Zad1.pdf

Fizyka Statystyczna A, 2023/2024
Zadania domowe seria 1

Termin oddania: 10 listopada, godzina 10.00 w sali 0.03 (po wykładzie)

1 Zadanie 1

Rozważmy układ dla którego spełnione jest równanie stanu $f(p, V, T) = 0$ (dla pewnej funkcji f). Pokazać, że
a)

$$\left(\frac{\partial V}{\partial p}\right)_T = \frac{1}{\left(\frac{\partial p}{\partial V}\right)_T};$$

b)

$$\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_T = -1.$$

2 Zadanie 2

Jeden mol jednoatomowego gazu doskonałego poddany jest procesowi cyklicznemu złożonemu z dwóch izobar p_1 i $p_2 = 2p_1$ i dwóch izochor V_1 i $V_2 = 2V_1$.

1. Naszkicuj przebieg procesu we współrzędnych (p, V)
2. Ile wynosi ciepło pobrane, a ile ciepło oddane przez gaz w trakcie jednego cyklu powyższego procesu? Molowe ciepło właściwe jednoatomowego gazu doskonałego $c_v = 3/2R$.
3. Ile wynosi praca wykonana przez gaz w powyższym procesie?
4. Jaka jest sprawność η tego procesu jako silnika? (Jak zmieniłby się wzór dla przypadku lodówki i pompy ciepła?)

3 Zadanie 3

Pokazać, ze jeśli ciepło własciwe c_x jest stałe w trakcie transformacji równowagowej jednego mola gazu doskonałego przy ustalonym parametrze $x(p, V) = const$, to równanie procesu ma postać:

$$pV^n = const$$

oraz wyznaczyć n. Jest to tzw. proces politropowy.
Hint: Użyj I zasady termodynamiki i wzoru na energię dla gazu doskonałego $dU = c_v dT$.

4 Zadanie 4

Równanie stanu gazu van der Waalsa dane jest przez

$$p = \frac{NRT}{V - Nb} - a\frac{N^2}{V^2},$$

Zadanie 1.

$$f(p, V, T) = 0 \qquad \left(\frac{\partial V}{\partial p}\right)_T$$

1) $$df = \left(\frac{\partial f}{\partial p}\right) dp + \left(\frac{\partial f}{\partial V}\right) dV + \left(\frac{\partial f}{\partial T}\right) dT = 0$$

$$\begin{cases} dp = \left(\frac{\partial p}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial T}\right)_V dT \\ dV = \left(\frac{\partial V}{\partial p}\right)_T dp + \left(\frac{\partial V}{\partial T}\right)_p dT \\ dT = \left(\frac{\partial T}{\partial p}\right)_V dp + \left(\frac{\partial T}{\partial V}\right)_p dV \end{cases}$$

$$\cancel{dV} = \left(\frac{\partial V}{\partial p}\right)_T \left[\left(\frac{\partial p}{\partial V}\right)_T \cancel{dV} + \left(\frac{\partial p}{\partial T}\right)_V \underset{\text{0, bo } T=\text{const}}{dT} \right] + \left(\frac{\partial V}{\partial T}\right)_p \underset{\to \text{0, bo } T=\text{const}}{dT}$$

$$\frac{1}{\left(\frac{\partial p}{\partial V}\right)_T} = \left(\frac{\partial V}{\partial p}\right)_T$$

2) $$df = \left(\frac{\partial f}{\partial p}\right) dp + \left(\frac{\partial f}{\partial V}\right) dV + \left(\frac{\partial f}{\partial T}\right) dT = 0$$

$$-\left(\frac{\partial f}{\partial p}\right) dp = \left(\frac{\partial f}{\partial V}\right) dV + \left(\frac{\partial f}{\partial T}\right) dT$$

$$-dp = \underbrace{\left(\frac{\partial p}{\partial f}\right)\left(\frac{\partial f}{\partial V}\right)}_{= -\left(\frac{\partial p}{\partial V}\right)_T} dV + \underbrace{\left(\frac{\partial p}{\partial f}\right)\left(\frac{\partial f}{\partial T}\right)}_{= -\left(\frac{\partial p}{\partial T}\right)_V} dT$$

