NATIONAL SCIENCE CENTRE
POLAND

From Hydrodynamics to Bubble Wall Velocity

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Based on:
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arXiv 2603.24583
with T. Krajewski, M. Lewicki, M. Merchand and M. Zych

Particle Physics and Cosmology Seminar
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Why study first-order transitions?

Bariogenesis

Necessary conditions:	SM
1) Baryon number violation	✓
2) C and CP violation	✗
3) Out-of-equilibrium dynamics	✗

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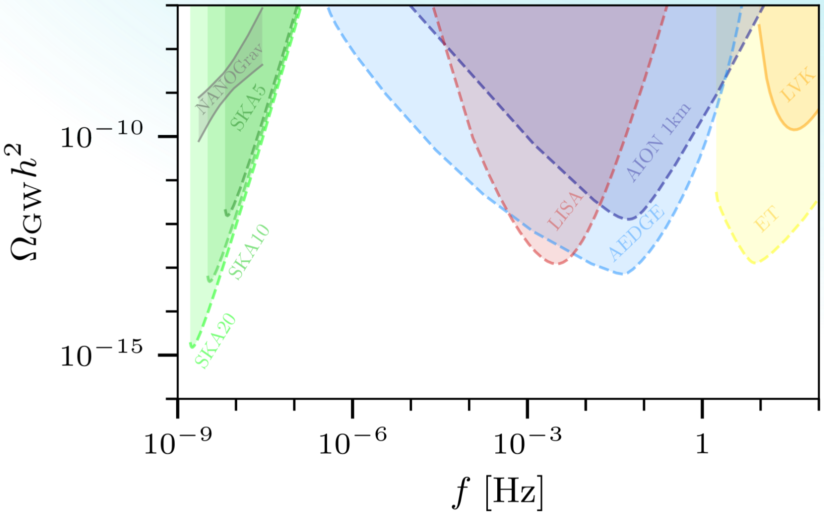
Gravitational waves



Cosmological first-order phase transition

Let us consider theory with Lagrangian density

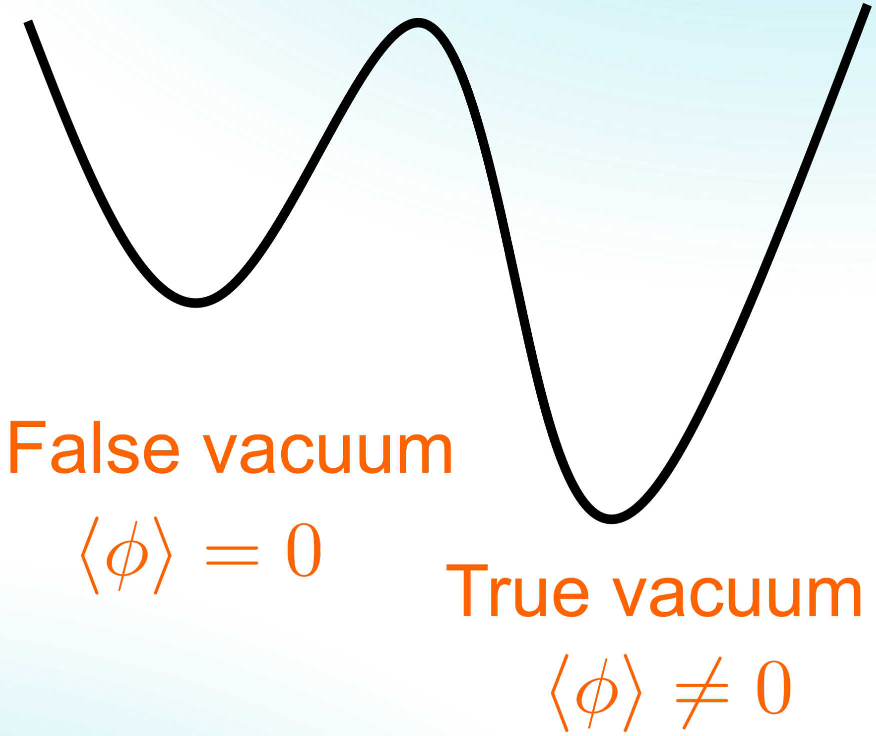
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi).$$

Assuming spherical symmetry the equation of motion reads

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = \frac{\partial V_{\text{eff}}(\phi, T)}{\partial \phi}.$$

where T is temperature.

