A kinetically non-canonical scalar quantum field theory in cosmology

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Outline

What is a non-canonical field theory?

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Non-canonical kinetic terms:

$$f\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi,\ \phi\right)\ \text{etc.}$$

Classical relativistic point particle

$$L = -\ mc\sqrt{\dot{x}_\mu \dot{x}^\mu} = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} = m\vec{v}\gamma$$

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[2] Garriga and Mukhanov 1999.

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Non-canonical field theories were already investigated, for example[1],[2].

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Making a non-canonical quantum field theory

Action

Let ϕ be a scalar field and let $X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Then a kinetically non-canonical theory of field ϕ has the action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{R}{16\pi G} + \underbrace{K(X,\phi) - V(\phi)}_{\mathcal{L}}\right), \tag{1}$$

Energy momentum tensor

The Hilbert energy momentum tensor of ϕ is:

$$T^{\mu\nu} = \frac{\partial K}{\partial X}\partial^{\mu}\phi\partial^{\nu}\phi - g^{\mu\nu}(K - V)\,. \tag{2}$$

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In the flat FLRW metric:

$$g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \mathrm{d}t^2 - a(t)^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,, \tag{3}$$

it gives the energy density and pressure:

$$\rho = \frac{\partial K}{\partial X}\dot{\phi}^2 - (K - V)\,, \tag{4}$$

$$p = \frac{\partial K}{\partial X}\frac{1}{3a^2}(\nabla\phi)^2 + (K - V)\,, \tag{5}$$

Equation of Motion

Varying S w.r.t. ϕ gives the following equation of motion ($H \equiv \frac{\dot{a}}{a}$):

$$\frac{\partial^2 K}{\partial X^2}\partial_\mu X \partial^\mu \phi + \frac{\partial^2 K}{\partial \phi \partial X} 2X + \frac{\partial K}{\partial X}\left[\partial_\mu \partial^\mu \phi + 3H\partial^0 \phi\right] - \frac{\partial K}{\partial \phi} + \frac{\partial V}{\partial \phi} = 0\,. \tag{6}$$

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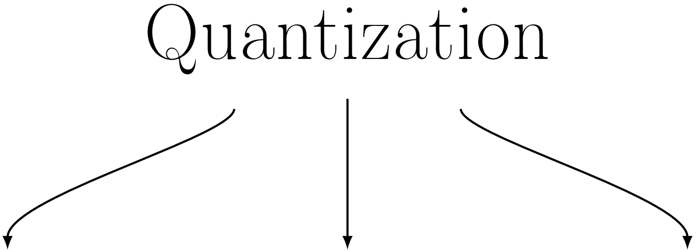
EoM of ϕ is nonlinear $\Rightarrow$ Solutions are not additive $\Rightarrow$ We can't construct a Hilbert space.
Even if we could, the canonical momentum of ϕ is:

$$\Pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \frac{\partial K}{\partial X}\dot{\phi}\,, \tag{7}$$

which after quantizing $\hat{\Pi}$ would be a nonlinear operator.

Quantization



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such that
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$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x})$

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$\phi(t, \vec{x}) = \phi_0(t) + \delta\phi(t, \vec{x})$

My current
approach
(valid, when $\delta\phi$ is "small")

Quantization

Expand action to quadratic terms in $\delta\phi$ (a "free theory").
Go to Fourier (momentum) space $\delta\phi \mapsto \phi_{\mathsf{k}}$.

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Go to Fourier (momentum) space $\delta\phi \mapsto \phi_{\mathrm{k}}$.
At times, when ϕ_{k} follows

$$\ddot{\phi}_{\mathrm{k}} + \omega_k^2 \phi_{\mathrm{k}} \approx 0\,, \tag{8}$$

and ω_k changes very slowly (adiabatically), we can quantize as

$$\hat{\phi}_{\mathrm{k}} = \hat{a}_k \frac{\alpha_k}{\sqrt{2\omega_k}} e^{i\int \omega_k t} + \hat{a}_k^\dagger \frac{\beta_k}{\sqrt{2\omega_k}} e^{-i\int \omega_k t} \tag{9}$$

Scalar Dirac–Born–Infeld theory

A DBI like action/Lagrangian features the kinetic term:

$$K(X) = -\Lambda^4 \left(\sqrt{1 - \frac{2X}{\Lambda^4}} - 1 \right) . \tag{10}$$

[3] Born and Infeld 1934a; Born and Infeld 1934b; Born and Infeld 1935; Dirac 1960.

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Constraints $X \leq \Lambda^4/2$.