$$\left(\frac{\partial p}{\partial V}\right)_T = -\left(\frac{\partial p}{\partial f}\right)\left(\frac{\partial f}{\partial V}\right)$$

$$\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_T = (-1)^3 \left[\cancel{\left(\frac{\partial p}{\partial f}\right)} \cancel{\left(\frac{\partial f}{\partial T}\right)} \cancel{\left(\frac{\partial T}{\partial f}\right)} \cancel{\left(\frac{\partial f}{\partial V}\right)} \cancel{\left(\frac{\partial V}{\partial f}\right)} \cancel{\left(\frac{\partial f}{\partial p}\right)} \right] = -1$$

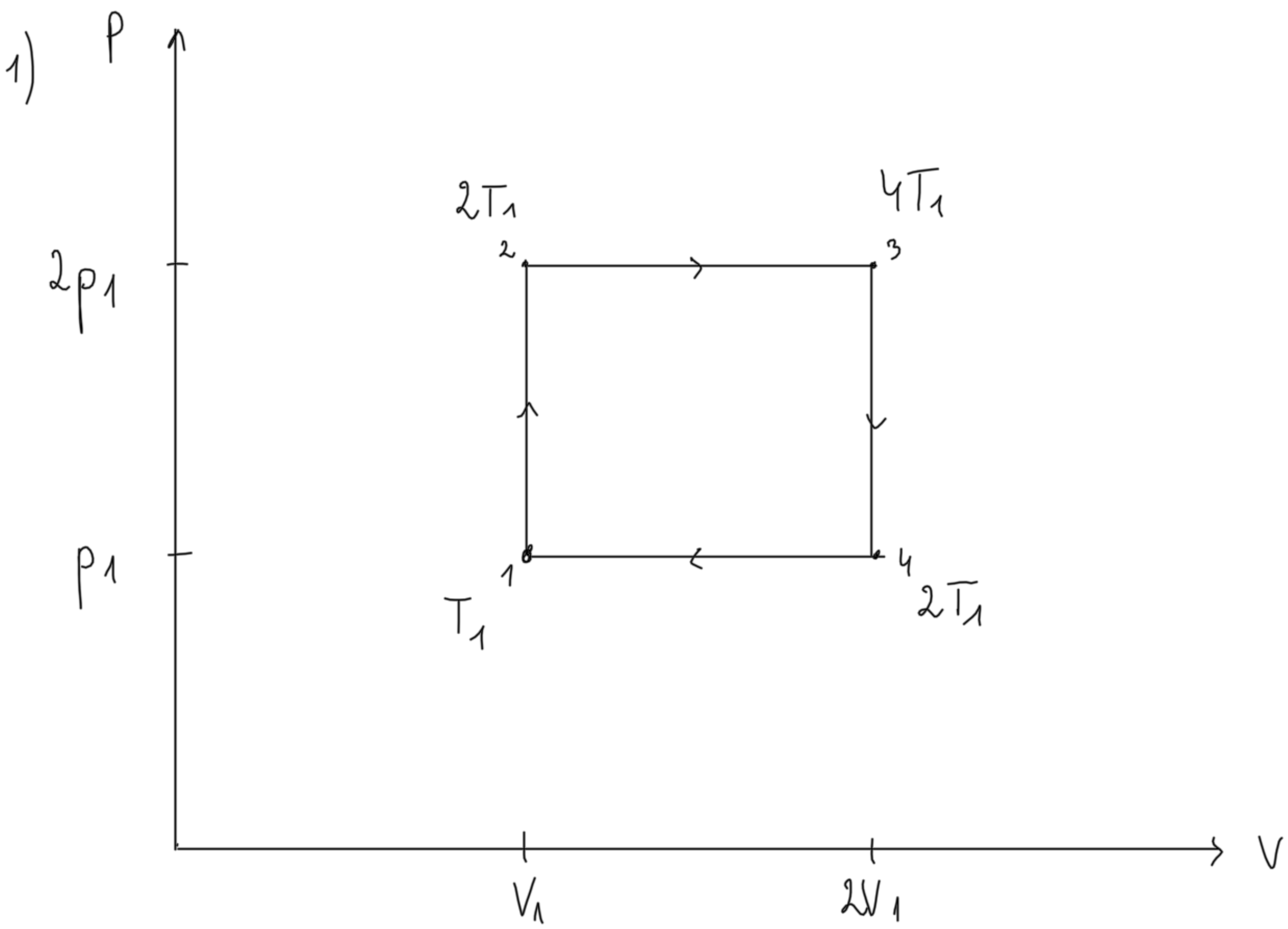
$$\left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial V}{\partial p}\right)_T = -1$$

Zadanie 2.