Effective potential $V_{\text{eff}}(\phi, T)$



Nucleation

New phase bubbles nucleate at a rate

$$\Gamma = A(T)\,\exp\left(-\frac{S_3}{T}\right),$$

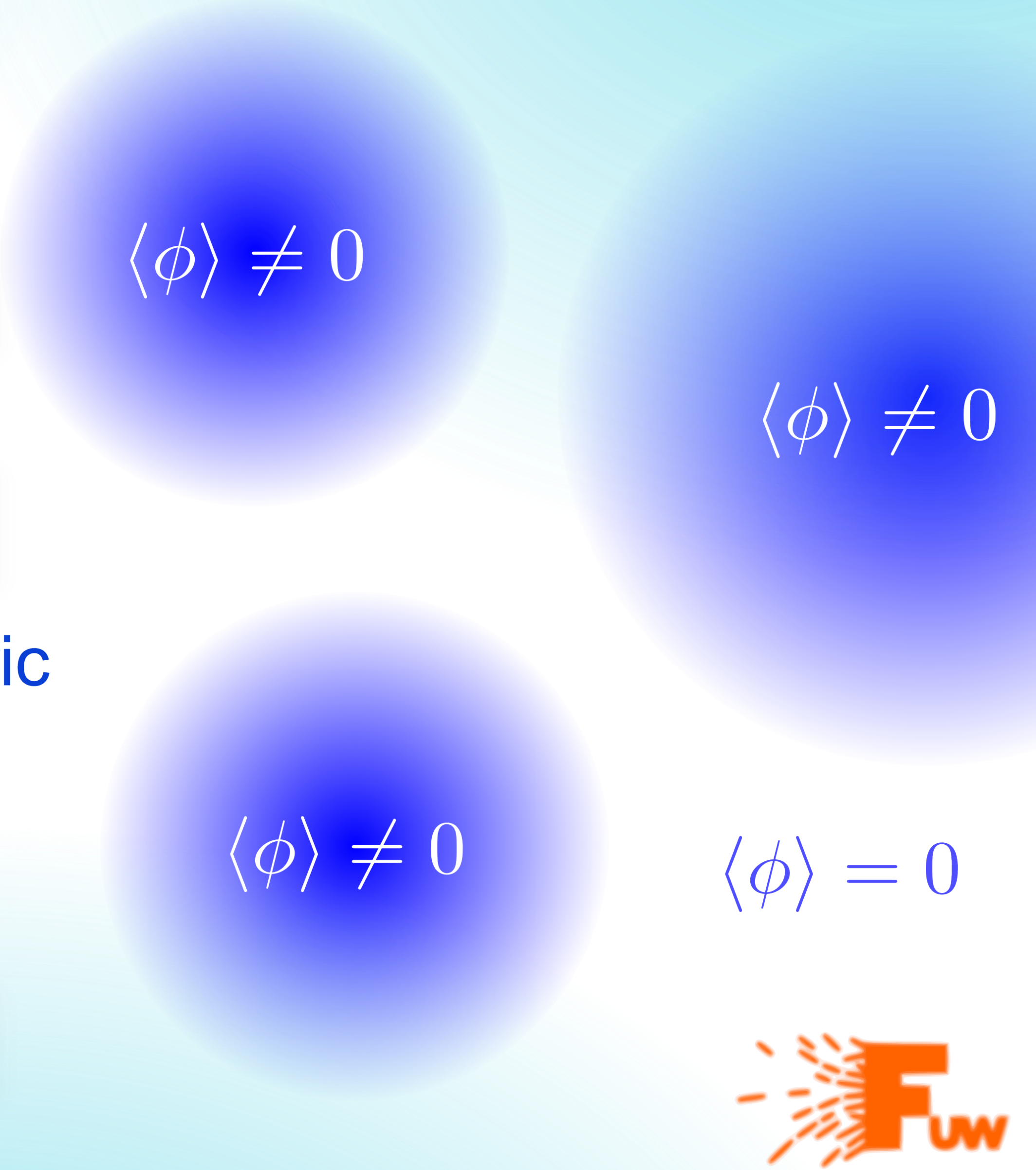
where

$$A(T) = T^4\left(\frac{S_3}{2\pi T}\right)^{\frac{3}{2}},$$

while S_3 is an $\mathbb{O}(3)$-symmetric solution to equation of motion (EOM).

Nucleation criterium:

$$\Gamma(T_n)/H^4 \approx 1$$

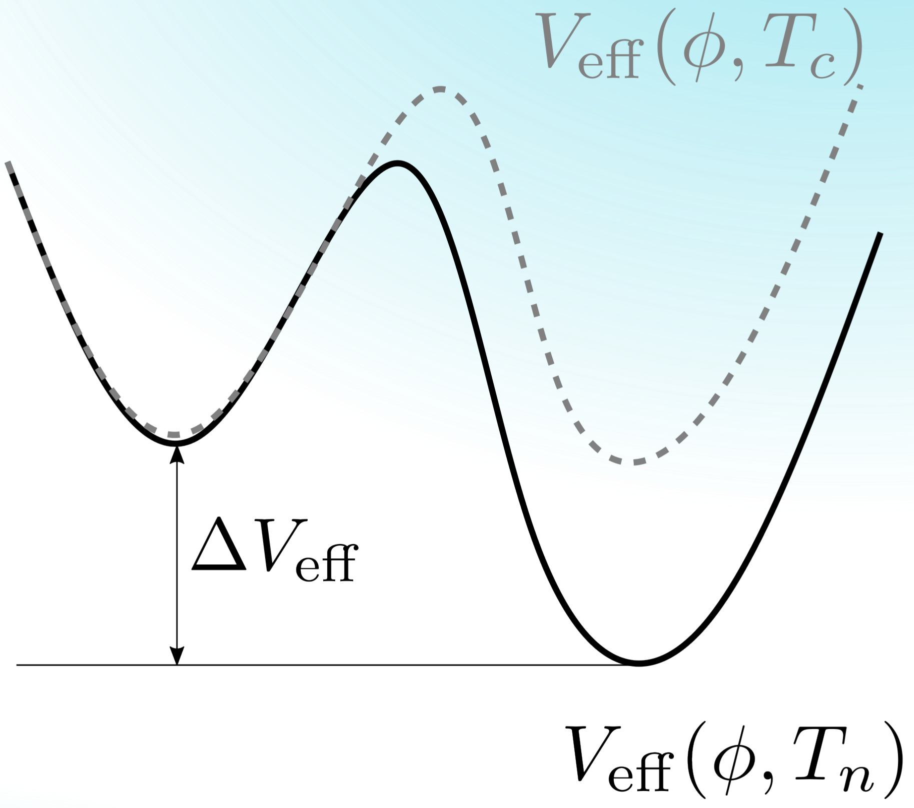


Transition Parameters

- Critical and nucleation temperatures: T_c, T_n
- Transition strength:

$$\alpha_\theta = \frac{\bar{\theta}_b - \bar{\theta}_s}{3\omega_s}, \quad \bar{\theta} = e - \frac{p}{c^2}$$

where p is the pressure, e and ω are the energy and enthalpy densities, and c is the sound speed in plasma.



- Terminal bubble wall velocity ξ_w

Primordial plasma equation of state

The Hemholtz free energy density $\mathcal{A}(T)$ is equal to minus the effective potential

$$\mathcal{A}_{\mathrm{b/s}}(T) = -V_{\mathrm{eff}}(\phi_{\mathrm{b/s}}, T).$$

The plasma pressure, enthalpy and entropy densities follow from the standard thermodynamic identities

$$p = -V_{\mathrm{eff}}, \qquad \omega = -T\frac{dV_{\mathrm{eff}}}{dT}, \qquad s = -\frac{dV_{\mathrm{eff}}}{dT}.$$

Primordial plasma equations of motion

In equilibrium, plasma energy-momentum tensor reads

$$T_{\rm fl}^{\mu\nu} = \omega u^\mu u^\nu - g^{\mu\nu} p,$$

where u^μ is the plasma four velocity. By taking the projectons of $\partial_\mu T_{\rm fl}^{\mu\nu}$ ∥ and ⊥ to the flow, one obtains

$$\partial_\mu (u^\mu \omega) - u_\mu \partial^\mu p = 0, \qquad \bar{u}^\nu u^\mu \omega \, \partial_\mu u_\nu - \bar{u}^\nu \partial_\mu p = 0,$$

with $\bar{u}^\mu$ such that $\bar{u}^2 = -1$ and $\bar{u}^\mu u_\mu = 0$.

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Assuming spherical symmetry, these equations become

$$\partial_t \tau + \frac{1}{r^2} \partial_r (r^2 (\tau + p) v) = 0,$$

$$\partial_t Z + \frac{1}{r^2} \partial_r \left(r^2 Z v \right) + \partial_r p = 0.$$

$$\tau = \omega \gamma^2 - p v$$

$$Z = \omega \gamma^2 v$$

Primordial plasma steady-state equations

The shock wave forming around expanding bubble lack the characteristic energy scale. Hence, the “stationary” solution must be a function of self-similar variable

$$\xi = r/t.$$

The steady-state equations for the shock wave are

$$2\frac{v}{\xi} = \gamma^2(1 - v\xi)\left[\frac{\tilde{\mu}^2}{c^2} - 1\right]\partial_\xi v, \quad \partial_\xi w = w\left(1 + \frac{1}{c^2}\right)\gamma^2\tilde{\mu}\,\partial_\xi v,$$

with $\tilde{\mu} = \frac{\xi - v}{1 - \xi v}$ and $c^2 \approx \frac{dV_{\mathrm{eff}}/dT}{T d^2 V_{\mathrm{eff}}/dT^2}$.

