Phase Space Quantum Mechanics Canonical Regime Part 1

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Classical Hamiltonian systems

Phase space

Phase space — a *Poisson manifold*, i.e. a smooth manifold M endowed with a two times contravariant antisymmetric tensor field $\mathcal{P}$ (a *Poisson tensor*) satisfying the below relation

$$\mathcal{L}_{\zeta_f}\mathcal{P} = 0,$$

for every vector fields ζ_f (*Hamiltonian fields*) defined as

$$\zeta_f := \mathcal{P}\mathrm{d}f\,, \quad f \in C^\infty(M).$$

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Algebra of observables

Algebra of observables — an algebra $\mathcal{A}_C = C^\infty(M)$, with respect to a point-wise product, of all (complex valued) smooth functions on M.

Admissible observables

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Lie structure on $C^\infty(M)$

A *Poisson bracket*

$$\{f, g\}_{\mathcal{P}} := \mathcal{P}(\mathrm{d}f, \mathrm{d}g), \quad f, g \in C^\infty(M)$$

introduces a structure of a Lie algebra on $C^\infty(M)$. Indeed, the Poisson bracket is a properly defined Lie bracket. In fact, it has the following properties

$$\begin{aligned} \{f, g\} &= -\{g, f\} && \text{(antisymmetry)}, \\ \{f, gh\} &= \{f, g\}h + g\{f, h\} && \text{(Leibniz's rule)}, \\ 0 &= \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} && \text{(Jacobi's identity)}. \end{aligned}$$

The algebra $C^\infty(M)$ endowed with the Poisson bracket is called a *Poisson algebra*.

Classical Hamiltonian system

Classical Hamiltonian system — a triple $(M, \mathcal{P}, H)$, where $H \in \mathcal{O}_C$ is some distinguished observable called a *Hamiltonian*.

Canonical coordinates

Local coordinates q^i, p_i $(i = 1, \ldots, N)$ in which a Poisson tensor $\mathcal{P}$ have (locally) a form

$$\mathcal{P} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i} \quad \text{i.e.} \quad \mathcal{P}^{ij} = \begin{pmatrix} 0_N & \mathbb{I}_N \\ -\mathbb{I}_N & 0_N \end{pmatrix}$$

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are called *canonical coordinates*.

In the canonical coordinates a Hamiltonian field ζ_f and a Poisson bracket take a form

$$\zeta_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i},$$
$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

Pure states, mixed states and expectation values of observables

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Mixed states

Mixed states — probability distribution functions defined on the phase space, i.e. functions $\rho \in C^\infty(M)$ such that

- $0 \leq \rho(\xi) \leq 1$ for $\xi \in M$,
- $\displaystyle\int_M \rho(\xi)\mathrm{d}\xi = 1$.

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- $\int_M \rho(\xi) \mathrm{d}\xi = 1$.

In this picture pure states can be defined as Dirac delta distributions, i.e.

$$\xi_0 \in M \longleftrightarrow \delta(\xi - \xi_0).$$

Expectation values of observables

Expectation value of observable $A \in \mathcal{A}_C$ in a state ρ:

$$\langle A \rangle_\rho := \int_M A(\xi)\rho(\xi)\mathrm{d}\xi \,.$$

For a pure state $\rho(\xi) = \delta(\xi - \xi_0)$:

$$\langle A \rangle_\rho = \int_M A(\xi)\delta(\xi - \xi_0)\mathrm{d}\xi \, = A(\xi_0).$$

Time evolution of classical Hamiltonian systems

The Hamiltonian H governs the time evolution of the system:

$$H \longrightarrow \zeta_H \longrightarrow \Phi_t^H$$

$\Longrightarrow$

$$\xi(t+\Delta t) = \Phi_{\Delta t}^H(\xi(t))$$
$$\Updownarrow$$
$$\dot{\xi} = \zeta_H$$
$$\Updownarrow$$
$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$
$$\text{(Hamilton equations)}$$

$$\rho(t) = (\Phi_{\Delta t}^H)^* \rho(t+\Delta t)$$
$$\Updownarrow$$
$$L(H,\rho) := \frac{\partial \rho}{\partial t} - \{H,\rho\} = 0$$
$$\text{(Liouville equation)}$$

For $\rho(\xi) = \delta(\xi - \xi_0)$ the Liouville equation induces the Hamilton equations.

