymbol{q}, \boldsymbol{p}) := K(\boldsymbol{S}\boldsymbol{q}, \boldsymbol{S}^{-T}\boldsymbol{p}) = H_{\mathrm{r}} + \frac{1}{2m}(p_5^2 + p_6^2), \tag{1}$$

$$H_{\mathrm{r}} = T_{\mathrm{r}} + U_{\mathrm{r}}, \tag{2}$$

$$T_{\mathrm{r}} := \frac{1}{2\mu_1}\left(p_1^2 + p_2^2\right) + \frac{1}{2\mu_2}\left(p_3^2 + p_4^2\right) + \frac{1}{m_3}\left(p_1 p_3 + p_2 p_4\right), \tag{3}$$

$$U_{\mathrm{r}} := -\frac{m_1 m_2}{\sqrt{(q_1 - q_3)^2 + (q_2 - q_4)^2}} - \frac{m_2 m_3}{\sqrt{q_3^2 + q_4^2}} - \frac{m_3 m_1}{\sqrt{q_1^2 + q_2^2}},$$

and

$$\mu_1 := \frac{m_1 m_3}{m_1 + m_3}, \qquad \mu_2 := \frac{m_2 m_3}{m_2 + m_3}.$$

Three body problem. Reduction II.

Elimination of the angular momentum.

$$H_{\mathrm{W}} = \frac{1}{2\mu_1}\left[v_1^2 + \frac{1}{u_1^2}\left(v_3u_2 - v_2u_3 - c\right)^2\right] + \frac{1}{2\mu_2}\left(v_2^2 + v_3^2\right) + \frac{1}{m_3}\left[v_1v_2 - \frac{v_3}{u_1}\left(v_3u_2 - v_2u_3 - c\right)\right] + U_{\mathrm{W}}$$

$$U_{\mathrm{W}} := -\frac{m_1m_2}{\sqrt{(u_1-u_2)^2 + u_3^2}} - \frac{m_2m_3}{\sqrt{u_2^2 + u_3^2}} - \frac{m_3m_1}{u_1},$$

Reduction II. Integrability

Tsygvintsev, A.: 2001, ‘The meromorphic non-integrability of the three-body problem’. *J. Reine Angew. Math.* **537**, 127–149.

Theorem

For $c \neq 0$ the system generated by H_{W} does not admit two additional meromorphic first integrals.

Boucher, D. and J.-A. Weil: ‘Application of J.-J. Morales and J.-P. Ramis’ theorem to test the non-complete integrability of the planar three-body problem.’. IRMA Lect. Math. Theor. Phys. 3, 163-177 (2003).

Reduction II. Partial integrability

Tsygvintsev, A. V.: 2007, 'On some exceptional cases in the integrability of the three-body problem.'. *Celest. Mech. Dyn. Astron.* **99**(1), 23–29.

$$\sigma := \frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{(m_1 + m_2 + m_3)^2}$$

Theorem

If

$$\sigma \notin \left\{ \frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{3^2} \right\},$$

the system generated by H_{W} does not admit any additional meromorphic first integral.

Partial integrability?

- We have a Hamiltonian system with n degrees of freedom. Find effective necessary conditions for the existence of $1 < m \leq m_{\max}$ additional first integrals ($m_{\max} \leq n$ or $m_{\max} \leq 2n - 1$?)
- We have a Hamiltonian system with n degrees of freedom which admits first integrals $F_1 = H, \ldots, F_k$. Find effective necessary conditions for the existence of additional first integrals F_{k+i}, $i = 1, 2, \ldots$

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Systems with homogeneous potentials

$$H = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + V(\boldsymbol{q}),$$

$V(\boldsymbol{q})$ homogeneous of degree $k \in \mathbb{Z}^\star$.

Darboux point

$$\operatorname{grad} V(\boldsymbol{d}) = \boldsymbol{d}.$$

Particular solution

$$\boldsymbol{q}(t) := \varphi(t)\boldsymbol{d}, \quad \boldsymbol{p}(t) := \dot{\varphi}(t)\boldsymbol{d}, \qquad \ddot{\varphi} = -\varphi^{k-1}.$$

Systems with homogeneous potentials

Phase curves

$$\mathbf{\Gamma}_{k,\varepsilon} := \left\{ (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n} \mid \ (\boldsymbol{q}, \boldsymbol{p}) = (\varphi \boldsymbol{d}, \psi \boldsymbol{d}), \frac{1}{2}\psi^2 + \frac{1}{k}\varphi^k = \varepsilon \right\}.$$

Morales-Ramis Theorem

Assume that homogeneous potential V of degree $k \in \mathbb{Z}^{\star}$ satisfies the following conditions:

1. there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$, and
2. matrix $V''(\boldsymbol{d})$ is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_n$;
3. the system is integrable in the Liouville sense with first integrals which are meromorphic in a connected neighbourhood U of phase curve $\boldsymbol{\Gamma}_{k,\varepsilon}$ with $\varepsilon \neq 0$, and independent on $U \setminus \boldsymbol{\Gamma}_{k,\varepsilon}$.