Matching conditions:

$$w_+\gamma_+^2 v_+ = w_-\gamma_-^2 v_-$$

$$w_+\gamma_+^2 v_+^2 + p_+ = w_-\gamma_-^2 v_-^2 + p_- \qquad + \quad ??$$

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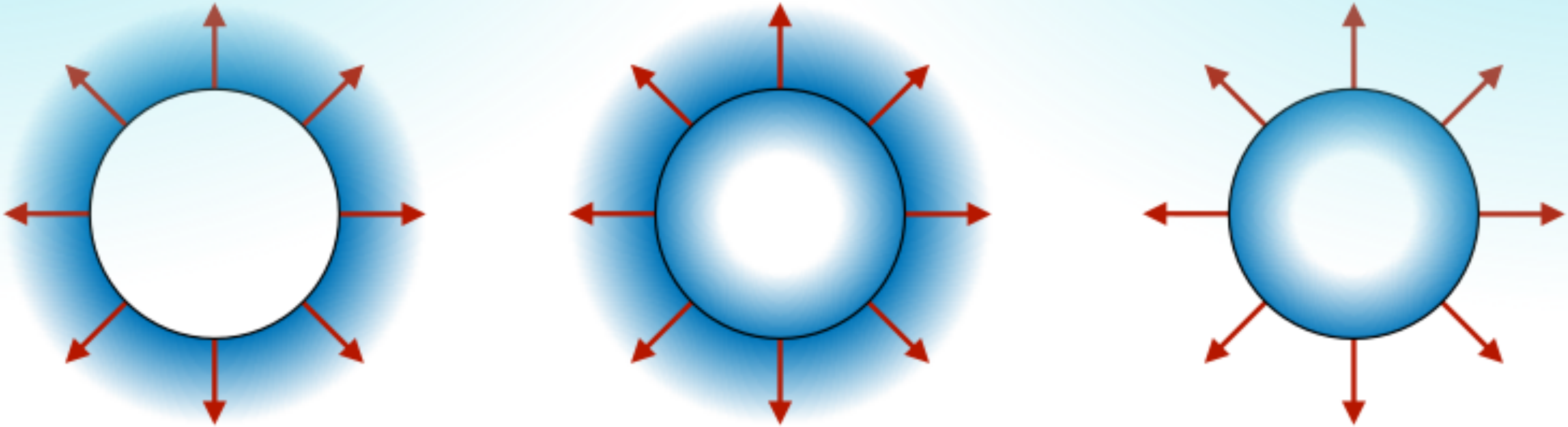
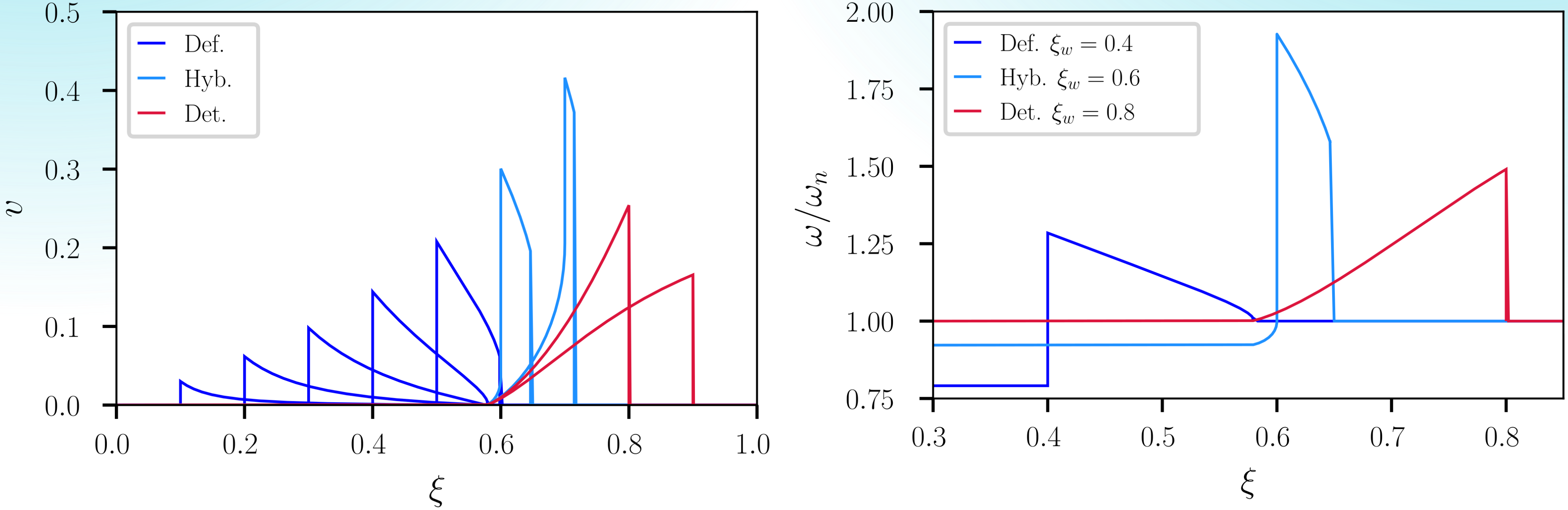
$$\begin{aligned} &w_+\gamma_+^2 v_+ = w_-\gamma_-^2 v_- \\ &w_+\gamma_+^2 v_+^2 + p_+ = w_-\gamma_-^2 v_-^2 + p_- \end{aligned} \qquad + \quad \xi_w$$

Solution types



Deflagration	Hybrid	Detonation
$\xi_w \leq c_s$	$c_s < \xi_w < v_J$	$v_J \leq \xi_w$

Stationary profiles



Computed using Bag equations of state with $\alpha_\theta = 0.1$.

Scalar field dynamics

The EOM of the scalar field in the bubble wall reads

$$\Box\phi + \frac{\partial V(\phi)}{\partial\phi} + \sum_i \frac{dm_i^2}{d\phi}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_i}\left(f_{\rm eq.} + \delta f_i\right) = 0$$

$$\Box\phi + \frac{\partial V_{\rm eff}(\phi, T)}{\partial\phi} + \sum_i \frac{dm_i^2}{d\phi}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_i}\delta f_i = 0$$

$$\Big\downarrow \; \int_{\rm Wall} dz\,\frac{d\phi}{dz}$$

Backreaction

Dissipative friction

$$\int_{\rm Wall} dz\,\frac{\partial V_{\rm eff}}{\partial T}\frac{dT}{dz} + \sum_i \int d\phi\,\frac{dm_i^2}{d\phi}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_i}\delta f_i = \Delta V_{\rm eff}$$

Local thermal equilibrium

Entropy conserved $\Longrightarrow$

New matching condition[1]