Time dependent expectation value of observables

A time dependent expectation value of an observable $A \in \mathcal{A}_C$ in a state $\rho(t)$, i.e. $\langle A \rangle_{\rho(t)}$, fulfills the following equation of motion

$$\langle A \rangle_{L(H,\rho)} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t} \langle A \rangle_{\rho(t)} - \langle \{A, H\} \rangle_{\rho(t)} = 0.$$

Time development of observables

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$$U_t^H = (\Phi_t^H)^* = e^{t\mathcal{L}_{\zeta_H}}.$$

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Time development of $A \in \mathcal{A}_C$:

$$A(t) = U_t^H A(0) = e^{t\mathcal{L}_{\zeta_H}} A(0) = e^{t\zeta_H} A(0) \iff \frac{\partial A}{\partial t}(t) - \{A(t), H\} = 0.$$

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Both presented approaches to the time development yield equal predictions concerning the results of measurements:

$$\langle A(0) \rangle_{\rho(t)} = \langle A(t) \rangle_{\rho(0)}.$$

Basics of deformation quantization

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Quantization of the Poisson algebra of observables

Quantization of $(M, \mathcal{P})$:
Deformation with respect to $\hbar$ of the classical algebra of observables $\mathcal{A}_C$ to some noncommutative quantum algebra of observables $\mathcal{A}_Q$, i.e.

$$\cdot \longrightarrow \star,$$

$$\{\cdot\,,\,\cdot\} \longrightarrow [\![\cdot\,,\,\cdot]\!] = \frac{1}{i\hbar}[\,\cdot\,,\,\cdot\,],$$

where $[\,\cdot\,,\,\cdot\,]$ is a $\star$-commutator.

$\star$-product

The deformed noncommutative multiplication on $\mathcal{A}_Q$ should satisfy such natural conditions

1. $f \star (g \star h) = (f \star g) \star h$ (associativity),
2. $f \star g = fg + o(\hbar)$,
3. $[\![f, g]\!] = \{f, g\} + o(\hbar)$,
4. $f \star 1 = 1 \star f = f$,
5. $f \star g = \sum_{k=0}^{\infty} \hbar^k B_k(f, g)$,

where $f, g, h \in \mathcal{A}_Q$ and $B_k \colon \mathcal{A}_Q \times \mathcal{A}_Q \to \mathcal{A}_Q$ are bilinear operators.

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From the construction of the $\star$-product it can be immediately seen that in the limit $\hbar \to 0$ the quantized algebra of observables reduces to the classical algebra of observables.

The bracket $[\![\cdot , \cdot]\!]$ is a well-defined Lie bracket. In fact, it satisfies

$$[\![f,g]\!] = -[\![g,f]\!] \qquad \text{(antisymmetry)},$$
$$[\![f, g \star h]\!] = [\![f,g]\!] \star h + g \star [\![f,h]\!] \qquad \text{(Leibniz's rule)},$$
$$0 = [\![f,[\![g,h]\!]]\!] + [\![h,[\![f,g]\!]]\!] + [\![g,[\![h,f]\!]]\!] \qquad \text{(Jacobi's identity)}.$$

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In this case

$$\mathcal{P} = \sum_{i=1}^{N} X_i \wedge Y_i = \sum_{i=1}^{N} (X_i \otimes Y_i - Y_i \otimes X_i),$$

where X_i, Y_i $(i = 1, \ldots, N)$ are some pair-wise commuting vector fields on M.

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Also

$$\{f, g\}_{\mathcal{P}} = f \left(\sum_{i=1}^{N} (X_i \otimes Y_i - Y_i \otimes X_i) \right) g = f \left(\sum_{i=1}^{N} (\overleftarrow{X}_i \overrightarrow{Y}_i - \overleftarrow{Y}_i \overrightarrow{X}_i) \right) g$$

$$= \sum_{i=1}^{N} (X_i(f) Y_i(g) - Y_i(f) X_i(g)).$$

Space of states and properties of canonical $\star_{\sigma,\alpha,\beta}$-products

In the rest of the presentation the case of $\star_{\sigma,\alpha,\beta}$-products related to the canonical Poisson tensor $\mathcal{P} = \partial_x \wedge \partial_p$ on a manifold $M = \mathbb{R}^2$ will be considered.

