Effective field theory approach to the ferromagnetic phase transition in the gas of fermions

P. Chankowski

Faculty of Physics, University of Warsaw

Based on:
Phys.Rev. **B104** 144425, (2021), arXiv:2108.00793 [cond-mat.quant-gas],
Acta.Phys.Pol. **B53** 9-A4, (2022), arXiv:2206.05076 [cond-mat.quant-gas],
Phys.Rev. **A107** (2023), 063311, arXiv:2303.09921 [cond-mat.quant-gas]
Acta.Phys.Pol. **B54** 11-A4, (2023), arXiv:2309.14782 [cond-mat.quant-gas],
Phys.Rev. **A** in press, arXiv:2510.22719 [cond-mat.quant-gas]
papers written with J. Wojtkiewicz, R. Bakshizada, S. Augustynowicz and O. Grocholski (IRFU, CEA, Université Paris-Saclay)

Stoner phase transition

A classic problem of many-body quantum mechanics and statistical physics is the gas of spin 1/2 fermions interacting through a short-range, repulsive two-body, spin independent potential.

Sufficiently strong such an interaction, cooperating with the Pauli exclusion principle, can induce spontaneous polarization P (magnetization) - this is called the Stoner phase transition.

Mechanism: having more fermions with the same spin projection decreases the interaction energy because same spin fermions (almost) do not interact - due to the Pauli exclusion the s-wave interaction/scattering is impossible. If the interaction energy is large - i.e. the interactions is sufficiently strong - decreasing the number of interacting pairs may be more important than the increase of the kinetic energy (increase of the Fermi level of one of the spin projections) generated by having unequal Fermi levels.

Two classes of theoretical models in which the effect can be studied

- Hubbard models - fermions on periodic lattices (a positive energy cost of having two fermions on the same sites and a hopping amplitude(s) as the basic model parameters)
- fermions moving freely in a fixed volume V and interacting by a short-range two-body potential $V_{\rm pot}(|\mathbf{r}_i - \mathbf{r}_j|)$

- itinerant ferromagnetism of some metals (Fe, Ni, Co) expected to result from the spin independent (screened) repulsive electrostatic interactions of electrons: $T_{\rm Curie} \sim$ electrostatic energy $\gg$ energy of spin-spin interactions
- the effect is studied in experiments with cold gases of fermionic atoms (in traps) - experimentally it is possible to tune the interaction strength (the scattering length) by exploiting the physics of the Feshbach resonance. More about this later.

Gas of interacting spin 1/2 fermions

Of interest are various thermodynamic characteristics: free energy, polarization (magnetization), heat capacity, susceptibility and the character of the phase transition (if it occurs).

The simplest is the case of $T = 0$ - the problem reduces then to finding the ground state energy density $E(P)/V$ of the system of $N = N_+ + N_-$ fermions as a function of $P = (N_+ - N_-)/N$, i.e. (in the traditional language) to solving $H\Psi = E\Psi$ with

$$\Psi(\dots, \mathbf{r}_i, \sigma_i, \dots, \mathbf{r}_j, \sigma_j, \dots) = -\Psi(\dots, \mathbf{r}_j, \sigma_j, \dots, \mathbf{r}_i, \sigma_i, \dots)\,,$$

and the Hamiltonian

$$H = H_0 + V_{\rm int} = -\frac{\hbar^2}{2m_f}\sum_{i=1}^{N} \nabla_i^2 + \frac{1}{2}\sum_{i\neq j} V_{\rm pot}(|\mathbf{r}_i - \mathbf{r}_j|)\,,$$

with some $V_{\rm pot}(|\mathbf{r}|)$, e.g. the hard sphere one or a more realistic one.

2^{nd} quantization language

The problem can be conveniently formulated in the 2^{nd} quantization (i.e. in the field theory) language

$$H_0 = -\frac{\hbar^2}{2m_f}\int d^3\mathbf{r}\sum_\sigma \hat\psi^\dagger_\sigma(\mathbf{r})\boldsymbol{\nabla}^2\hat\psi_\sigma(\mathbf{r}) = \sum_{\mathbf{p},\sigma=\pm}\frac{\hbar^2\mathbf{p}^2}{2m_f}\,a^\dagger_{\mathbf{p},\sigma}a_{\mathbf{p},\sigma}\,,$$

$$V_{\rm int} = \frac{1}{2}\int d^3\mathbf{r}\int d^3\mathbf{r}'\,\hat\psi^\dagger_\sigma(\mathbf{r})\hat\psi^\dagger_{\sigma'}(\mathbf{r}')\,V_{\rm pot}(|\mathbf{r}-\mathbf{r}'|)\,\hat\psi_{\sigma'}(\mathbf{r}')\hat\psi_\sigma(\mathbf{r})$$
$$= \frac{1}{2V}\sum_{\mathbf{q}}\sum_{\mathbf{p},\sigma}\sum_{\mathbf{p}',\sigma'} a^\dagger_{\mathbf{p}+\mathbf{q},\sigma}a^\dagger_{\mathbf{p}'-\mathbf{q},\sigma'}\,\tilde V_{\rm pot}(|\mathbf{q}|)\,a_{\mathbf{p}',\sigma'}a_{\mathbf{p},\sigma}\,.$$

p - with periodic boundary cond's in $V = L^3$ (ultimately the TMD limit is to be taken). $\hat\psi(\mathbf{r}) = \int d^3\mathbf{p}/(2\pi)^3 e^{i\mathbf{p}\cdot\mathbf{r}}a_\sigma(\mathbf{p})$.

Parametrization

$\tilde{V}_{\rm pot}(|\mathbf{q}|)$ of the hard sphere potential is ill defined. It is better ("because such is the power of tradition") to trade $V_{\rm pot}$ for some observables, e.g. scattering length(s) a_0, $a_1, \ldots$ and effective ranges $r_0, r_1, \ldots$. of the elastic fermion-fermion scattering.
Usual justification: properties of the diluted gas (at very low T) should be mainly determined by rare binary periferic collisions of (almost free) low energy particles.
If $V_{\rm pot}$ is characterized by a length scale R, one expects $a_\ell \sim r_\ell \sim R$ and can, therefore, try to compute E/V (or other thermodynamic quantities) as a series in $\rho^{1/3}R$ or $k_{\rm F}R$

$$k_{\rm F} = \left(\frac{6\pi^2}{g_s}\frac{N}{V}\right)^{1/3} = \left(\frac{6\pi^2}{g_s}\rho\right)^{1/3}, \qquad g_s = 2s + 1\,.$$

Problematic: the interaction must be strong (naive perturbative expansions may not be applicable).

In the 50' of the 20$^{\rm th}$ c. T.D. Lee, C.N. Yang & K. Huang computed, using the 'pseudopotential' method (see Kesio's *Statistical Mechanics*), the first order corrections to the H spectrum of interacting (through the hard sphere $V_{\rm pot}$) spin 1/2 fermions. In particular

$$\frac{E(P)}{V} = \frac{3}{5}\frac{k_{\rm F}^3}{6\pi^2}\frac{\hbar^2 k_{\rm F}^2}{2m_f}\left\{(1+P)^{5/3} + (1-P)^{5/3} + \frac{20}{9\pi}(k_{\rm F}\,a_0)\,(1-P^2)\right\}.$$

In this (mean field) approximation also the free energy $F(T, V, \mathcal{H}, N_+, N_-)$ of the system in the external magnetic field $\mathcal{H}$ can be obtained. At low T ($k_{\rm B}T \ll \varepsilon_{\rm F} \equiv \hbar^2 k_{\rm F}^2/2m_f$) spontaneous polarization/magnetization appears (the global minimum of F starts to continously move from $P = 0$ to $P \neq 0$) when

$$k_{\rm F}\,a_0 \geq \frac{\pi}{2}\left[1 + \frac{\pi^2}{12}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F}}\right)^2 + \dots\right] \qquad (\textit{Stoner's condition})$$

The phase transition turns out to be continuous (it constitutes a textbook example of such a transition):

History cont'd

In the 60', 70' of the 20^{th} c. computations of E/V for $P = 0$ became for Many-Body Quantum Mechanics like $g - 2$ for QED. Carried out by several people (De Dominicis, Martin, Efimov, Amusia, Bishop, Baker) using the methods based on Green's functions of many-body QM with general interaction $V_{\rm pot}(\mathbf{r}_i - \mathbf{r}_j)$ and reexpressing the result (a hard part) in terms of the scattering lengths etc. (see the Fetter & Walecka textbook). E/V (for $P = 0$) has been obtained in this way first up to $(k_{\rm F}R)^2$ (i.e. $(k_{\rm F}a_0)^2$) and later up to $(k_{\rm F}R)^3$ (i.e. $(k_{\rm F}a_0)^3$, $k_{\rm F}^3 a_1^3$, $k_{\rm F}^3 a_0^2 r_0$).