2 izobary : p_1, $p_2 = 2p_1$

2 izochory : V_1, $V_2 = 2V_1$

1)



2) $c_v = \frac{3}{2} R$

$\Delta U = 0$

$đW = -pdV$

$dU = đQ + đW$

$Q_{in}, Q_{out} = ?$

gaz dosk. $\rightarrow pV = nRT \xrightarrow{1 \text{ mol}} pV = RT$

$p_1 V_1 = RT_1$

$$dQ = dU + pdV \quad \underset{pV}{\overset{nRT}{\|}}$$

$$dQ = C_V dT + d(pV) - Vdp$$

$$dQ = C_V dT + R dT - V dp$$

$$\left(\frac{\partial Q}{\partial T}\right)_p = C_V + R = C_p = \frac{3}{2}R + R = \frac{5}{2}R$$

$$Q_{12} = C_V \Delta T = C_V T_1 = \frac{3}{2}R \cdot \frac{p_1 V_1}{R} = \frac{3}{2} p_1 V_1$$

$$Q_{23} = C_p \Delta T = C_p \cdot 2T_1 = \frac{5}{2}R \cdot 2\frac{p_1 V_1}{R} = 5 p_1 V_1$$

$$Q_{34} = C_V \Delta T = C_V \cdot (-2T_1) = \frac{3}{2}R \cdot \left(-2\frac{p_1 V_1}{R}\right) = -3 p_1 V_1$$

$$Q_{41} = C_p \Delta T = C_p \cdot (-T_1) = \frac{5}{2}R\left(-\frac{p_1 V_1}{R}\right) = -\frac{5}{2} p_1 V_1$$

$$Q_{pob} = Q_{12} + Q_{23} = \frac{13}{2} p_1 V_1$$

$$Q_{odd} = -\frac{11}{2} p_1 V_1$$

$\Delta W_{12} = 0 \qquad \Delta W_{34} = 0$

3) $\Delta W_{23} = -p\Delta V = -2p_1(2V_1 - V_1) = -2p_1 V_1$

$\Delta W_{41} = -p\Delta V = -p_1(V_1 - 2V_1) = p_1 V_1$ ← praca wykonana przez gaz

4) $W_{całk} = \Delta W_{23} + \Delta W_{41} = -p_1 V_1$

$$\eta = \frac{|W_{całk}|}{Q_{pob}} = \frac{p_1V_1}{\frac{13}{2}p_1V_1} = \frac{2}{13}$$

lodówka: $\eta = \frac{|Q_{odd}|}{|W_{całk}|} = \frac{\frac{11}{2}p_1V_1}{p_1V_1} = \frac{11}{2}$

pompa ciepła: $\eta = \frac{|Q_{pob}|}{|W_{całk}|} = \frac{\frac{13}{2}p_1V_1}{p_1V_1} = \frac{13}{2}$

Zadanie 3.

$dU = C_V dT \Rightarrow_{n=1}$ $\quad dU = đQ + đW$ $\qquad$ gaz doskonały

$c_x = \left(\frac{\partial Q}{\partial T}\right)_x = \text{const.}$ $\qquad dU = đQ - pdV$

$$đQ = dU + pdV$$

$$c_x\, dT = C_V dT + pdV$$

$$c_x\, dT - C_V dT - pdV = 0$$

$$(C_x - C_V)\, dT - pdV = 0$$

$$pV = nRT \Rightarrow dT = \frac{d(pV)}{R}$$

$(C_x - C_V)\frac{d(pV)}{R} - pdV = 0$ $\qquad R = C_p - C_V$ (wyprowadzone w poprzednim zadaniu)