Morales-Ramis Theorem

Then each (k, λ_i) belongs to an item of the following list

case	k	λ
1.	± 2	arbitrary
2.	k	$p + \frac{k}{2}p(p-1)$
3.	k	$\frac{1}{2}\left(\frac{k-1}{k} + p(p+1)k\right)$
4.	3	$-\frac{1}{24} + \frac{1}{6}(1+3p)^2, \quad -\frac{1}{24} + \frac{3}{32}(1+4p)^2$
		$-\frac{1}{24} + \frac{3}{50}(1+5p)^2, \quad -\frac{1}{24} + \frac{3}{50}(2+5p)^2$
5.	4	$-\frac{1}{8} + \frac{2}{9}(1+3p)^2$
6.	5	$-\frac{9}{40} + \frac{5}{18}(1+3p)^2, \quad -\frac{9}{40} + \frac{1}{10}(2+5p)^2$

Morales-Ramis Theorem

case	k	λ	
7.	-3	$\frac{25}{24} - \frac{1}{6}(1+3p)^2$,	$\frac{25}{24} - \frac{3}{32}(1+4p)^2$
		$\frac{25}{24} - \frac{3}{50}(1+5p)^2$,	$\frac{25}{24} - \frac{3}{50}(2+5p)^2$
8.	-4	$\frac{9}{8} - \frac{2}{9}(1+3p)^2$	
9.	-5	$\frac{49}{40} - \frac{5}{18}(1+3p)^2$,	$\frac{49}{40} - \frac{1}{10}(2+5p)^2$

where p is an integer.

A better version of the Morales-Ramis Theorem

V is not integrable if

1. $V''(\boldsymbol{d})$ has a Jordan block of size $s \geq 3$;
2. if $V''(\boldsymbol{d})$ has a Jordan block of size $s = 2$ with λ from item 2 of the Morales-Ramis table in item

Duval Guillaume and Andrzej J. Maciejewski: 2009, 'Jordan obstruction to the integrability of Hamiltonian systems with homogeneous potentials'. *Annales de l'Institut Fourier*. to appear.

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Our theorem

Assume that a homogeneous potential V of degree $k \in \mathbb{Z}^\star$ satisfies the following conditions:

1. there exists a non-zero $\boldsymbol{d} \in \mathbb{C}^n$ such that $V'(\boldsymbol{d}) = \boldsymbol{d}$, and
2. matrix $V''(\boldsymbol{d})$ is diagonalisable with eigenvalues $\lambda_1, \ldots, \lambda_n = k - 1$;
3. the system admits commuting additional first integrals $F_1 = H, \ldots, F_m$, $1 \leq m \leq n - 1$, which are meromorphic and independent in a connected neighbourhood U of $\boldsymbol{\Gamma}_{k,\varepsilon}$, $\varepsilon \neq 0$.

Then among pairs (k, λ_i), at least m, let us say, (k, λ_{n-i}) with $i = 0, \ldots, m - 1$, belong to the Morales-Ramis table.

Our theorem. Continuation

Moreover, if the system admits one more additional first integral F_{m+1} which is meromorphic on U, and such that $F_1, \ldots, F_{m+1}$ pairwise commute and are independent on $U \setminus \boldsymbol{\Gamma}_{k,\varepsilon}$, then either

A1. there exist $1 \leq i \leq n-m$ such that pair (k, λ_i), belongs to the Morales-Ramis table, or

A2. there exist $1 \leq i < j \leq n-m$ such that

$$\frac{1}{2k}\sqrt{(k-2)^2 + 8k\lambda_i} = \frac{1}{2k}\sqrt{(k-2)^2 + 8k\lambda_j} + p, \qquad (4)$$

for some $p \in \mathbb{Z}$.

The second possibility

$$V = (q_1^2 + q_2^2)^2 + \frac{1}{2}\lambda q_3^2(q_1^2 + q_2^2) + \frac{1}{4}q_3^4, \quad \lambda \in \mathbb{C}.$$

- First integral $F = q_1 p_2 - q_2 p_1$.
- Darboux point $\boldsymbol{d} = (0, 0, 1)$.
- $V''(\boldsymbol{d}) = \mathrm{diag}(\lambda, \lambda, 3)$.