In 1970 Kanno in a little noticed paper has obtained the analytic formula for the order $(k_{\rm F}a_0)^2$ contribution to E/V for arbitrary P assuming the hard-spheres interaction potential.

In 2000 Furnstahl & Hammer proposed a much simpler approach exploiting the effective field theory (EFT) method and using it easily rederived the known (up to $(k_{\rm F}R)^3$) result for E/V ($P=0$).

In 2005 Duine and McDonald have computed (implicitly using the EFT method combined with the oldfashioned thermal perturbation theory) the temperature dependent free energy up to $(k_{\rm F}a_0)^2$ and analysed the phase transion in this approximation. At low T it is (as a function of $k_{\rm F}a_0$) first order (will show it later).

In 2020 the complete order $(k_F R)^4$ contribution to E/V for $P=0$ has been computed making the full use of the EFT approach by Wellenhofer, Drishler and Schwenk

Our story begins...

In 2021 we (being unaware of the Kanno's and other results) have used the EFT to compute $E(P)/V$ for $s = 1/2$ at the second order reproducing (semianalytically) the Kanno's result. This approach makes it clear that it is universally valid (not only for hard spheres interaction).

Then we have obtained from L.He a series of his papers ($\sim$ 2012) in which he, using the EFT method, performed an all orders resummation of a class of contributions (proportional to $(k_{\rm F} a_0)^n$) to $E(P)/V$ and claimed that the phase transition (at $T = 0$) is continuous. He also claims his predictions (also concerning the critical value of $k_{\rm F} a_0$) are in good agreement with numerical results (Quantum Monte Carlo method) of Pilati et al. (obtained for concrete interaction potentials)

To be competitive we have set ourselves to compute the complete order $(k_{\rm F} R)^3$ correction to $E(P)/V$ using the EFT...

Replace the (spatially) nonlocal interaction $V_{\rm pot}(|\mathbf{r}_i - \mathbf{r}_j|)$ by an infinite series of local interactions of decrasing length dimensions:

$$V^{\rm eff}_{\rm int} = V^{(C_0)}_{\rm int} + (V^{(C_2)}_{\rm int} + V^{(C_2)\dagger}_{\rm int}) + V^{(C_2')}_{\rm int} + \dots$$

$$V^{(C_0)}_{\rm int} = C_0 \sum_{\sigma'<\sigma} \int d^3\mathbf{r}\, \hat{\psi}^\dagger_{\sigma'}\hat{\psi}_{\sigma'}\hat{\psi}^\dagger_{\sigma}\hat{\psi}_{\sigma}\,,$$

$$V^{(C_2)}_{\rm int} = -\frac{C_2}{8} \sum_{\sigma'<\sigma} \int d^3\mathbf{r}\, \hat{\psi}^\dagger_{\sigma}\hat{\psi}^\dagger_{\sigma'}(\hat{\psi}_{\sigma'}\boldsymbol{\nabla}^2\hat{\psi}_{\sigma} - 2\boldsymbol{\nabla}\hat{\psi}_{\sigma'}\cdot\boldsymbol{\nabla}\hat{\psi}_{\sigma} + \boldsymbol{\nabla}^2\hat{\psi}_{\sigma'}\hat{\psi}_{\sigma})\,,$$

$$V^{(C_2')}_{\rm int} = -\frac{C_2'}{4} \sum_{\sigma'<\sigma} \int d^3\mathbf{r}\, (\boldsymbol{\nabla}\hat{\psi}^\dagger_{\sigma}\hat{\psi}^\dagger_{\sigma'} - \hat{\psi}^\dagger_{\sigma}\boldsymbol{\nabla}\hat{\psi}^\dagger_{\sigma'})\cdot(\boldsymbol{\nabla}\hat{\psi}_{\sigma'}\hat{\psi}^\dagger_{\sigma} - \hat{\psi}_{\sigma'}\boldsymbol{\nabla}\hat{\psi}^\dagger_{\sigma})\,.$$

Organizing principles: same symmetries as the "fundamental" interaction and the power counting rules (limit the number of terms contributing at a given order) which assume that the "fundamental" interaction $V_{\rm pot}(|\mathbf{r}|)$ is characterized by a single scale R.

Feynman diagrams (i.e. the Dyson expansion; used also in the past with the original nonlocal $V_{\rm int}$ but is then more complicated): the form of the propagator depends on the ground state of H_0 around which one expands: if one deals with $N = N_+ + N_-$ fermions, the ground state is $|0_{N_+N_-}\rangle$ characterized by $p_{\rm F+}$ and $p_{\rm F-}$ and

$$i\tilde{G}_{\pm\pm}(\omega, \mathbf{k}) = i\left[\frac{\theta(|\mathbf{k}| - p_{\rm F\pm})}{\omega - \omega_{\mathbf{k}} + i0} + \frac{\theta(p_{\rm F\pm} - |\mathbf{k}|)}{\omega - \omega_{\mathbf{k}} - i0}\right]$$

with $\omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m_f$ (propagation of "particles" and "holes"). If one considers scattering, the ground state is $|0_{00}\rangle$ and the

$$i\tilde{G}_{\pm\pm}(\omega, \mathbf{k}) = \frac{i}{\omega - \omega_{\mathbf{k}} + i0}$$

(no antiparticles).

The EFT approach: power counting rules

"Fundamental scale" R characterizing $V_{\rm pot}(|\mathbf{r}|)$
the scale of the problem: $|\mathbf{k}|$ for a scattering problem or $k_{\rm F}$ for E/V computation
Contribution of a diagram to an amplitude is of order $|\mathbf{k}|^\nu$ or $k_{\rm F}^\nu$, where

$$\nu = 2 + 3L + \sum_i V_i(d_i - 2)\,,$$

L the number of loops in the graph,
V_i the number of vertices of type i with d_i derivatives in the graph
Coupling C_i of the vertex of type i

$$C_i \sim \frac{4\pi\hbar^2}{m_f}\,R^{-\Delta_i}\,,$$

$\Delta_i = 5 - d_i - (3/2)n_i$ (n_i - number of lines in the vertex of type i)

The EFT approach

Locality of $V_{\text{int}}^{\text{eff}}$ implies UV divergences (absent in the original theory). After regularization (by an UV cutoff Λ or DIMREG) they are eliminated by trading the "bare" (cutoff dependent) couplings C_0, C_2, C_2', $\dots$ for some measurable quantities for which one takes the scattering lenghts a_0, $a_1, \dots$ and the effective ranges $r_0, \dots$

Once C_0, C_2, $C_2', \dots$ are expressed in terms of a_0, $a_1, \dots$, etc., one can compute (employing the same regularization) other pysical quantities like E/V or (later) temperature dependent free energy F - all UV divergences will cancel out - this is the essence of renormalization. (No "field renormalization" is needed - "field renormalization" is a fake also in relativistic theories...)

One could introduce "renormalized" couplings C_{0R}, etc. (similar to α_s of QCD) and parametrize observables like $\mathcal{A}$, E/V, through them but ultimately one wants to express one observable through other one(s). So, parametrizing E/V through a_0 etc. need not be related to our (not clear) intuition how energy of the gas is determined by two-body collisions...