$K(X) \to X$ as $\Lambda \to \infty$.

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DBI scalar

I investigate a real scalar field ϕ with Lagrangian density:

$$\mathcal{L} = -\Lambda^4\left(\sqrt{1-\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}}-1\right)-V(\phi)\,. \tag{11}$$

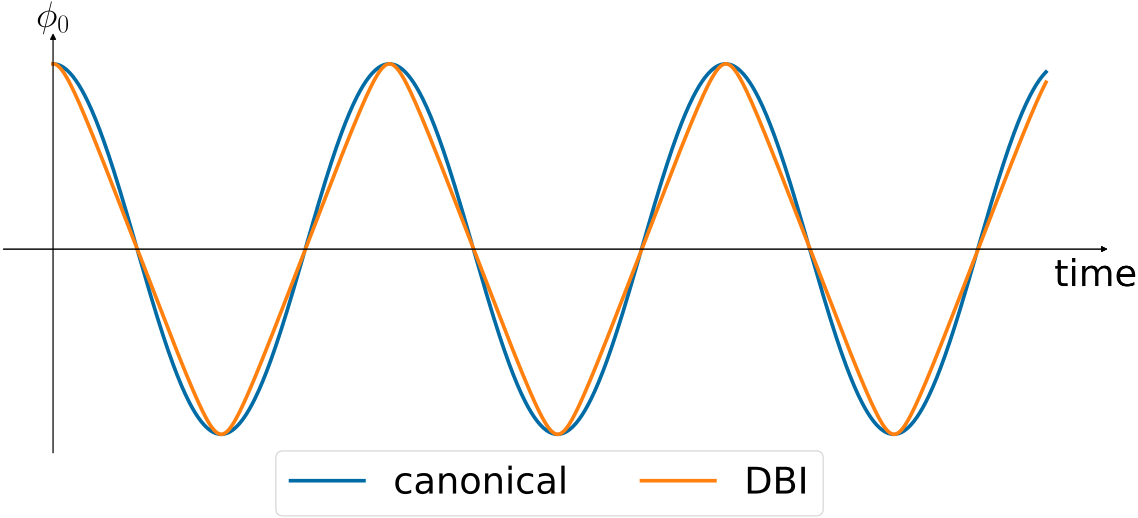
I analyse its behaviour during and after Inflation, looking at self production of ϕ particles.
I treat ϕ as an spectator field, not an inflaton.

Decomposing $\phi(t,\vec{x}) = \phi_0(t) + \delta\phi(t,\vec{x})$ and $\delta\phi \mapsto \phi_{\vec{k}}$:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0\left(1 - \frac{\dot{\phi}_0^2}{\Lambda^4}\right) + V'(\phi_0)\left(1 - \frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{3/2} \simeq 0\,, \tag{12}$$

$$\ddot{\phi}_k + 3\left[H\left(1 - 3\frac{\dot{\phi}_0^2}{\Lambda^4}\right) - \frac{V(\dot{\phi}_0)}{\Lambda^4}\left(1 - \frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{\frac{1}{2}}\right]\dot{\phi}_k + \\ + \left[V''(\phi_0)\left(1 - \frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{\frac{3}{2}} + \left(1 - \frac{\dot{\phi}_0^2}{\Lambda^4}\right)\frac{k^2}{a^2}\right]\phi_k \simeq 0\,. \tag{13}$$

ρ_0, p_0, $\delta\rho$ and δp can be constructed from $T^{\mu\nu}$.



Behaviour of ϕ_0 ($H = 0$, quadratic potential)

Behaviour during Inflation

During Inflation $H = H_I$ is very large and constant. Analysing the EoM of ϕ_0 I found that:

$$\dot{\phi}_0 = \pm\sqrt{\frac{\Lambda^4}{1 + C_0 e^{6H_I t}}} + \mathcal{O}\left(\frac{V'(\phi_0)}{H_I}\right), \tag{14}$$

which means ϕ_0 changes very slowly.