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Canonical $\star_{\sigma,\alpha,\beta}$-products

The $\star_{\sigma,\alpha,\beta}$-product takes the form

$$f \star_{\sigma,\alpha,\beta} g = f \exp\left(i\hbar\sigma \overleftarrow{\partial}_x \overrightarrow{\partial}_p - i\hbar\bar{\sigma} \overleftarrow{\partial}_p \overrightarrow{\partial}_x + \hbar\alpha \overleftarrow{\partial}_x \overrightarrow{\partial}_x + \hbar\beta \overleftarrow{\partial}_p \overrightarrow{\partial}_p\right) g$$
$$= \sum_{n,m,r,s=0}^{\infty} (-1)^m (i\hbar)^{n+m} \hbar^{r+s} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} \cdot (\partial_x^{n+r} \partial_p^{m+s} f)(\partial_x^{m+r} \partial_p^{n+s} g).$$

Particular cases of the $\star_{\sigma,\alpha,\beta}$-product

The well-known particular cases of the $\star_{\sigma,\alpha,\beta}$-product are

1. for $\sigma = \alpha = \beta = 0$, the Kupershmidt-Manin product,
2. for $\sigma^{ij} = \frac{1}{2}\delta^{ij}, \alpha = \beta = 0$, the Moyal (or Groenewold) product

$$f \star_{\frac{1}{2}} g = f \exp\left(\frac{1}{2} i\hbar(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\right) g$$
$$= f \exp\left(\frac{1}{2} i\hbar \partial_x \wedge \partial_p\right) g.$$

3. for $\sigma = \frac{1}{2}$, $\alpha = \frac{2\lambda - 1}{2\omega}$, $\beta = \omega^2\alpha$ where $\omega, \lambda \in \mathbb{R}$ and $\omega > 0$

$$f \star g = f \exp\left(\hbar\lambda \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \hbar\bar{\lambda} \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a\right) g,$$

where the new coordinates $a(x,p) = (\omega x + ip)/\sqrt{2\omega}$, $\bar{a}(x,p) = (\omega x - ip)/\sqrt{2\omega}$ called *holomorphic coordinates* were used.

Theorem

Let $f, g \in \mathcal{A}_Q$ be such that $f \star_{\sigma,\alpha,\beta} g$ and $g \star_{\sigma,\alpha,\beta} f$ are integrable functions. Then there holds

$$\iint (f \star_{\sigma,\alpha,\beta} g)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint (g \star_{\sigma,\alpha,\beta} f)(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

Moreover, for the Moyal $\star$-product (the case of $\sigma = \frac{1}{2}$ and $\alpha = \beta = 0$) there holds

$$\iint (f \star_{\frac{1}{2}} g)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

Space of states

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Space of states — in the case $\alpha = \beta = 0$ the Hilbert space $L^2(\mathbb{R}^2)$. It is possible to extend the $\star_\sigma$-product to the $\star_\sigma$-product between functions from $L^2(\mathbb{R}^2)$, as to make from $L^2(\mathbb{R}^2)$ a Hilbert algebra with respect to the $\star_\sigma$-multiplication.

Space of states for the general $\star_{\sigma,\alpha,\beta}$-multiplication

For $\alpha \neq 0, \beta \neq 0$ the space of states for the general $\star_{\sigma,\alpha,\beta}$-multiplication can be defined by

$$\mathcal{H} = S_{\alpha,\beta}(L^2(\mathbb{R}^2)).$$

where for appropriate $f \in L^2(\mathbb{R}^2$:

$$S_{\alpha,\beta}f(x,p) = \frac{1}{2\pi\hbar\sqrt{\alpha\beta}} \iint f(x',p')e^{-\frac{1}{2\hbar\alpha}(x-x')^2}e^{-\frac{1}{2\hbar\beta}(p-p')^2}\mathrm{d}x'\,\mathrm{d}p'\,.$$

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The scalar product and the $\star_{\sigma,\alpha,\beta}$-product on $\mathcal{H}$ can be defined by

$$\langle\Psi|\Phi\rangle_{\mathcal{H}} = \langle S_{\alpha,\beta}^{-1}\Psi|S_{\alpha,\beta}^{-1}\Phi\rangle_{L^2}, \quad \Psi,\Phi\in\mathcal{H},$$
$$\Psi\star_{\sigma,\alpha,\beta}\Phi = S_{\alpha,\beta}^{-1}\Psi\star_\sigma S_{\alpha,\beta}^{-1}\Phi, \quad \Psi,\Phi\in\mathcal{H}.$$

Hence, $\mathcal{H}$ is also a Hilbert algebra.

Theorem

The scalar product on $\mathcal{H}$ can be written in a form

$$\langle\Psi|\Phi\rangle_{\mathcal{H}} = \iint (\mathcal{F}\Psi(\xi,\eta))^* \mathcal{F}\Phi(\xi,\eta)\mathrm{d}\mu(\xi,\eta),$$

where

$$\mathrm{d}\mu(\xi,\eta) = e^{\frac{1}{\hbar}\alpha\xi^2}e^{\frac{1}{\hbar}\beta\eta^2}\mathrm{d}\xi\,\mathrm{d}\eta.$$

It is possible to define a left and right $\star_{\sigma,\alpha,\beta}$-product of a function $A \in \mathcal{A}_Q$ with functions from some subspace of $\mathcal{H}$ receiving again a function from $\mathcal{H}$.