Computing the elastic scattering amplitude with EFT

The standard formula is[1]

$$\langle \mathbf{k}_1'\sigma_1', \mathbf{k}_2'\sigma_2' | \mathrm{T}\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\, \hat{V}^I_{\rm int}(t)\right) | \mathbf{k}_1\sigma_1, \mathbf{k}_2\sigma_2\rangle$$
$$\equiv \delta_{ab} - \frac{i}{\hbar}(2\pi)^4\delta^{(4)}(k_1' + k_2' - k_1 - k_2)\,\mathcal{A}\,,$$

("four-vector" notation: $k^0 \equiv \omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m_f\,,$
$\hat{V}^I_{\rm int}(t) = e^{iH_0t/\hbar}\hat{V}_{\rm int}e^{-iH_0t/\hbar}$).
In the CMS: $\mathbf{k}_1 = \mathbf{k} = -\mathbf{k}_2$. Taking $\sigma = +$, $\sigma' = -$ allows to extract the QM textbook scattering amplitude $f(|\mathbf{k}|, \theta)$

$$f(|\mathbf{k}|, \theta) = -\frac{m_{\rm red}}{2\pi\hbar^2}\,\mathcal{A}(\theta, C_0(\Lambda), \Lambda)\,. \qquad (m_{\rm red} = m_f/2)$$

[1] It is based on strong assumptions - usually not satisfied in relativistic theories but here they are right!

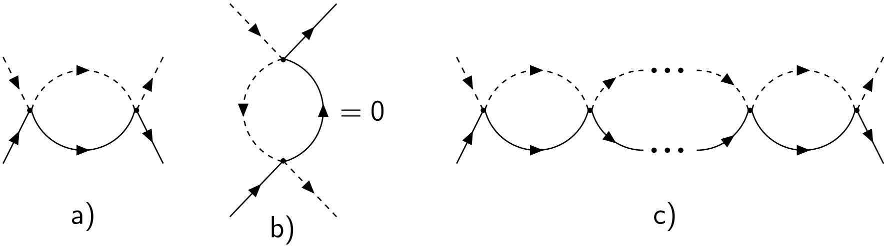

Figure: Elastic scattering of two fermions having different spin projections within the EFT - contributions of $V_{\rm int}^{(C_0)}$ (time flows from the left to the right). Diagram *a*) has $\nu = 1$. Diagram *b*) as well as all "self" interaction diagrams modifying the dispersion relation $\omega_{\mathbf{k}} = \hbar\mathbf{k}^2/2m_f$, vanish owing to the lack of antiparticles and normal ordering (this is why the standard formula for the S-matrix works). Diagram with a single vertex $\propto C_2$ or C_2' ($L = 0$) have $\nu = 2$ and contribute only in higher orders in k.

Computing the scattering amplitude: the main result

Taking only $V_{\rm int}^{(C_0)}$ into account one gets (computation is easier than in the relativistic case - all frequency integrals easily taken directly by the residue method) one gets

$$\mathcal{A} = C_0\left\{1 + \left(\frac{C_0}{i\hbar}\right)\left(\frac{m_f}{i\hbar}\,I_0\right) + \left(\frac{C_0}{i\hbar}\right)^2\left(\frac{m}{i\hbar}\,I_0\right)^2 + \dots\right\},$$

(terms proportional to C_2 etc. omited) where

$$I_0 = \int^{\Lambda}\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{1}{\mathbf{q}^2 - \mathbf{k}^2 - i0} = \frac{i}{4\pi}\,k + \frac{1}{2\pi^2}\,\Lambda - \frac{1}{2\pi^2}\,\frac{k^2}{\Lambda} + \dots,$$

Determination of the bare couplings $C_0, \ldots$

Matching $f = -(m_f/4\pi\hbar^2)\mathcal{A}$ onto

$$f(k,\theta) = -a_0\left[1 - ia_0k + \left(\frac{1}{2}\,a_0r_0 - a_0^2\right)k^2 + \ldots\right] - a_1^3k^2\cos\theta + \ldots,$$

obtained by expanding $k\cot\delta_0 = -1/a_0 + \frac{1}{2}\,r_0k^2 + \ldots,$
$k\cot\delta_1 = -3/(k^2a_1^3) + \ldots,$ $k\cot\delta_2 = \ldots$ in the general textbook partial wave expansion formula

$$f(k,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\,\frac{1}{k\cot\delta_\ell - ik}\,P_\ell(\cos\theta)\,,$$

gives

$$C_0 = \frac{4\pi\hbar^2}{m_f}\,a_0\left(1 + \frac{2}{\pi}\,a_0\Lambda + \frac{4}{\pi^2}\,a_0^2\Lambda^2 + \ldots\right) = C_{0R} + \delta C_R^{(1)}(\Lambda) + \ldots$$

Couplings $C_2 \propto a_0^2r_0$, $C_2' \propto a_1^3$ can be determined in the same way.

Basic formula for E/V and the order $k_{\rm F}a_0$ correction

$$e^{-iT(E-E^{(0)})/\hbar} = \langle 0_{N_+N_-}|\mathrm{T}_t \exp\left(-\frac{i}{\hbar}\int_{-T/2}^{T/2} dt\, V^I_{\rm int}(t)\right)|0_{N_+N_-}\rangle$$

$(T\to\infty)$. Corrections $\Delta E/V$ given by momentum space connected vacuum diagrams.

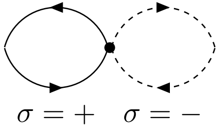

Figure: Order C_0, i.e. $k_{\rm F}a_0$ contribution to E/V. This is UV finite and reproduces the Lee - Yang - Huang result:

$$\frac{E^{(1)}}{V} = C_0\,\frac{p^3_{{\rm F}+}p^3_{{\rm F}-}}{36\pi^4} = \left[\frac{3}{5}\,\frac{k^3_{\rm F}}{6\pi^2}\,\frac{\hbar^2k^2_{\rm F}}{2m_f}\right]\frac{20}{9\pi}\,(k_{\rm F}a_0)\,(1-P^2)$$

The order $(k_{\rm F}a_0)^2$ correction to E/V

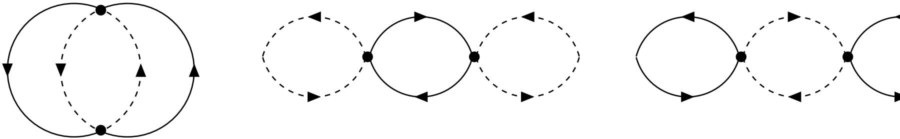

Figure: Order C_0^2 (i.e. $(k_{\rm F}a_0)^2$) contributions to E/V. Only the first diagram is nonzero. It is UV divergent.

$$\frac{E^{(2)}}{V} = \frac{32 m_f C_0^2}{\hbar^2(2\pi)^7}\int_0^{s_{\rm max}} ds\, s^2 \int d^3\mathbf{t}\,\theta(p_{{\rm F}-} - |\mathbf{t}+\mathbf{s}|)\theta(p_{{\rm F}+} - |\mathbf{t}-\mathbf{s}|)\, g(t,s)$$

$g(t,s) = -\Lambda + g_{\rm finite}(t,s) + \mathbf{t}^2/\Lambda + \dots$ obtained analyticaly. The divergence disappears after expressing the bare coupling C_0 the sum $E^{(1)}/V + E^{(2)}/V$ through $k_{\rm F}a_0$ and the cutoff Λ up to the order $(k_{\rm F}a_0)^2$

Ferromagnetic Phase Transition at $T = 0$: 1st and 2nd order results

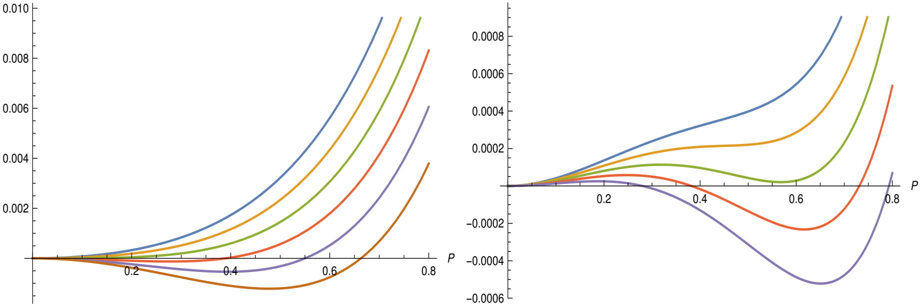

Figure: The difference $(E(P) - E(0))/V$ (in units $(3/5)(k_{\rm F}^3/6\pi^2)\varepsilon_{\rm F}$) as a function of the polarization $P = (N_+ - N_-)/N$ for different values of $k_{\rm F} a_0$. Left: first order approximation - the transition is continuous, $(k_{\rm F} a_0)_{\rm cr} = \pi/2 = 1.57$. Right: second order approximation - the transition is first order, $(k_{\rm F} a_0)_{\rm cr} = 1.054$.