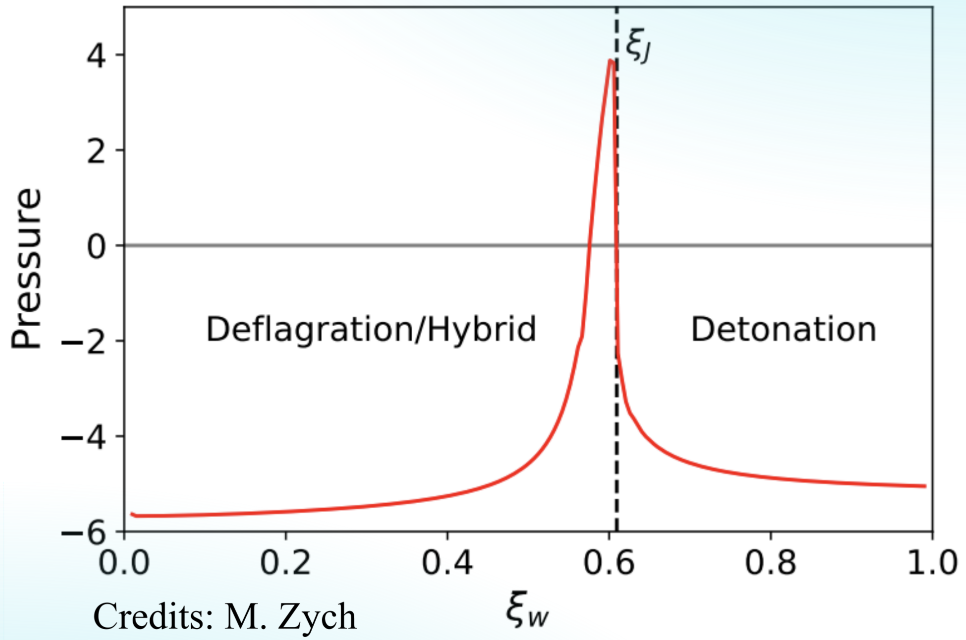
$$s_+ v_+ \gamma_+ = s_- v_- \gamma_-$$

In LTE ξ_w is fixed by

- tansition strength α_θ
- enthalpy ratio $\Psi_n = \frac{\omega_b(T_n)}{\omega_s(T_n)}$
- sound speed $c_s,\ c_b$

No stable detonations exist in LTE, only run-away solutions with $\xi_w \to 1$.

Net pressure on the wall



Credits: M. Zych

[1] W.-Y. Ai et al. *Model-independent bubble wall velocities in local thermal equilibrium*, JCAP **07** (2023), arXive: 2303.10171.

Benchmark model

As a benchmark scenario, we use the real singlet extension of the Standard Model, with a discrete symmetry $s \to -s$

$$V(H,s) = \mu_h^2 |H|^2 + \lambda_h |H|^4 + \frac{1}{2}\lambda_{hs}|H|^2 s^2 + \frac{1}{2}\mu_s^2 s^2 + \frac{1}{4}\lambda_s s^4 .$$

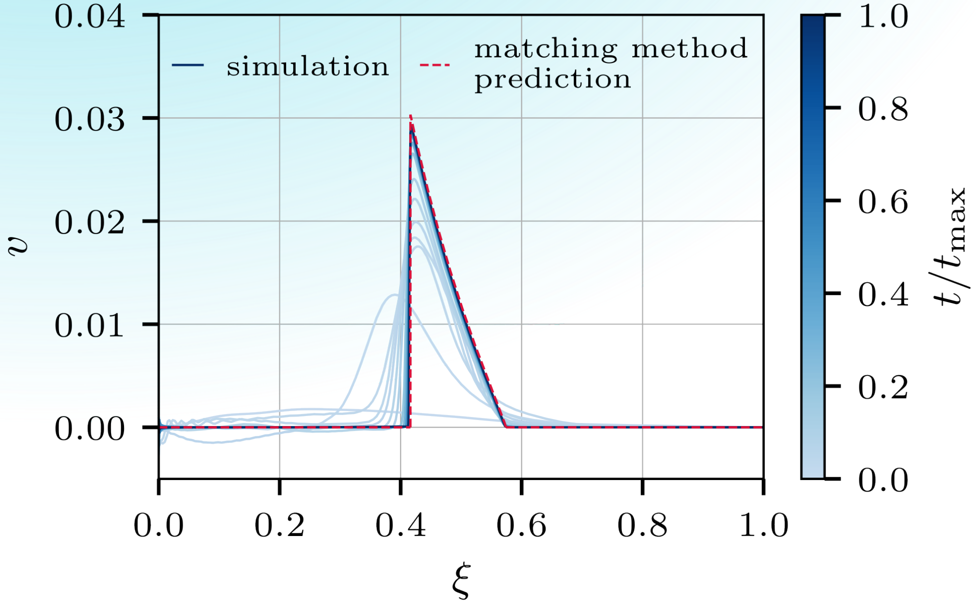
The leading-order finite-temperature corrections to this potential are captured by simple replacement rules

$$\mu_h^2 \to \mu_h^2 + \frac{1}{48}T^2 \left(9g^2 + 3g'^2 + 12y_t^2 + 24\lambda_h + 2\lambda_{hs}\right),$$

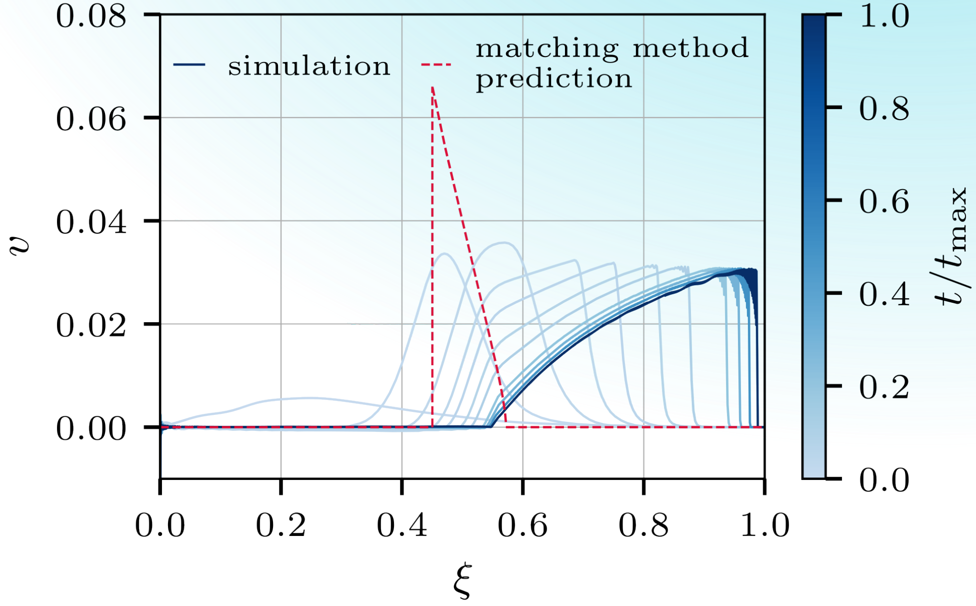
$$\mu_s^2 \to \mu_s^2 + \frac{1}{12}T^2 \left(2\lambda_{hs} + 3\lambda_s\right).$$

Stable LTE profile or run-away?

Real-time EOM solutions fall into two categories:[2]



Stable deflagration/hybrid



Ultrarelativistic detonation (run-away in LTE)

[2] T. Krajewski at al., *Hydrodynamical constraints on the bubble wall velocity,* Phys. Rev. D. , (2023), arXive:2303.18216. Plots curtesy of M. Zych.