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Back to the three body problem. Steps

$$H_\mathrm{r} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}_\mathrm{r} \boldsymbol{p} + U_\mathrm{r}(\boldsymbol{q}).$$

1. $$U_\mathrm{r}'(\boldsymbol{c}) = \boldsymbol{M}_\mathrm{r}^{-1}\boldsymbol{c},$$
2. spectr $\boldsymbol{M}_\mathrm{r} U_\mathrm{r}''(\boldsymbol{c}) = (-2, 1, \lambda_3, \lambda_4)$
3. If H_r integrable, then

$$\lambda_3, \lambda_4 \in \mathcal{M}_{-1} := \left\{ -\frac{1}{2}p(p-3) \mid p > 1, \quad p \in \mathbb{N} \right\} =$$
$$\{1, 0, -2, -5, -9, -14, \ldots\}$$

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Steps

4. If H_{r} admits an additional first integral, then, either $\lambda_3 \in \mathcal{M}_{-1}$, or $\lambda_4 \in \mathcal{M}_{-1}$, or

$$E(\lambda_3, \lambda_4) := \frac{1}{2k}\left[\sqrt{(k-2)^2 + 8k\lambda_3} - \sqrt{(k-2)^2 + 8k\lambda_4}\right] \in \mathbb{Z}.$$

Euler solution

Parametrisation

$$U_{\mathrm{r}}'(\boldsymbol{e}) = \boldsymbol{M}_{\mathrm{r}}^{-1}\boldsymbol{e},$$

$$\boldsymbol{e} := (a, 0, a(1+\rho), 0), \qquad a > 0, \quad \rho > 0.$$

$$\begin{cases} -\dfrac{m_2}{\rho^2} + m_3 + \dfrac{\alpha}{m}(m_2\rho - m_3) = 0, \\ \dfrac{m_1}{\rho^2} + \dfrac{m_3}{(1+\rho)^2} - \dfrac{\alpha}{m}[(m_1 + m_3)\rho + m_3] = 0, \end{cases}$$

$\alpha = a^3$.

Euler solution

$$P(\rho) := m_1 + m_2 + (2m_1 + 3m_2)\rho + (m_1 + 3m_2)\rho^2 - (m_1 + 3m_3)\rho^3 - (2m_1 + 3m_3)\rho^4 - (m_1 + m_3)\rho^5 = 0.$$

Characteristic polynomial

$$d(z) := \det(\boldsymbol{M}_{\mathrm{r}} U_{\mathrm{r}}''(\boldsymbol{e}) - z\boldsymbol{E}_4).$$

Euler solution

Characteristic polynomial

$$d(z) := \det(\boldsymbol{M}_{\mathrm{r}} U_{\mathrm{r}}''(\boldsymbol{e}) - z\boldsymbol{E}_4).$$

$$m_2 = \frac{m_1(1+\rho)^2(\rho^3-1) + m_3\rho^3(3+\rho(3+\rho))}{1+3\rho(1+\rho)}.$$

$$d(z) = (z+2)(z-1)(z-\lambda)(z+2\lambda),$$

$$\lambda := 2\frac{\rho^2(1+\rho)[2+\rho(3+2\rho)][m_3+(m_1+m_3)\rho]}{[1+\rho(2+\rho)(1+\rho^2)][m_3\rho^2+m_1(1+\rho)^2]}.$$

Euler solution

Conclusions

$$\lambda_3 = \lambda > 0, \quad \lambda_4 = -2\lambda < 0,$$

$$\lambda_3, \lambda_4 \in \mathcal{M}_{-1} \quad \Longleftrightarrow \quad (\lambda_3, \lambda_4) = (1, -2).$$

$$\operatorname{spectr} \boldsymbol{M}_{\mathrm{r}} U''(\boldsymbol{e}) = (-2, 1, 1, -2),$$

$$\boldsymbol{M}_{\mathrm{r}} U''(\boldsymbol{e}) \simeq \operatorname{diag}(J_2(-2), J_2(1)) \quad \Longrightarrow \quad H_{\mathrm{r}} \text{ is not integrable!}$$