The order $(k_{\mathrm{F}} a_0)^3$ correction to E/V for $P \neq 0$

There are two nonzero 4-loop diagrams contributing in the order C_0^3 to E/V

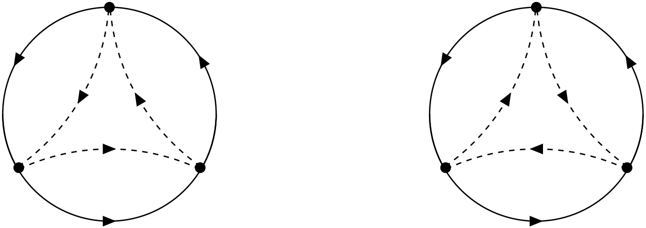

Figure: The particle-particle and the particle-hole ("mercedes-like") and several vanishing (when E/V is concerned, but not if Ω/V) diagrams:

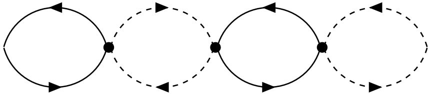

Figure: The "audi"-type, order C_0^3 contribution to $\Omega^{(3)}$.

Order C_0^3 diagrams which do not contribute to E/V

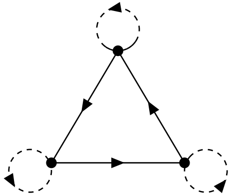

Figure: Order C_0^3, "mitsubishi-type" diagrams

Figure: No-fancy-name diagrams.

Some details of the third order computation

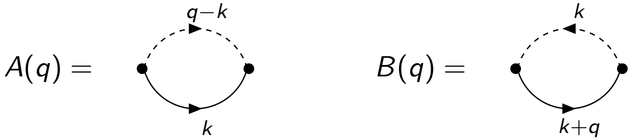

Figure: Two "elementary" one-loop diagrams - out of these blocks the "mercedes-like" and higher order ring diagrams can be composed.

$$A(q) \propto \int_{\mathbf{k}} \left[\frac{\theta^{>+}_{\mathbf{k}-\mathbf{q}}\,\theta^{>-}_{\mathbf{k}}}{q^0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}-\mathbf{q}} + i0} - \frac{\theta^{<+}_{\mathbf{k}-\mathbf{q}}\,\theta^{<-}_{\mathbf{k}}}{q^0 - \omega_{\mathbf{k}} - \omega_{\mathbf{k}-\mathbf{q}} - i0} \right]$$
$$\equiv A_{pp} + A_{hh}$$

$$\frac{E^{(3)pp+ph}}{V} = C_0^3 \int \frac{d^4q}{(2\pi)^4} \left[A_{pp} + A_{hh}\right]^3 + C_0^3 \int \frac{d^4q}{(2\pi)^4} \left[B_{ph} + B_{hp}\right]^3 .$$

Some details of the third order computation cont'd

From the order C_0^3 ("mercedes-like") particle-particle ring one gets $A_{hh}A_{pp}^2$ and $A_{hh}^2 A_{pp}$ and similarly from the particle-hole ring ("mercedes-like").

$$E^{(3)}/V \propto [G^{(1)} + G^{(2)}] + \propto [K^{(1)} + K^{(2)}]$$

$$G^{(1)} \propto \int_0^{s_{\rm max}} ds\, s^2 \int d^2\mathbf{t}\, \theta(p_{\rm F-} - |\mathbf{t}+\mathbf{s}|)\theta(p_{\rm F+} - |\mathbf{t}-\mathbf{s}|)\, [g(t,s)]^2$$

In $G^{(2)}$ opposite thetas and $[h(t,s)]^2$. In $K^{(1)}$ and $K^{(2)}$ mixed thetas products of functions $f_\sigma^{(1,2)}(\mathbf{t}\cdot\mathbf{s}, s)$. These functions have been obtained analytically. Only $G^{(1)}$ UV divergent - all divergences shown analytically to cancel out when bare coupling C_0 gets expressed in terms of a_0 and Λ. The remaining integrals can be done numerically with *Mathematica*

Third order computation cont'd

We have checked that we reproduce the known result for $P = 0$.
We have also computed the contribution to E/V of the operators $V_{\rm int}^{(C_2)}$ and $V_{\rm int}^{(C_2')}$ which are of order $k_{\rm F}^3 a_0^2 r_0$ and $(k_{\rm F} a_1)^3$ thus completing E/V to the $(k_{\rm F} R)^3$ order.
We have also generalized to $s > 1/2$ spins (additional "Mercedes-like" diagrams)

The N-th order *pp*-ring gives contributions $A_{hh}A_{pp}^{N-1}$ ($G^{(1)}$ for $N = 3$), $A_{hh}^2 A_{pp}^{N-2}$ ... $A_{hh}^{N-1}A_{pp}$ ($G^{(2)}$ $N = 3$).
L. He first did the resummation of $A_{hh}A_{pp}^{N-1}$ terms only. In the next paper based on the approach of N. Kaiser he managed to resum the complete $(A_{hh} + A_{pp})^N$ contribution of the *pp* rings.
We have checked that at the third order the contribution of *ph* ring (neglected in the He's resummation) is not much smaller than that of the third order *pp* ring.

We have checked numerically the impact the order $(k_{\rm F} a_0)^3$ terms have on the phase transition (at $T = 0$) to the ordered state:

Ferromagnetic Phase Transition at $T = 0$: 2nd and 3rd order results

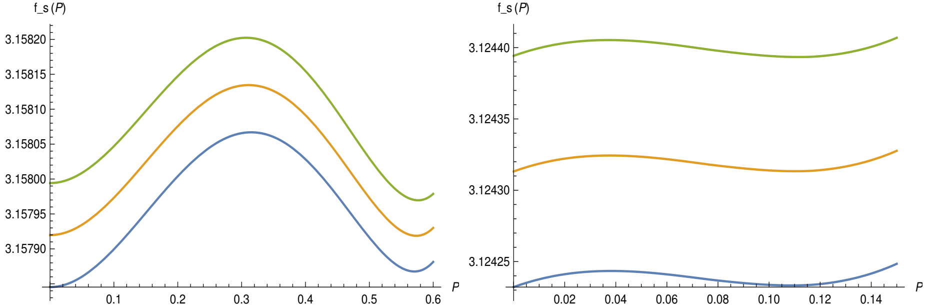

Figure: $E(P)/V$ (in units $(3/5)(k_{\rm F}^3/6\pi^2)\varepsilon_{\rm F}$) as a function of the polarization $P = (N_+ - N_-)/N$ for values of $k_{\rm F} a_0$ close to the transition. Left: 2nd order approximation - the transition is first order, $(k_{\rm F} a_0)_{\rm cr} = 1.054$. Right: 3rd order approximation (for $a_1 = r_0 = 0$) - the transition is very weakly first order, $(k_{\rm F} a_0)_{\rm cr} = 0.991$.