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Theorem

For any function $A \in \mathcal{A}_Q$ there holds

$$\begin{aligned} A_L \star_{\sigma,\alpha,\beta} &= A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}), \\ A_R \star_{\sigma,\alpha,\beta} &= A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}), \end{aligned}$$

where

$$\begin{aligned} \hat{q}_{\sigma,\alpha} &:= x + i\hbar\sigma\partial_p + \hbar\alpha\partial_x = x_L \star_{\sigma,\alpha,\beta} \,, \\ \hat{p}_{\sigma,\beta} &:= p - i\hbar\bar{\sigma}\partial_x + \hbar\beta\partial_p = p_L \star_{\sigma,\alpha,\beta} \,, \end{aligned}$$

and

$$\begin{aligned} \hat{q}^*_{\bar{\sigma},\alpha} &:= x - i\hbar\bar{\sigma}\partial_p + \hbar\alpha\partial_x = x_R \star_{\sigma,\alpha,\beta} \,, \\ \hat{p}^*_{\bar{\sigma},\beta} &:= p + i\hbar\sigma\partial_x + \hbar\beta\partial_p = p_R \star_{\sigma,\alpha,\beta} \,. \end{aligned}$$

(σ, α, β)-ordered operator functions

The symbol $A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ denotes a (σ, α, β)-*ordered operator function* defined by

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) := A(-i\hbar\partial_\xi, i\hbar\partial_\eta) e^{\frac{i}{\hbar}(\xi\hat{q} - \eta\hat{p} + (\frac{1}{2} - \sigma)\xi\eta) + \frac{1}{2\hbar}(\alpha\xi^2 + \beta\eta^2)} \Big|_{\xi=\eta=0}.$$

Examples of operator functions

- For $A(x,p) = x^2 + p^2$

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = (\hat{q}^2 + \hat{p}^2)_{\sigma,\alpha,\beta} = \hat{q}^2 + \hat{p}^2 - \hbar\alpha - \hbar\beta$$
$$= \hat{q}^2 + \hat{p}^2 + i(\alpha+\beta)\hat{q}\hat{p} - i(\alpha+\beta)\hat{p}\hat{q}.$$

In particular, the case when $\alpha = \beta = 0$ gives

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$$(\hat{q}^2 + \hat{p}^2)_{\sigma,\alpha,\beta} = \hat{q}^2 + \hat{p}^2.$$

- For $A(x,p) = xp$

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = (\hat{q}\hat{p})_{\sigma,\alpha,\beta} = \hat{q}\hat{p} - i\hbar\sigma = \bar{\sigma}\hat{q}\hat{p} + \sigma\hat{p}\hat{q}.$$

In particular, the case when $\sigma = 0, \frac{1}{2}, 1$ gives

$$(\hat{q}\hat{p})_{\sigma=0,\alpha,\beta} = \hat{q}\hat{p} \qquad \text{(normal ordering)},$$
$$(\hat{q}\hat{p})_{\sigma=\frac{1}{2},\alpha,\beta} = \frac{1}{2}\hat{q}\hat{p} + \frac{1}{2}\hat{p}\hat{q} \qquad \text{(Weyl ordering)},$$
$$(\hat{q}\hat{p})_{\sigma=1,\alpha,\beta} = \hat{p}\hat{q} \qquad \text{(anti-normal ordering)}.$$

Adjoint of $\star_{\sigma,\alpha,\beta}$-multiplication

It is possible to introduce adjoint of left and right $\star_{\sigma,\alpha,\beta}$-multiplication in a standard way

$$\langle (A_L \star_{\sigma,\alpha,\beta})^\dagger \Psi_1 | \Psi_2 \rangle_{\mathcal{H}} = \langle \Psi_1 | A_L \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}},$$
$$\langle (A_R \star_{\sigma,\alpha,\beta})^\dagger \Psi_1 | \Psi_2 \rangle_{\mathcal{H}} = \langle \Psi_1 | A_R \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}}.$$

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$$\langle (A_R \star_{\sigma,\alpha,\beta}\,)^\dagger \Psi_1 | \Psi_2 \rangle_{\mathcal{H}} = \langle \Psi_1 | A_R \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}} \,.$$

From this it then follows that

$$(A_L \star_{\sigma,\alpha,\beta}\,)^\dagger = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}) = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}),$$
$$(A_R \star_{\sigma,\alpha,\beta}\,)^\dagger = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}) = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}).$$

Pure states, mixed states and expectation values of observables

Pure states

Pure states are functions $\Psi_{\text{pure}} \in \mathcal{H}$ which satisfy the following conditions

1. $\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta} = (\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta})^\dagger$ (hermiticity),
2. $\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} = \frac{1}{\sqrt{2\pi\hbar}} \Psi_{\text{pure}}$ (idempotence),
3. $\|\Psi_{\text{pure}}\|_{\mathcal{H}} = 1$ (normalization).