Behaviour during Inflation

Neglecting $\dot{\phi}_0^2$:

$$\ddot{\phi}_k + 3H_I\dot{\phi}_k + \left[V''(\Phi_0) + k^2 a^{-2}\right]\phi_k \simeq 0\,, \tag{15}$$

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After redefinition $\psi_k = a\phi_k$ and $\mathrm{d}t = a\mathrm{d}\eta$:

$$\partial_\eta^2\psi_k + \left[k^2 - \frac{V''(\Phi_0)}{H_I^2\eta^2}\right]\psi_k = 0\,. \tag{16}$$

Solutions (Bunch–Davies vacuum):

$$\psi_k^{\pm}(\eta) = \frac{1}{\mathcal{N}} e^{\mp i\frac{2H_I^2}{V''(\Phi_0)}k\eta}\left(1 \mp \frac{i}{\frac{2H_I^2}{V''(\Phi_0)}k\eta}\right)\,. \tag{17}$$

Stochastic behaviour of ϕ_0

Superhorizon modes ($k \lesssim aH_I$) contribute to the effective value of ϕ_0 inside a horizon. They exit the horizon at an arbitrary phase, making ϕ_0 walk randomly.

[4] Graham and Scherlis 2018.

Stochastic behaviour of ϕ_0

Superhorizon modes ($k \lesssim aH_I$) contribute to the effective value of ϕ_0 inside a horizon. They exit the horizon at an arbitrary phase, making ϕ_0 walk randomly.
Following[4], I assume effective ϕ_0 to follow a Langevin equation

$$\dot{\phi}_0 = \pm\sqrt{\frac{\Lambda^4}{1 + C_0 e^{6H_I t}}} + \mathcal{O}\left(\frac{V'(\phi_0)}{H_I}\right) + f(t)\,, \tag{18}$$

with a Gaussian noise $f(t)$ with correlation function

$$\langle f(t_1) f(t_2)\rangle = \frac{H_I^3}{4\pi^2}\delta(t_1 - t_2)\,. \tag{19}$$

[4] Graham and Scherlis 2018.

Stochastic behaviour of ϕ_0

After some time ($\sqrt{\ldots}$ in $\dot{\phi}_0$ vanishes) its probability distribution function becomes:

$$\varrho(\phi_0) \propto \exp\left(-\mathcal{O}\left(\frac{V(\phi_0)}{H_I^4}\right)\right). \tag{20}$$

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$$\langle\phi_0\rangle = 0\,. \tag{21}$$

$$V(\phi_0) = \frac{1}{2}m^2\phi_0^2 \Rightarrow \ \sigma \propto \frac{H_I^2}{m} \qquad V(\phi_0) = \frac{1}{4!}\lambda\phi_0^4 \Rightarrow \ \sigma \propto \frac{H_I}{\lambda^{\frac{1}{4}}}\,. \tag{22}$$

Therefore $|\phi_0|$ of order $\mathcal{O}(\sigma)$ during Inflation is not unreasonable.

Is this consistent with ϕ being an spectator field?

Both cases $V(\phi_0) = \frac{1}{2}m^2\phi_0^2$ and $V(\phi_0) = \frac{1}{4!}\lambda\phi_0^4$ give:

$$V(\phi_0) \lesssim \mathcal{O}\left(H_I^4\right) , \tag{23}$$

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From Friedman equation:

$$H^2 = \frac{8\pi}{3m_{Pl}^2}\rho_{tot} , \tag{24}$$

the inflation has a $\mathcal{O}\left(H_I^2 m_{Pl}^2\right)$ contribution to the total energy density, so contribution from ϕ_0 is negligible in comparison. It is a spectator.

Simulating DBI in the early Universe

Potential

I assume the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 + V_0\,. \tag{25}$$

In particular m^2 may be negative (symmetry breaking potential), so V_0 is there to keep $V \geq 0$ and:

$$|\mathrm{VeV}| = \sqrt{\frac{-6m^2}{\lambda}}\,. \tag{26}$$

And for initial conditions I take a vacuum, in a similar way to[5].

[5] Greene et al. 1997; Felder, Kofman, and Linde 2001; Kamada and Sakurai 2026.

Dimensionless variables

My model has a scaling symmetry.

$$\varphi = \frac{\phi}{\phi_0(t=0)} \qquad \tau = \sqrt{\lambda}\phi_0(t=0)t \qquad \tilde{\partial}_\nu = \frac{1}{\sqrt{\lambda}\phi_0(t=0)}\partial_\nu$$

$$\mu^2 = \frac{m^2}{\lambda\phi_0^2(t=0)} \qquad \eta^4 = \frac{\Lambda^4}{\lambda\phi_0^4(t=0)} \qquad \mathscr{H} = \frac{1}{\sqrt{\lambda}\phi_0(t=0)}H$$

$$\mathcal{V}_0 = \frac{1}{\lambda\phi_0^4(t=0)}V_0 \qquad \kappa^2 = \frac{k^2}{\lambda\phi_0^2(t=0)}$$