Impact of the order $(k_{\rm F}a_1)^3$ and $k_{\rm F}^3a_0^2r_0$ terms

The Barcelona J^3-team (J. Pera, J. Casulleras, J. Boronat, without R. Lewandowski) has completed the same computation of the $(k_{\rm F}R)^3$ terms slightly later and analysed more thorouhgly the dependence of the phase transition on the a_1 and r_0 parametrs. Concluded that then its first order character is not necessarily weakened. (A concrete "fundamental" potential $V_{\rm pot}(|\mathbf{r}|)$ of course predicts uniquely all a_0, a_1, r_0 etc. E.g. the "hard sphere" potential gives $a_1 = a_0$ and $r_0 = (2/3)a_0$.)

It seemed that the picture clarifies: if only a_0 is included the transition is continuous, as predicted by L. He, but at fixed orders of the perturbative expansion it can only look as weaker and weaker first order one - continuous transition is related to fluctuations at all length scales and seeing it requires some kind of a resummation... So we have set ourselves to compute $F(T, V, N_+, N_-)$ up to $(k_{\rm F}R)^3$...

$T \neq 0$: EFT computation of gas free energy $F(T, V, N_+, N_-)$

Spin preserved by the interaction - natural to introduce μ_+ and μ_- and working within the Grand Canonical Ensemble ($\beta \equiv 1/k_{\rm B}T$)

$$\hat{\rho} = \frac{1}{\Xi_{\rm stat}} e^{-\beta\hat{K}}, \quad \text{with} \quad \hat{K} = \hat{H}_0 - \sum_\sigma \mu_\sigma \hat{N}_\sigma + \hat{V}_{\rm int} \equiv \hat{K}_0 + \hat{V}_{\rm int},$$

compute the statistical sum (encodes complete thermodynamics)

$$\Xi_{\rm stat}(T, V, \mu_+, \mu_-) = \mathrm{Tr}\left(e^{-\beta\hat{K}}\right) = e^{-\beta\Omega(T,V,\mu_+,\mu_-)}.$$

$\Omega(T, V, \mu_+, \mu_-) = -Vp(T, \mu_+, \mu_-)$ is the Grand Canonical Potential. In the 2$^{\rm nd}$ quantization formalism

$$\hat{K}_0 = \sum_{\mathbf{p},\sigma} (\varepsilon_{\mathbf{p}} - \mu_\sigma)\, a^\dagger_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} = \sum_\sigma \int \frac{d^3\mathbf{p}}{(2\pi)^3} (\varepsilon_{\mathbf{p}} - \mu_\sigma)\, a^\dagger_\sigma(\mathbf{p})\, a_\sigma(\mathbf{p}),$$

with $\varepsilon_{\mathbf{p}} \equiv \hbar^2\mathbf{p}^2/2m_f$, etc.

$$\frac{\Xi_{\rm stat}}{\Xi^{(0)}_{\rm stat}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_n \cdots \int_0^\beta d\tau_1 {\rm Tr}\left(\hat{\rho}^{(0)}\, {\rm T}_\tau[\hat{V}^I_{\rm int}(\tau_n) \ldots \hat{V}^I_{\rm int}(\tau_1)]\right).$$

(where $\hat{V}^I_{\rm int}(\tau_2) = e^{\tau_2 \hat{K}_0} \hat{V}_{\rm int} e^{-\tau_2 \hat{K}_0}$ and $\hat{\rho}^{(0)} = e^{-\beta \hat{K}_0}/\Xi^{(0)}_{\rm stat}$) can be computed perturbatively. The Grand Potential $\Omega = -(1/\beta) \ln \Xi_{\rm stat}$ is, therefore, given by

$$\Omega = \Omega^{(0)} - \frac{1}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta d\tau_n \ldots \int_0^\beta d\tau_1 {\rm Tr}\left(\hat{\rho}^{(0)}\, {\rm T}_\tau[\hat{V}^I_{\rm int}(\tau_n) \ldots \hat{V}^I_{\rm int}(\tau_1)]\right)^{\rm con}$$

i.e. by the sum of connected "vacuum" diagrams.
$\Omega^{(0)}(T, V, \mu_+, \mu_-)$: the textbook expression

$$\Omega^{(0)}(T, V, \mu_+, \mu_-) = -\frac{1}{\beta} \sum_{\sigma=\pm} V \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 + e^{-\beta(\varepsilon_{\mathbf{k}} - \mu_\sigma)}\right).$$

Computing Ω perturbatively

Dyson expansion leads to Feynman diagrams with the (momentum space) propagators

$$-\tilde{G}_{\pm}(\omega_n^{\mathrm{F}}, \mathbf{k}) = \frac{-1}{i\omega_n^{\mathrm{F}} - (\varepsilon_{\mathbf{k}} - \mu_{\pm})}\,.$$

Integrals over frequencies are replaced by sums over the (fermionic) Matsubara frequencies $\omega_n^{\mathrm{F}} = (\pi/\beta)(2n+1)$.

$$-\mathcal{G}_{\pm}(\tau, \mathbf{r}) = \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{-i\omega_n^F \tau}\, e^{i\mathbf{k}\cdot\mathbf{r}} \left(-\tilde{\mathcal{G}}_{\pm}(\omega_n^F, \mathbf{k})\right),$$

$$\mathcal{G}_{\pm}(0, \mathbf{0}) = \int_{\mathbf{k}} \left[1 + e^{\beta(\varepsilon_{\mathbf{k}} - \mu_{\pm})}\right]^{-1} = \frac{N_{\pm}(T, \mu_{\pm})}{V}$$

Recovering the textbook (Mean Field) results

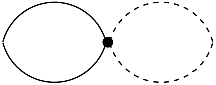

Figure: The order C_0 correction $\Omega^{(1)}$.

$$\Omega^{(1)} = \frac{1}{\beta}\int_0^\beta d\tau \int d^3\mathbf{x}\, C_0\, \mathcal{G}^{(0)}_{++}(0,\mathbf{0})\, \mathcal{G}^{(0)}_{--}(0,\mathbf{0}) = C_0\, V\, \frac{N_+}{V}\, \frac{N_-}{V}\,.$$

One wants to minimize $F(T, V, N_+, N_-) = \Omega + N_+\mu_+ + N_-\mu_-$ (obtained by inverting $-(\partial\Omega/\partial\mu_\pm)_{T,V} = N_\pm$) wrt. N_+ and N_- at $N = N_+ + N_-$ fixed (equilibrium). To the first order its simple:

$$F(N_+, N_-) = \Omega^{(0)}(\mu_+^{(0)}, \mu_-^{(0)}) + N_+\mu_+^{(0)} + N_-\mu_-^{(0)} + \Omega^{(1)}(\mu_+^{(0)}, \mu_-^{(0)}).$$

The external magnetic field $\mathcal{H}$ can be taken into account (just by shifting in $\hat{K}_0$ the chemical potentials). One recovers in this way the textbook (Kesio) MF results shown before. One can go ahead!

Obtaining $F(N_+, N_-)$

At first sight it seems that to get $F(N_+, N_-)$ out of $\Omega(\mu_+, \mu_-)$ one needs to adjust (numerically) in each order μ_+ and μ_- so that $-(\partial\Omega/\partial\mu_\pm)_{T,V}$ match $N_\pm$. But in the systematic expansion it reduces to determining μ_+, and μ_- from $-(\partial\Omega^{(0)}/\partial\mu_\pm)_{T,V}$ and omitting in the Dyson expansion those diagrams which were vanishing in computing E/V (i.e. in order C_0^3 the "audi-type", "mitsubishi-type" and "no-fancy-name" diagrams)!