Mixed states

Mixed states $\Psi_{\mathrm{mix}} \in \mathcal{H}$ are defined as linear combinations of some families of pure states $\Psi_{\mathrm{pure}}^{(\lambda)}$

$$\Psi_{\mathrm{mix}} := \sum_{\lambda} p_\lambda \Psi_{\mathrm{pure}}^{(\lambda)},$$

where $0 \leq p_\lambda \leq 1$ and $\sum_\lambda p_\lambda = 1$.

Quantum distribution functions

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Marginal distributions

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$$P(x) := \int \rho(x,p)\mathrm{d}p\,, \qquad P(p) := \int \rho(x,p)\mathrm{d}x\,,$$

are probabilistic distribution functions.

Expectation value of observables

The expectation value of an observable $A \in \mathcal{A}_Q$ in an admissible state $\Psi \in \mathcal{H}$:

$$\langle A \rangle_\Psi = \iint (A \star_{\sigma,\alpha,\beta} \rho)(x,p) \mathrm{d}x \, \mathrm{d}p \,,$$

where

$$\rho = \frac{1}{\sqrt{2\pi\hbar}} \Psi.$$

Time evolution of quantum Hamiltonian systems

The time evolution of a quantum Hamiltonian system is governed by a Hamiltonian H. It will be assumed that $H \in \mathcal{O}_Q$ and that H is self-adjoint in $\mathcal{H}$, i.e. $H = H^*$ and $H_{L,R}\,\star_{\sigma,\alpha,\beta} = (H_{L,R}\,\star_{\sigma,\alpha,\beta})^\dagger$.

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The time evolution of a quantum Hamiltonian system is governed by a Hamiltonian H. It will be assumed that $H \in \mathcal{O}_Q$ and that H is self-adjoint in $\mathcal{H}$, i.e. $H = H^*$ and $H_{L,R} \star_{\sigma,\alpha,\beta} = (H_{L,R} \star_{\sigma,\alpha,\beta})^\dagger$.

Time evolution equation

The time evolution of a quantum distribution function ρ:

$$L(H,\rho) := \frac{\partial \rho}{\partial t} - [|H,\rho|] = 0$$

$$\Updownarrow$$

$$i\hbar \frac{\partial \rho}{\partial t} - [H,\rho] = 0.$$

Stationary states

Stationary states Ψ satisfy

$$[H, \Psi] = 0.$$

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For pure states the above equation is equivalent to a pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$H \star_{\sigma,\alpha,\beta} \Psi = E\Psi, \quad \Psi \star_{\sigma,\alpha,\beta} H = E\Psi,$$

for some $E \in \mathbb{R}$.

The formal solution of the time evolution equation takes the form

$$\rho(t) = U(t) \star_{\sigma,\alpha,\beta} \rho(0) \star_{\sigma,\alpha,\beta} U(-t),$$

where

$$U(t) = e_{\star_{\sigma,\alpha,\beta}}^{-\frac{i}{\hbar}tH} := \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{i}{\hbar}t\right)^k \underbrace{H \star_{\sigma,\alpha,\beta} \dots \star_{\sigma,\alpha,\beta} H}_{k}$$

is an unitary function in $\mathcal{H}$ as H is self-adjoint.

Time dependent expectation value

A time dependent expectation value of an observable $A \in \mathcal{A}_Q$ in a state $\rho(t)$, i.e. $\langle A \rangle_{\rho(t)}$, fulfills the following equation of motion

$$\langle A \rangle_{L(H,\rho)} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t} \langle A \rangle_{\rho(t)} - \langle [\![A, H]\!] \rangle_{\rho(t)} = 0.$$

Time development of observables

The time development of $A \in \mathcal{A}_Q$:

$$A(t) = U(-t) \star_{\sigma,\alpha,\beta} A(0) \star_{\sigma,\alpha,\beta} U(t) \iff \frac{\partial A}{\partial t}(t) - [\![A(t), H]\!] = 0.$$

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Both presented approaches to the time development yield equal predictions concerning the results of measurements:

$$\langle A(0) \rangle_{\rho(t)} = \langle A(t) \rangle_{\rho(0)} .$$

The end
of
Part 1