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$$\Box\phi + \frac{\partial V(\phi)}{\partial \phi} + \sum_i \frac{dm_i^2}{d\phi} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_i} (f_{\text{eq.}} + \delta f_i) = 0$$

$$\Box\phi + \frac{\partial V_{\text{eff}}(\phi, T)}{\partial \phi} + \sum_i \frac{dm_i^2}{d\phi} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_i} \delta f_i = 0$$

$$\Big\downarrow \quad \int_{\text{Wall}} dz \frac{d\phi}{dz}$$

Backreaction

Dissipative friction

$$\int_{\text{Wall}} dz \frac{\partial V_{\text{eff}}}{\partial T} \frac{dT}{dz} + \underbrace{\sum_i \int d\phi \frac{dm_i^2}{d\phi} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_i} \delta f_i}_{\mathcal{F}} = \Delta V_{\text{eff}}$$

Out-of-equilibrium interactions

The δf^a can be computed by solving Boltzmann equations

$$\mathcal{L}_{\rm Luv.}[f^a_{\rm eq.} + \delta f^a] = -\mathcal{C}[\delta f^b]$$

where $\mathcal{C}$ is a collision term and $\mathcal{L}_{\rm Luv.} = p^\mu \partial_\mu + \frac{1}{2}(\partial_\mu m_a^2)\partial_{p^\mu}\,.$

Exact solutions:

- Local thermal equilibrium $(\mathcal{C} \to \infty,\ \delta f^a = 0)$
- Ballistic limit $(C \to 0,\ \text{maximal}\ \delta f^a)$

Numerical methods for computing dissipative friction

Grad expansion

Expanding δf^a in momenta

$$\delta f = (f'_{\rm eq.}) \times [\mu + T\,(\,\delta u\, u^\mu + \delta\tau\, \bar{u}^\mu)p_\mu + \ldots]\,,$$

+ Minimizing moments

Expansion in orthogonal polynomial baisis

Solving Boltzmann eq. in orthogonal basis (Chebyshev or sph. charmonics)

+ Minimizing Lagrangian

Hydrodynamic simulations

Solving eq. of motion on the lattice with friction term

$$\mathcal{F}_{\rm ans.} = \eta\, u^\mu \partial_\mu \phi.$$

Effective dissipation model

Dissipative friction is often modelled with ansatz

$$\mathcal{F}_{\text{ans.}} = \eta\, u^\mu \partial_\mu \phi,$$

where η is a constant.

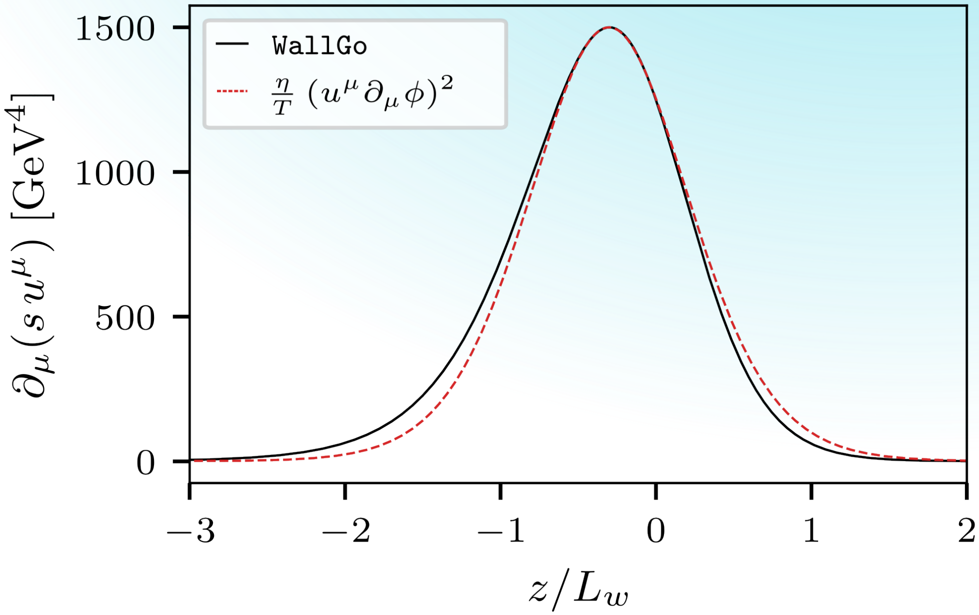
The friction $\mathcal{F}$ fixes the entropy rate

$$\partial_\mu (s u^\mu) = \frac{1}{T} \left(u^\mu \partial_\mu \phi\right) \times \mathcal{F},$$

and thus

$$\partial_\mu (s u^\mu)_{\text{ans.}} = \frac{\eta}{T} \left(u^\mu \partial_\mu \phi\right)^2.$$

Cross-check with `WallGo`[3]



[3] A. Ekstedt et al., *How fast does the WallGo? A package for computing velocities in first-order phase transitions*, arXive: 2411.04970.

New matching condition beyond LTE

By integrating entropy current we can compute entropy produced in the wall:

$$\Delta S = \int_{\rm wall} dz \frac{\eta}{T}(u^\mu \partial_\mu \phi)^2 .$$

Using the ansatz for the background field profile

$$\phi(z) = \frac{v_n}{2}\left(1 - \tanh\left(\frac{z}{L_w}\right)\right) ,$$

we can estimate ΔS as

$$\Delta S \approx \tilde{\eta} \frac{\omega_+}{T_+} \gamma_+^2 v_+^2 ,$$

$$\tilde{\eta} = \frac{\eta\, v_n^2}{3\, \omega_+ L_w} \approx \frac{\eta\, v_n^2}{3\, \omega_s(T_n) L_w} .$$