Computing $F^{(2)}$

$$F^{(2)} = -\frac{1}{2}\, C_0^2\, V\, \frac{1}{\beta} \sum_{l\in\mathbb{Z}} \int_{\mathbf{q}} [A(\omega_l^B, \mathbf{q})]^2$$

$$A(\omega_l^B, \mathbf{q}) = \int_{\mathbf{p}} \frac{\{\mathbf{p}\}}{i\omega_l^B - [\mathbf{p}]}\,,$$

$$\frac{F^{(2)}}{V} = -\frac{1}{2}\, C_0^2 \int_{\mathbf{q}} \int_{\mathbf{p}_1} \int_{\mathbf{p}_2} \frac{\{\mathbf{p}_1\}\{\mathbf{p}_2\}}{[\mathbf{p}_1] - [\mathbf{p}_2]} \left(\frac{1}{1 - e^{\beta[\mathbf{p}_1]}} - \frac{1}{1 - e^{\beta[\mathbf{p}_2]}} \right).$$

$$N^{\mathbf{p}}_{--} \equiv n_+(\mathbf{p})\, n_-(\mathbf{q}-\mathbf{p})\,,$$

$$N^{\mathbf{p}}_{++} \equiv [1 - n_+(\mathbf{p})]\, [1 - n_-(\mathbf{q}-\mathbf{p})]\,,$$

$$n_\pm(\mathbf{p}) = \left[1 + \exp\{\beta(\varepsilon_{\mathbf{p}} - \mu_\pm^{(0)})\}\right]^{-1},$$

$$\{\mathbf{p}\} \equiv n_+(\mathbf{p}) + n_-(\mathbf{q}-\mathbf{p}) - 1\,,$$

$$[\mathbf{p}] \equiv \varepsilon_{\mathbf{p}} - \mu_+^{(0)} + \varepsilon_{\mathbf{q}-\mathbf{p}} - \mu_-^{(0)}\,.$$

With the helpn of the relation

$$\{\mathbf{p}\} = N^{\mathbf{p}}_{--} - N^{\mathbf{p}}_{++} = N^{\mathbf{p}}_{--}\left(1 - e^{\beta[\mathbf{p}]}\right).$$

the final form of $F^{(2)}$ is

$$\frac{F^{(2)}}{V} = -C_0^2 \int_{\mathbf{q}} \int_{\mathbf{p}_1} \int_{\mathbf{p}_2} N^{\mathbf{p}_1}_{--} \frac{n_+(\mathbf{p}_2) + n_-(\mathbf{q} - \mathbf{p}_2) - 1}{[\mathbf{p}_1] - [\mathbf{p}_2]},$$

"-1" gets subtracted when C_0 in $F^{(1)}/V$ is replaced by $C_{0R}(1 + \Lambda \text{ terms})$ and the finite part can be converted into

$$\frac{F^{(2)}}{V} = C_0^{\mathrm{R}} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} n_+(\mathbf{k}_1)\, n_-(\mathbf{k}_2)\, L(\mathbf{k}_1, \mathbf{k}_2).$$

Computing $F^{(2)}$ cont'd

$$L(\mathbf{k}_1,\mathbf{k}_2) = -\frac{C_0^{\mathrm{R}} m_f}{(2\pi)^2\hbar^2|\mathbf{k}_1+\mathbf{k}_2|}\int_0^\infty dp\, p\,[n_+(p)+n_-(p)] \times \ln\left|\frac{(p-\Delta_+)(p-\Delta_-)}{(p+\Delta_+)(p+\Delta_-)}\right|,$$

with

$$\Delta_\pm = \frac{1}{2}|\mathbf{k}_1+\mathbf{k}_2| \pm \frac{1}{2}|\mathbf{k}_1-\mathbf{k}_2|\,.$$

We have invented a method allowing efficiently evaluate numerically $F^{(2)}/V$ and have checked that at $T=0$ we recover $E^{(2)}/V$ for all values of P.

Thermal profile of the transition in the 2nd order

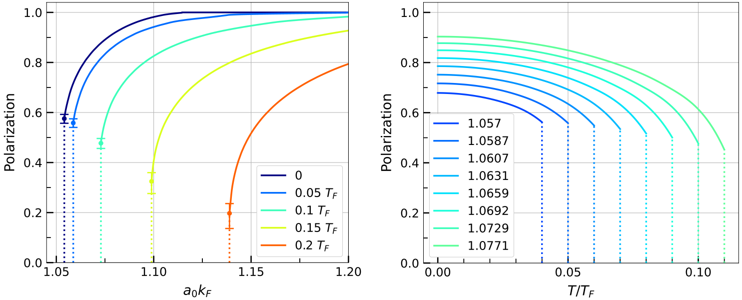

Figure: Polarization obtained using the free energy F computed up to the $(k_{\rm F} a_0)^2$ order. Left panel: as a function of the "gas parameter" $k_{\rm F} a_0$. Right panel: as a function of the temperature.

We have also reproduced the results of MacDonald and Duine and wanted to go ahead and compute $F^{(3)}$...

Computing $F^{(3)}$ - particle-particle mercedes diagram

but we got stuck for a longer time. Formally

$$\frac{F^{(3)pp}}{V} = \frac{1}{3}\, C_0^3\, \frac{1}{\beta} \sum_l \int_{\mathbf{q}} [A(\omega_l^{\mathrm{B}}, \mathbf{q})]^3 \,.$$

This can be converted into

$$\frac{F^{(3)pp}}{V} = \frac{C_0^3}{3} \int_{\mathbf{q}} \int_{\mathbf{p}_1} \int_{\mathbf{p}_2} \int_{\mathbf{p}_3} \left(N_{--}^{\mathbf{p}_1} \frac{\{\mathbf{p}_2\}}{[\mathbf{p}_1] - [\mathbf{p}_2]} \frac{\{\mathbf{p}_3\}}{[\mathbf{p}_1] - [\mathbf{p}_3]} + \text{two terms} \right),$$

An algebraic identity transforms the (symmetrized!) integrand into the symmetrized sum of two terms which in the $T = 0$ limit reduce to the $G^{(1)}$ and $G^{(2)}$ contributions to $E^{(3)}/V$. We have also demonstrated how the divergences cancel out.
This seemed to reduce to the integral like that for $F^{(2)}$ but with $L(\mathbf{k}_1, \mathbf{k}_2)$ replaced by $(L(\mathbf{k}_1, \mathbf{k}_2))^2$. But we could not recover the $T = 0$ result... The problem was in the P-value prescription implicit in the imaginary time formulation

Resolution

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \left[\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} + \frac{1}{(z-x)(z-y)}\right].$$

The integrand is algebraically zero, but

$$3\int_0^1 dx \left(\mathrm{P}\int_0^1 dy \, \frac{1}{x-y}\right)^2 = 3\int_0^1 dx \, \ln^2 \frac{1-x}{x} = 3\,\frac{\pi^2}{3} \neq 0.$$

To get the correct result (zero) one has to regularize by setting $x \to x + i\epsilon$, $y \to y + 2i\epsilon$, $z \to z + 3i\epsilon$ (the sign of ϵ is irrelevant; the integrand is still algebraically zero but its singularities are now off the integration axes). It is then straightforward to find that the application of the Sochocki formula $1/(x \pm i0) = P(1/x) \mp i\pi\delta(x)$ to the regularized integral leads to (the terms linear in the Dirac deltas neatly cancel out)

$$3\int_0^1 dx \left(\mathrm{P}\int_0^1 dy \, \frac{1}{x-y}\right)^2 + \int_0^1 dx \left(i\pi \int_0^1 dy \, \delta(x-y)\right)^2 = 0\,.$$

$$\frac{F^{(3)pp}}{V} = C_0^3 \int_{\mathbf{q}} \int_{\mathbf{p}_1} N^{\mathbf{p}_1}_{--} \left[\left(\mathrm{P} \int_{\mathbf{p}_2} \frac{\{\mathbf{p}_2\}}{[\mathbf{p}_1] - [\mathbf{p}_2]} \right)^2 + \frac{1}{3} \left(i\pi \int_{\mathbf{p}_2} \{\mathbf{p}_2\}\, \delta([\mathbf{p}_1] - [\mathbf{p}_2]) \right)^2 \right].$$