$$\frac{(C_x - C_V)}{(C_p - C_V)}(Vdp + pdV) - pdV = 0$$

$$\frac{(C_x - C_V)}{(C_p - C_V)}Vdp + \left(\frac{C_x - C_V}{C_p - C_V} - 1\right)pdV = 0$$

$$V dp + \left(1 - \frac{C_p - C_v}{C_x - C_v}\right) p dV = 0 \quad / : Vp$$

$$\frac{dp}{p} + \left(1 - \frac{C_p - C_v}{C_x - C_v}\right) \frac{dV}{V} = 0$$

$$\ln p + \ln V^{\left(1 - \frac{C_p - C_v}{C_x - C_v}\right)} = const.$$

$$\ln \left(p V^{\left(1 - \frac{C_p - C_v}{C_x - C_v}\right)} \right) = const.$$

$$p V^{\left(1 - \frac{C_p - C_v}{C_x - C_v}\right)} = const. \quad \Rightarrow \quad \underbrace{n = 1 - \frac{C_p - C_v}{C_x - C_v}}$$

Zadanie 4.

$$p = \frac{NRT}{V - Nb} - a \frac{N^2}{V^2} \qquad a, b = const.$$

$$C_v = N T^2 \qquad U(T, V, N),\ S(T, V, N) = ?$$

$$dU = \text{đ}Q - p dV = T dS - p dV \qquad C_v = \left(\frac{\partial \overset{dU + p dV \,\leftarrow 0 \text{ dla } V = const.}{Q}}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV \qquad dS = \frac{\text{đ}Q}{T}$$

$$dS = \frac{dU + p dV}{T}$$

z ćwiczeń: $\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial p}{\partial T}\right)_V - p$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left[T \left(\frac{\partial p}{\partial T}\right)_V - p \right] dV$$

$$dU = C_v dT + \left[T \frac{\partial}{\partial T} \left(\frac{NRT}{V - Nb} - a \frac{N^2}{V^2} \right)_V - \frac{NRT}{V - Nb} + a \frac{N^2}{V^2} \right] dV$$

$$dU = C_v dT + \left[T \frac{NR}{V - Nb} - \frac{NRT}{V - Nb} + a \frac{N^2}{V^2} \right] dV$$

$$dU = C_V dT + a\frac{N^2}{V^2}\,dV = NT^2 dT + a\frac{N^2}{V^2}\,dV$$

$$U = \int NT^2 dT + \int a\frac{N^2}{V^2}\,dV = \underline{\frac{1}{3}NT^3 - \frac{aN^2}{V} + \overset{\text{const.}}{U_0}}$$

$$dS = \frac{dU}{T} + \frac{p\,dV}{T} = \frac{NT^2 dT}{T} + \frac{aN^2}{TV^2}\,dV + \frac{p}{T}\,dV$$

$$dS = NT\,dT + \left(\cancel{\frac{aN^2}{TV^2}} + \frac{NR}{V-Nb} - \cancel{a\frac{N^2}{TV^2}}\right)dV$$

$$dS = NT\,dT + \frac{NR}{V-Nb}\,dV$$

$$\underline{S = \frac{1}{2}NT^2 + NR\ln(V-Nb) + \overset{\text{const.}}{S_0}}$$

Zadanie 5. średnia i wariancja

a) rozkład jednorodny

$$\begin{cases} p(x) = \frac{1}{2a} & x\ (-a, a) \\ p(x) = 0 & \text{w przeciwnym wypadku} \end{cases}$$

$$\langle x \rangle = \int_{-a}^{a} \frac{1}{2a}\,x\,dx = \frac{1}{2a}\cdot\frac{1}{2}x^2\Big|_{-a}^{a} = \frac{1}{4a}\left(a^2 - (-a)^2\right) = \underline{\underline{0}}$$

$$\operatorname{var} x = \int_{-a}^{a}\frac{1}{2a}x^2\,dx - \left(\int_{-a}^{a}\frac{1}{2a}x\,dx\right)^2 = \frac{1}{2a}\cdot\frac{1}{3}x^3\Big|_{-a}^{a} =$$

$$= \frac{1}{6a}\left(a^3 - (-a)^3\right) = \frac{1}{6a}\cdot 2a^3 = \underline{\underline{\frac{a^2}{3}}}$$