The new matching codition is

$$\boxed{s_- \gamma_- v_- - s_+ \gamma_+ v_+ = \Delta S.}$$

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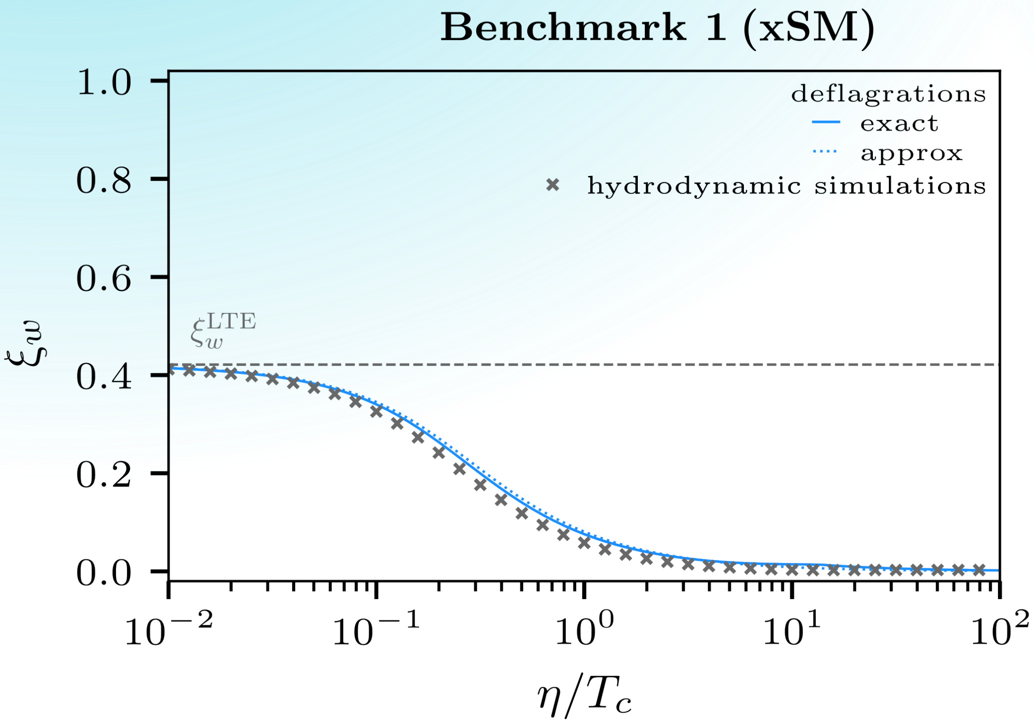
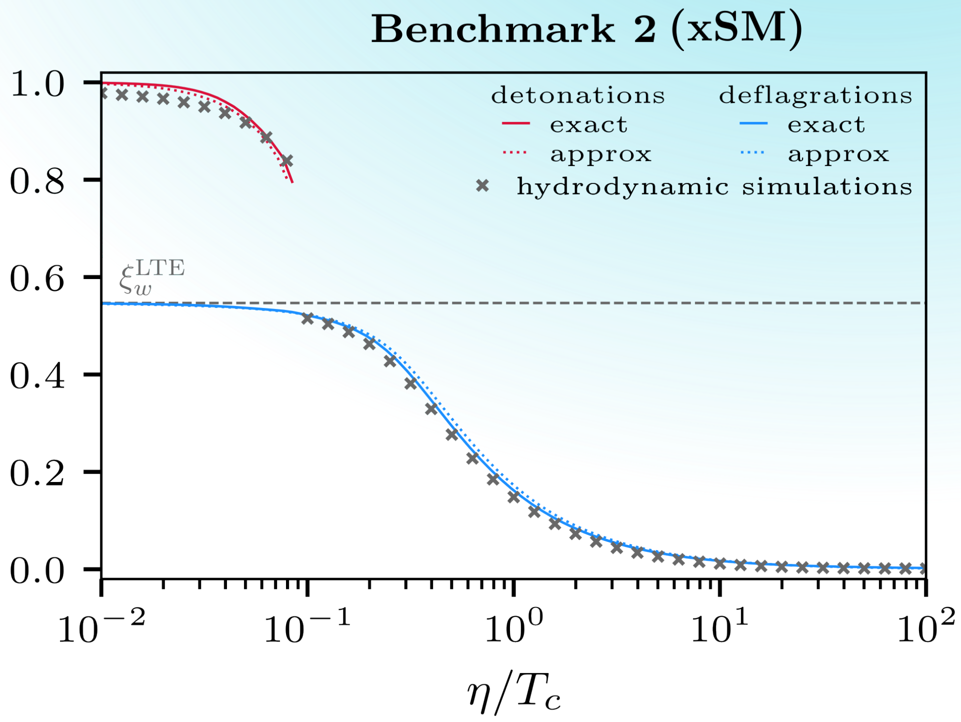
$$\boxed{s_- \gamma_- v_- - s_+ \gamma_+ v_+ = \tilde{\eta} \frac{\omega_+}{T_+} \gamma_+^2 v_+^2 .}$$

Cross-check with the real-time lattice simulations


Benchmark 1 (xSM)

Only deflagration/hybrid branch in LTE


Benchmark 2 (xSM)

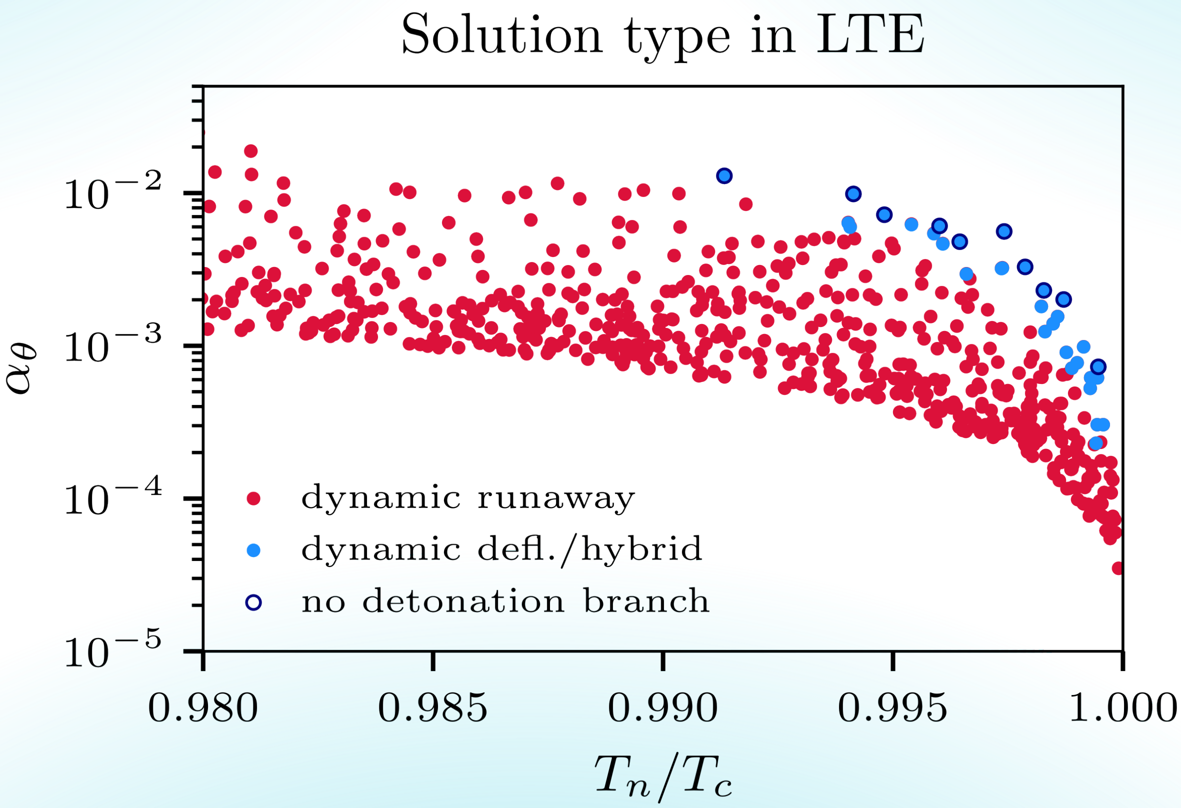
Deflagration/hybrid and detonation branches coexist in LTE