Cancellation of the UV divergences:
$\{\mathbf{p}_2\} \equiv n_+(\mathbf{p}_2) + n_-(\mathbf{q} - \mathbf{p}_2) - 1 \to n_+(\mathbf{p}_2) + n_-(\mathbf{q} - \mathbf{p}_2)$ in the first term and $C_0 \to C_0^{\mathrm{R}}$. The result takes the form

$$\frac{F^{(3)pp}_{\mathrm{fin}}}{V} = C_0^{\mathrm{R}} \int_{\mathbf{k}_1} \int_{\mathbf{k}_1} n_+(\mathbf{k}_1)\, n_-(\mathbf{k}_2) \left[L^2(\mathbf{k}_1, \mathbf{k}_2) + \frac{1}{3} \left(iL_\delta(\mathbf{k}_1, \mathbf{k}_2) \right)^2 \right],$$

with

$$L_\delta(\mathbf{k}_1, \mathbf{k}_2) = \pi \frac{C_0^{\mathrm{R}} m_f}{(2\pi)^2 \hbar^2 |\mathbf{k}_1 + \mathbf{k}_2|} \int_{p_{\mathrm{min}}}^{p_{\mathrm{max}}} dp\, p \left[n_+(p) + n_-(p) - 1 \right],$$

in which $p_{\mathrm{min}} = |\Delta_-|$ and $p_{\mathrm{max}} = \Delta_+$.

$$\frac{F^{(N)pp}}{V} = (-1)^{N+1}\,\frac{C_0^N}{N}\,\frac{1}{\beta}\sum_l \int_{\mathbf{q}} [A(\omega_l^{\mathrm{B}}, \mathbf{q})]^N\,.$$

After similar steps gives

$$\frac{F^{(N)pp}}{V} = \frac{C_0^{\mathrm{R}}}{N}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} n_+(\mathbf{k}_1)\,n_-(\mathbf{k}_2)\,\frac{(L+iL_\delta)^N-(L-iL_\delta)^N}{2iL_\delta}\,.$$

Real and independent of the sign of L_δ (in the prescription to handle the P-value integrals the sign of ϵ is arbitrary). Reproduces $N=2$ and $N=3$ and $N=1$.

Summation over N can also be done![2]

$$\frac{F^{pp}}{V} = C_0^{\mathrm{R}}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} n_+(\mathbf{k}_1)\,n_-(\mathbf{k}_2)\,\frac{\arctan(L_\delta/(1-L))}{L_\delta}\,.$$

[2]By comparing with the expansion of $\arctan t = (1/2i)\ln[(1+it)/(1-it)]$ in powers of t.

Before showing the results...

To what corresponds keeping terms with a_0 while neglecting those with r_0, a_1 etc.? That is, assuming that $a_0 \gg |r_0|, |a_1|$, etc?
It turns out that if the "sausage" diagrams contributing to the scattering amplitude $\mathcal{A}$ are resummed, the elastic scattering amplitude has in the s-channel a simple pole for $|\mathbf{k}| = i/a_0$, i.e. for energy $E = -\hbar^2/2m_f a_0$. There is a bound (bosonic) state! This means that $a_0 \gg |r_0|, |a_1|$ corresponds to an attractive interaction $V_{\rm pot}(\mathbf{x}_i - \mathbf{x}_j)$, rather than to a repulsive one.
Keeping only a_0 terms corresponds rather to experiments with cold fermionic atoms gases which interact mostly attractively and $a_0 \gg |r_0|, |a_1|$ is obtained near a Feshbach resonance.

Feshbach resonance in a nutshell

Mutual interaction of two atoms depends on their total J - assume there are two such states. But $\mathcal{H}$ couples to some other two states

$$\hat{H} = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + V_1(\mathbf{r}) & 0 \\ 0 & -\frac{\hbar^2}{2m}\nabla^2 + V_2(\mathbf{r}) \end{pmatrix} + \mathcal{H}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{H} = \begin{pmatrix} -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) - \mathcal{H} & \delta V(\mathbf{r}) \\ \delta V(\mathbf{r}) & -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + \mathcal{H} \end{pmatrix}$$

($V(\mathbf{r})$ and $\delta V(\mathbf{r})$ tend to 0 as $|\mathbf{r}| \to \infty$).
If δV neglected, there are two independent scattering "channels". For $-\mathcal{H} < E < \mathcal{H}$ one is "open" and another one "closed". Owing to $\delta V(\mathbf{r})$, the energy of the particle scattering in the "open" channel can fit the bound state in the other chanel - this can be arranged for by tuning $\mathcal{H}$. a_0 then decreases to $-\infty$ (on the so called BCS side of the resonance), jumps from $-\infty$ to $+\infty$ (the BCS-BEC cross-over), and and then decreases from $+\infty$ (on the BEC side).

On the BEC side just after crossing the Feshbach resonance a_0 is positive and large. Experimentalist say "we can in this way realize a repulsive gas of fermions" (or: "we have realized a textbook Stoner model Hamiltonian"...) and relying on the Mean Field formula (linear in a_0)

$$\frac{E(P)}{V} = \frac{3}{5}\,\frac{k_{\rm F}^3}{6\pi^2}\,\frac{\hbar^2 k_{\rm F}^2}{2m_f}\left\{(1+P)^{5/3} + (1-P)^{5/3} + \frac{20}{9\pi}\,(k_{\rm F}\,a_0)\,(1-P^2)\right\}$$

search for the ferromagnetic state.
There was a claim of the Ketterle group that they found it. But later revised their interpretation - the gas of fermions on the BEC side is in a metastable state and the bound states form - the true equilibrium state is very different than the state of a gas of fermions (Bose condensation of bosonic dimers occurs).

Recovering and generalizing (to $T \neq 0$) the result of L. He

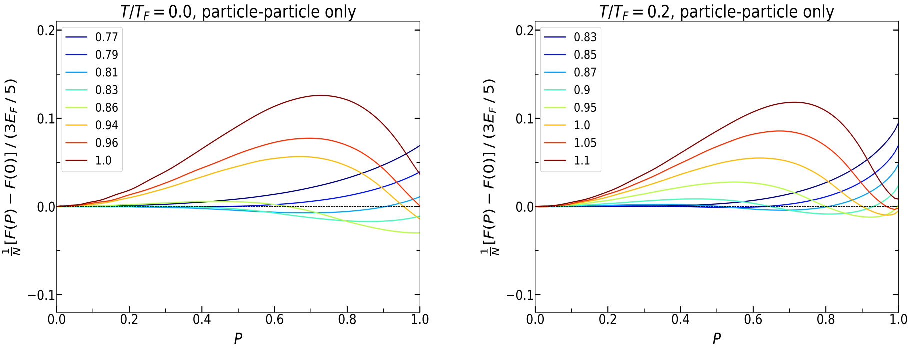

Figure: The difference $(F(P) - F(0))^{pp}/N$ (in units $(3/5)\varepsilon_{\rm F}$) as a function of P for different values of the gas parameter $k_{\rm F} a_0$. Only pp rings resummed. Left plot ($T = 0$) corresponds to L. He's result.