b) r. Laplace'a

$$p(x) = \frac{1}{2a} e^{-\frac{|x|}{a}} \qquad x \in \mathbb{R}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \frac{1}{2a} e^{-\frac{|x|}{a}} x\,dx = \int_{-\infty}^{0} \frac{1}{2a} e^{\frac{x}{a}} x\,dx + \int_{0}^{\infty} \frac{1}{2a} e^{-\frac{x}{a}} x\,dx =$$

$$= \int_{0}^{\infty} \frac{1}{2a} e^{-\frac{x}{a}} (-x)\,dx + \int_{0}^{\infty} \frac{1}{2a} e^{-\frac{x}{a}} x\,dx = \underline{\underline{0}}$$

$$\text{var } X = \int_{-\infty}^{\infty} \frac{1}{2a} e^{-\frac{|x|}{a}} x^2\,dx = \int_{-\infty}^{0} \frac{1}{2a} e^{\frac{x}{a}} x^2\,dx + \int_{0}^{\infty} \frac{1}{2a} e^{-\frac{x}{a}} x^2\,dx =$$

$$= \frac{1}{2a} \left(\int_{0}^{\infty} e^{-\frac{x}{a}} x^2\,dx + \int_{0}^{\infty} e^{-\frac{x}{a}} x^2\,dx \right) = \frac{1}{a} \int_{0}^{\infty} e^{-\frac{x}{a}} x^2\,dx =$$

$$= \underline{\underline{2a^2}}$$

c) r. Maxwella

$$p(x) = \sqrt{\frac{2}{\pi}} \frac{x^2}{a^3} e^{-\frac{x^2}{2a^2}} \qquad x \geqslant 0$$

$$\langle x \rangle = \int_{0}^{\infty} \sqrt{\frac{2}{\pi}} \frac{x^2}{a^3} e^{-\frac{x^2}{2a^2}} x\,dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^3} \int_{0}^{\infty} x^3 e^{-\frac{x^2}{2a^2}} dx =$$

$$= \left\{ y = \frac{x^2}{2a^2} \qquad dy = \frac{1}{\cancel{2}a^2} \cdot \cancel{2} x\,dx \Rightarrow dx = \frac{dy \cdot a^2}{x} \right\}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^3} \int_{0}^{\infty} \underbrace{x^{\cancel{3}\,2}}_{2a^2 y} \cdot e^{-y} \cdot \frac{dy \cdot a^2}{\cancel{x}} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^3} \int_{0}^{\infty} 2a^4 y e^{-y} dy =$$

$$= \sqrt{\frac{2}{\pi}} \cdot 2a \int_0^\infty y e^{-y}\, dy = a \cdot 2\sqrt{\frac{2}{\pi}} \left[\left(-y e^{-y}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-y}\right) dy \right] =$$

$$= 2a\sqrt{\frac{2}{\pi}} \left(-e^{-y}\right)\Big|_0^\infty = \underline{\underline{2\sqrt{\frac{2}{\pi}}\, a}}$$

$$\operatorname{var} X = \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{x^2}{a^3} e^{-\frac{x^2}{2a^2}} x^2\, dx - \left(2\sqrt{\frac{2}{\pi}}\, a\right)^2 = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a^3} \int_0^\infty x^4 e^{-\frac{x^2}{2a^2}}\, dx -$$

$$- \frac{8a^2}{\pi} \overset{\text{wolfram alpha}}{=} \cancel{\sqrt{\frac{2}{\pi}}} \cdot 3 \cdot \cancel{\sqrt{\frac{\pi}{2}}}\, \cancel{\sqrt{\frac{1}{a^2}}} \cdot a^{\cancel{5}\,2} - \frac{8a^2}{\pi} = 3a^2 - \frac{8a^2}{\pi} = \underline{\underline{a^2\left(3 - \frac{8}{\pi}\right)}}$$

Zadanie 6.

$$p = \frac{n - i + 1}{n}$$

$$E = \sum_i \frac{1}{p_i} = \sum_i \frac{n}{n - i + 1} = \sum_{i=1}^{6} \frac{6}{7 - i}$$

$$E = 1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = \underline{\underline{14{,}7}}$$