Euler solution

Partial integrability

If H_r admits an additional first integral, then, either $\lambda_3 = \lambda \in \mathcal{M}_{-1}$, or $\lambda_4 = -2\lambda \in \mathcal{M}_{-1}$, or

$$e(\lambda) := E(\lambda_3, \lambda_4) = \frac{1}{2}\left(\sqrt{9+16\lambda} - \sqrt{9-8\lambda}\right) \in \mathbb{Z},$$

$$e(\lambda) \in \mathbb{Z} \quad \Longrightarrow \quad \lambda \in (0, 9/8].$$

Lagrange solution

Parametrisation

$$U'_{\mathrm{r}}(\boldsymbol{c}) = \boldsymbol{M}_{\mathrm{r}}^{-1}\boldsymbol{c}, \tag{5}$$

$$\boldsymbol{c} := \sqrt[3]{m}(a_1, b_1, a_2, b_2)^T, \tag{6}$$

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 1.$$

Lagrange solution

Spectrum

$$\operatorname{spectr}(\boldsymbol{M}_{\mathrm{r}} U_{\mathrm{r}}''(\boldsymbol{c})) = (-2, 1, \lambda_-, \lambda_+),$$

$$\lambda_\pm := -\frac{1}{2} \pm \frac{3}{2}\sqrt{1 - 3Q},$$

$$Q := \frac{1}{m^2}(m_1 m_2 + m_2 m_3 + m_3 m_1).$$

Lagrange solution

Partial integrability conditions

If H_{r} admits an additional first integral, then either $\lambda_{\pm} \in \mathcal{M}_{-1}$, or

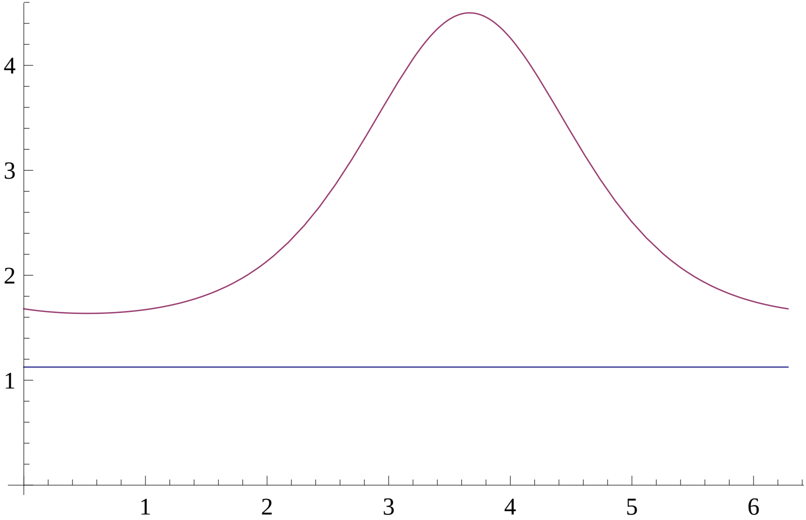
$$E(\lambda_+, \lambda_-) := \frac{1}{2k}\left[\sqrt{(k-2)^2 + 8k\lambda_+} - \sqrt{(k-2)^2 + 8k\lambda_-}\right] = \frac{1}{2}\left(\sqrt{13+12a} - \sqrt{13-12a}\right) \in \mathbb{Z},$$

where $k = -1$, $a := \sqrt{1-3Q}$.

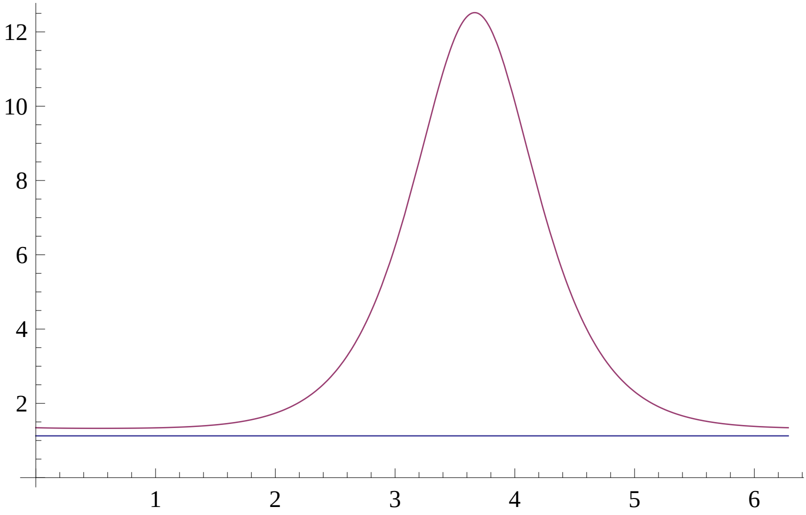
Tsygvintsev Condition

$$Q \in \left\{ \frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{3^2} \right\}.$$

Dependence $\lambda(s)$ for $Q = 8/27$



Dependence $\lambda(s)$ for $Q = 2/9$



Summary

Theorem

The reduced planar three body problem governed by H_r does not admit any additional meromorphic first integral.