In words

For $T = 0$ the minimum of $(F(P) - F(0))^{pp}/V$ starts to move away from $P = 0$ for $(k_{\rm F} a_0)_{\rm cr} = 0.79$ - the transition is continuous $(k_{\rm F} a_0)_{\rm cr}(T)$: 0.85 for $T = 0.2\,T_{\rm F}$, 0.92 for $0.3\,T_{\rm F}$ and 1.12 for $0.5\,T_{\rm F}$

The minimum is back at $P = 0$ for $k_{\rm F} a_0 > 0.96$ (at $T = 0$) the first order "reentrant" transition to the paramagnetic state - the consequence of the existence of the maximum of the energy density (at $T = 0$) for $P = 0$ treated as a function of $k_{\rm F} a_0$ (next plots) seen indeed in experiments with cold gases. Results from the appearance for $k_{\rm F} a_0 > 1.34$ of the simple pole in the "in-medium" particle-particle elastic scattering amplitude which can be interpreted as being due to the existence of the "in-medium" positive energy bound state of two fermions (of opposite spin projections).

$$\frac{F^{(3)ph}}{V} = -C_0^{\mathrm{R}} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} [1 - n_+(\mathbf{k}_1)]\, n_-(\mathbf{k}_2) \left[K^2(\mathbf{k}_1, \mathbf{k}_2) + \frac{1}{3} \left(iK_\delta(\mathbf{k}_1, \mathbf{k}_2) \right)^2 \right]$$

where

$$K(\mathbf{k}_1, \mathbf{k}_2) = \frac{C_0^{\mathrm{R}} m_f}{(2\pi)^2 \hbar^2 |\mathbf{k}_1 + \mathbf{k}_2|} \left(\int_0^\infty dp\, p\, n_+(p) \ln\left| \frac{p - \Delta_1}{p + \Delta_1} \right| + \int_0^\infty dp\, p\, n_-(p) \ln\left| \frac{p - \Delta_2}{p + \Delta_2} \right| \right),$$

$$K_\delta(\mathbf{k}_1, \mathbf{k}_2) = \pi \frac{C_0^{\mathrm{R}} m_f}{(2\pi)^2 \hbar^2 |\mathbf{k}_1 + \mathbf{k}_2|} \left(\int_{|\Delta_1|}^\infty dp\, p\, n_+(p) - \int_{|\Delta_2|}^\infty dp\, p\, n_-(p) \right),$$

in which

$$\Delta_1 \equiv \frac{\mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{|\mathbf{k}_1 + \mathbf{k}_2|}, \qquad \Delta_2 \equiv \frac{\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{|\mathbf{k}_1 + \mathbf{k}_2|}.$$

It is UV finite. Numerical convergence of integrations more delicate. Reproduces for $T = 0$ the known contribution to F/V

$$\sum_{N=1}^{\infty}\frac{F^{(N)ph}}{V} = -\, C_0^{\mathrm{R}}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}[1-n_+(\mathbf{k}_1)]\, n_-(\mathbf{k}_2)\, \frac{\arctan(K_\delta/(1-K))}{K_\delta}\,.$$

From this sum one has to subtract two first terms of the series:

$$\frac{F^{ph}}{V} = -C_0^{\mathrm{R}}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}[1-n_+(\mathbf{k}_1)]\, n_-(\mathbf{k}_2)\left[\frac{\arctan(K_\delta/(1-K))}{K_\delta} - 1 - K\right].$$

Interaction (free) energy

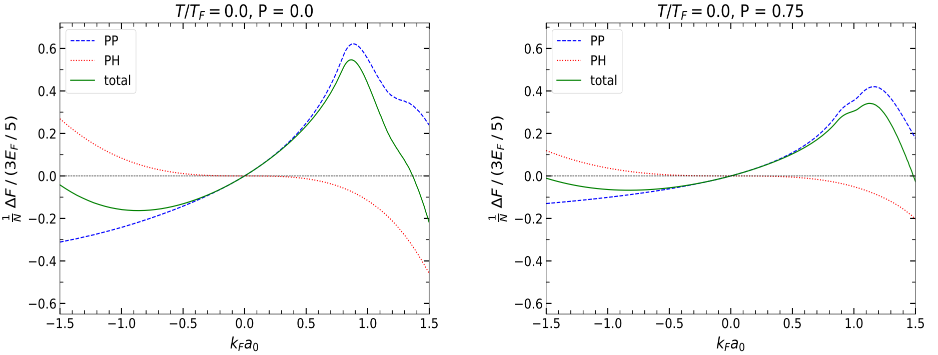

Figure: Dependence on $k_{\rm F}a_0$ of the resumed contributions of the particle-particle rings (dashed blue) and of the particle-hole rings (dotted red lines) and of their sum (solid green lines) for $T = 0$ and two values of the polarization P.

Interaction (free) energy cont'd

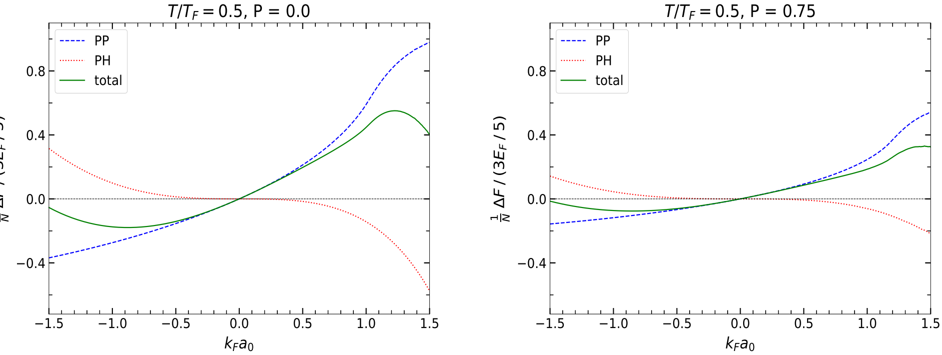

Figure: Dependence on $k_{\rm F} a_0$ of the resumed contributions of the particle-particle rings (dashed blue) and of the particle-hole rings (dotted red lines) and of their sum (solid green lines) for $T = 0.5\,T_{\rm F}$ and two values of the polarization P.

Phase transition?

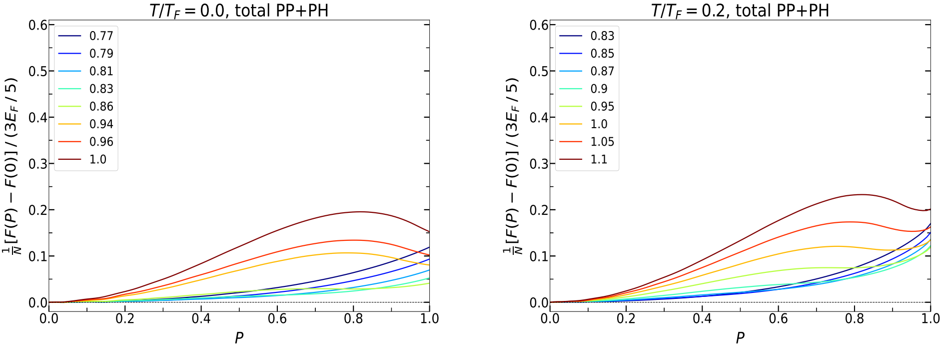

Figure: The difference $(F(P) - F(0))^{pp+ph}/N$ (in units $(3/5)\varepsilon_{\rm F}$ for $T = 0$ and $T = 0.2\ T_{\rm F}$ as a function of the polarization $P = (N_+ - N_-)/N$ for different values (indicated in the panels) of the gas parameter $k_{\rm F} a_0$.

Disappeared!

Conclusions

Computations in the imaginary time formalism are easier!
We have computed F up to the order $(k_{\rm F}a_0)^3$. It remains to add the order $(k_{\rm F}a_1)^3$ and $k_{\rm F}^3a_0^2r_0$ terms to complete the order $(k_{\rm F}R)^3$.

The question about the existence and the character (order) of the FM phase transtistion in systems of fermions with truly repulsive interaction is open but there are no indications that it does not exist.

The question about the FM phase transition in systems of cold fermionic atoms (with attractive interaction despite positive a_0) is unclear - our result suggests it may well not exist. (The qualitative argument using the Pauli principle obviously does not apply to this case!)

Conclusions cont'd

A general theoretical question emerges: how to obtain a thermodynamic function (like F) of a metastable (from the TMD point of view) state using the machinery of statistical mechanics? Here it is likely that the QFT (EFT or not) with its nonseparable Hilbert space and implicit TMD limit is the right framework - when the perturbative expansion is constructed around the groud state of noniteracting fermions (but I have no idea whether this can be done nonperturbatively)

What about the QMC results indicating that a phase transition occurs? How simulating a system of a few tens of particles in a finite volume obtain properties of a metastable state? In these computations they infer the existence of a phase transition (and its continuous character) by finding that χ diverges. But using the Kanno 2nd order analytic result one also gets $\chi = \infty$ at $k_{\rm F} a_0 = 1.058$ whereas the transition is first order and for $k_{\rm F} a_0 = 1.054$.

So the situation remains unclear