LTE selection rule

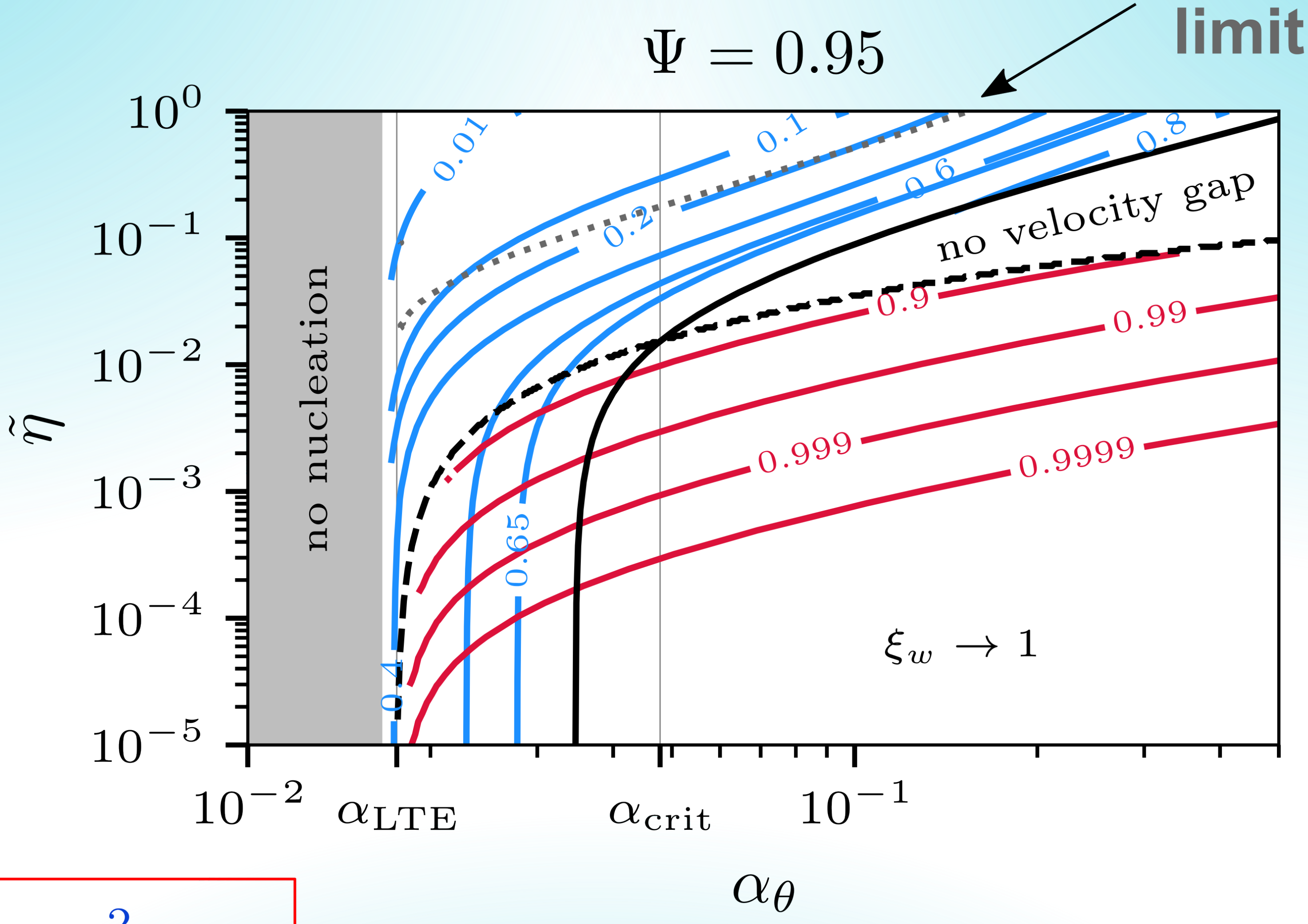
$$\left(\alpha_\theta, \Psi_n\right)$$

Deflagration/hybrid

Runaway detonation



Wall velocity beyond LTE



Ballistic limit

LTE limit

$$\tilde{\eta} \approx \frac{\eta\, v_n^2}{3\,\omega_s(T_n)L_w}.$$

How to compute $\tilde{\eta}$?

1) Using Grad expansion, we find an equation for chemical potential $\mu(z)$ for a single species

$$\frac{d\mu(z)}{dz} + b\frac{T}{\gamma_w \xi_w}\mu(z) = \frac{c}{2T^2}\phi(z)\phi'(z),$$

where b and c are slowly varying functions of m/T.

2) Next, we evaluate friction

$$\mathcal{F}^{(\mu)} = \frac{c}{2} N_{\rm eff}\, T^2\, \phi(z)\mu(z).$$

3) Finally, we fit our ansatz $\mathcal{F}_{\rm ans.}$ to $\mathcal{F}^{(\mu)}$. The fitted η is used to compute $\tilde{\eta}$ from which we obtain ξ_w.

Scalling of $\tilde{\eta}$

In the natural units the eq. for chemical potential reads

$$\mu'(\tilde{z}) + \frac{b}{\gamma_w \xi_w}\mu(\tilde{z}) = \frac{c}{2T^2}\phi\phi'(\tilde{z}).$$

with $\tilde{z} = T\,z.$ Thus,

$$\mu \sim (v_n/T_n)^2,$$

$$\mathcal{F}^{(\mu)} \propto T^2\phi\,\mu \sim T_n^2\,v_n(v_n/T_n)^2,$$

while

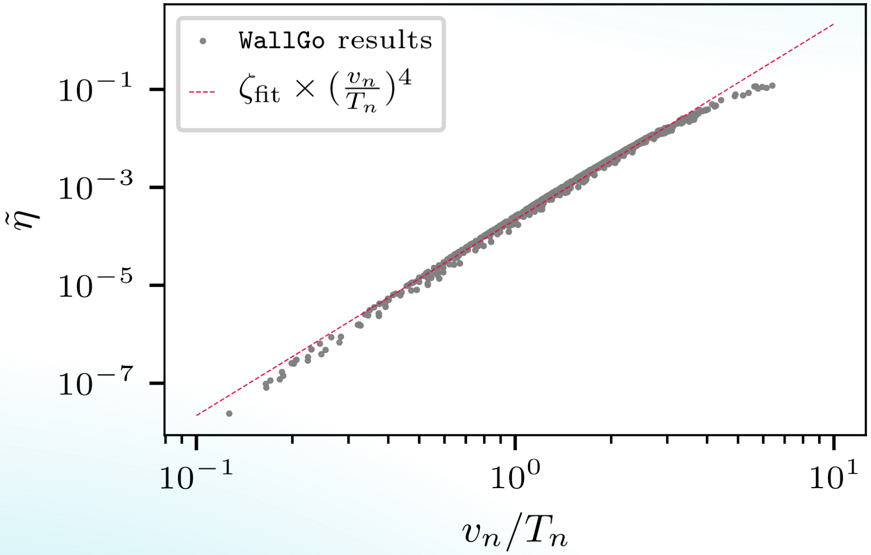
$$\mathcal{F}_{\text{ans.}} = \eta\,\gamma_w\xi_w T\phi'(\tilde{z}) \sim \eta\,T_n v_n.$$

Hence,

$$\eta \sim T_n(v_n/T_n)^2,$$

$$\tilde{\eta} \sim v_n^2/T_n^3\eta \sim (v_n/T_n)^4.$$

Cross-check with `WallGo`[3]



[3] A. Ekstedt et al., *How fast does the WallGo? A package for computing velocities in first-order phase transitions*, arXive: 2411.04970.

Error estimation

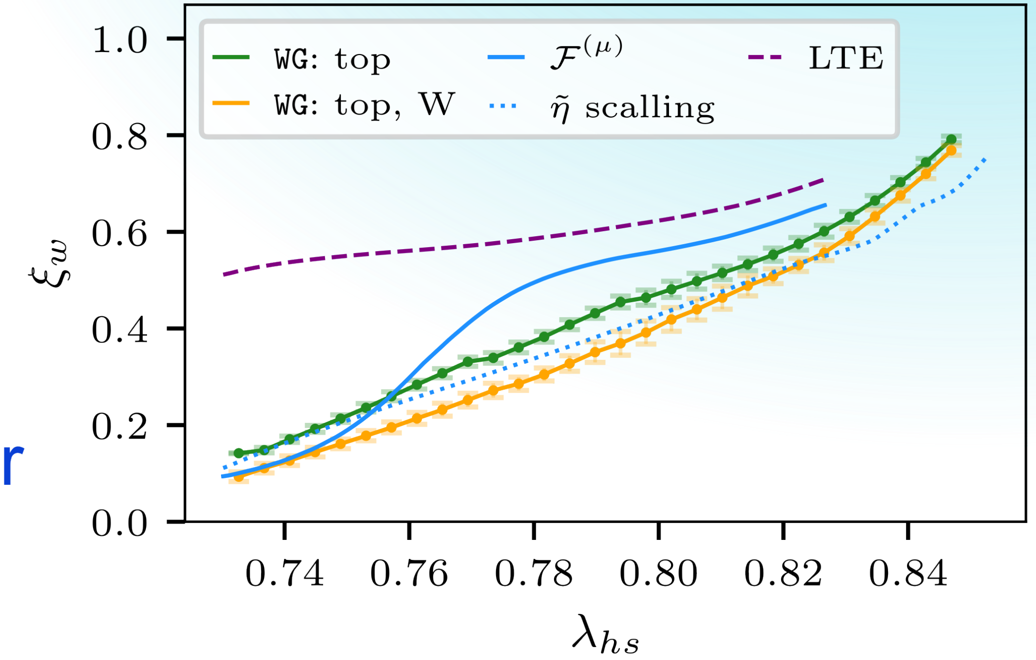
The coefficient $\tilde{\eta}$ obeys simple scalling relation

$$\tilde{\eta} = \zeta \left(\frac{v_n}{T_n}\right)^4 .$$

Assuming the friction is induced solely by the top quark, we evaluated prefactor

$$\zeta_{\rm xSM} \approx 2.5 \times 10^{-4} .$$

Cross-check with `WallGo`[3]



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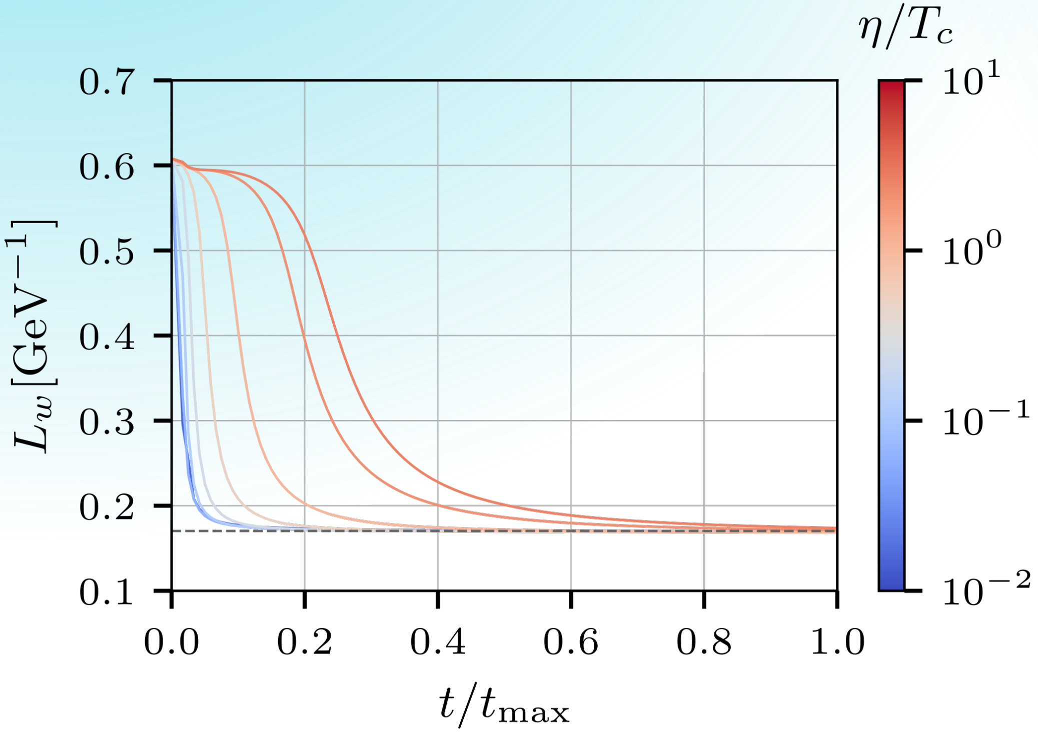
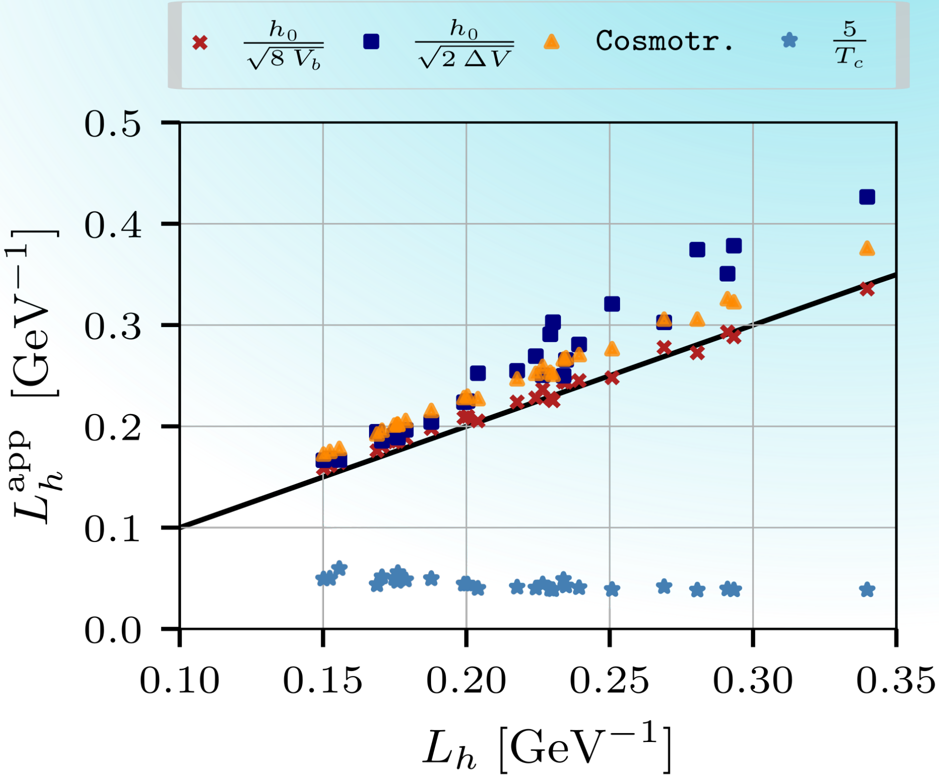
Summary

- Quantitative study of first-order phase transitions requires the computation of the terminal wall velocity ξ_w.
- The terminal wall velocity is determined by the entropy balance across the bubble wall.
- In some cases, multiple stationary states can coexist, and one must perform real-time simulations to determine ξ_w.
- **The slow deflagration/hybrid shock waves form if no stable detonation branch exists.**
- Out-of-equilibrium friction scales as $(v_n/T_n)^4$, which can be used to independently fix the terminal wall velocity.

Estimation of the Wall Width



Evolution of the wall width L_w in the real-time hydrodynamic simulations



Different analytical estimations of L_w in LTE limit