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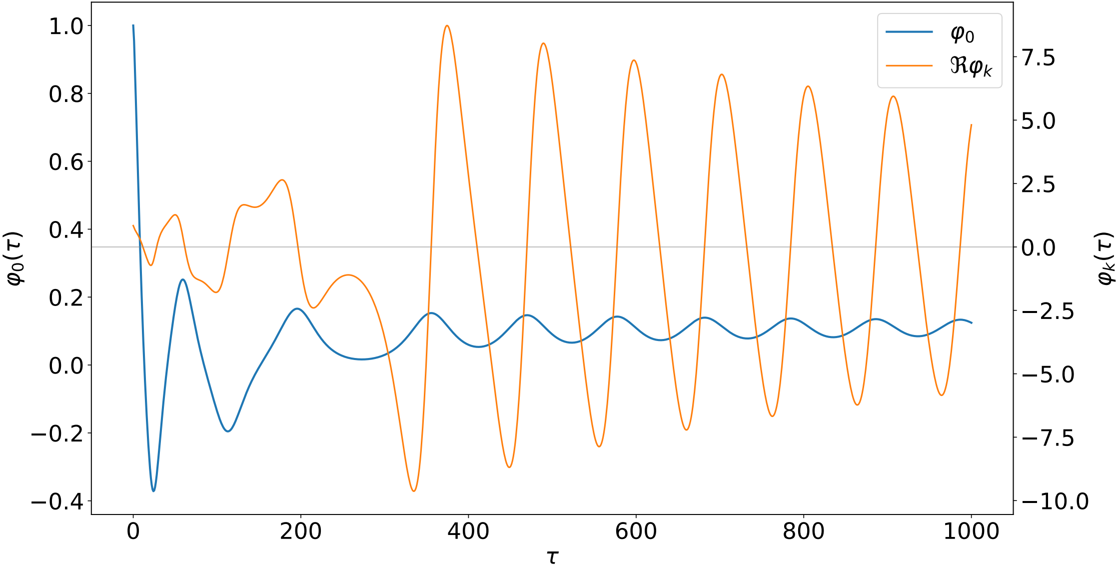
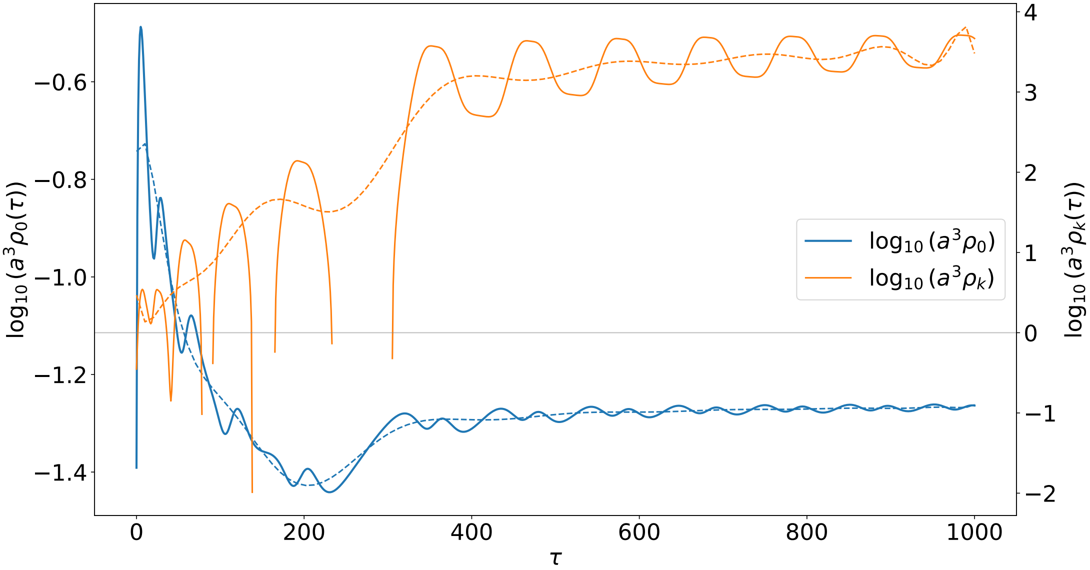
When $\mu^2 < 0$, starting with $|\phi_0| \gg |\mathrm{VeV}|$ implies the physically interesting parameter space is:

$$\mu^2 \in \left(-\frac{1}{6}, 0\right) \tag{27}$$



Sample field mode evolution just after Inflation, $\mu^2 = -0.0021$, $\eta = 0.32$, $\kappa = 0.0039$.



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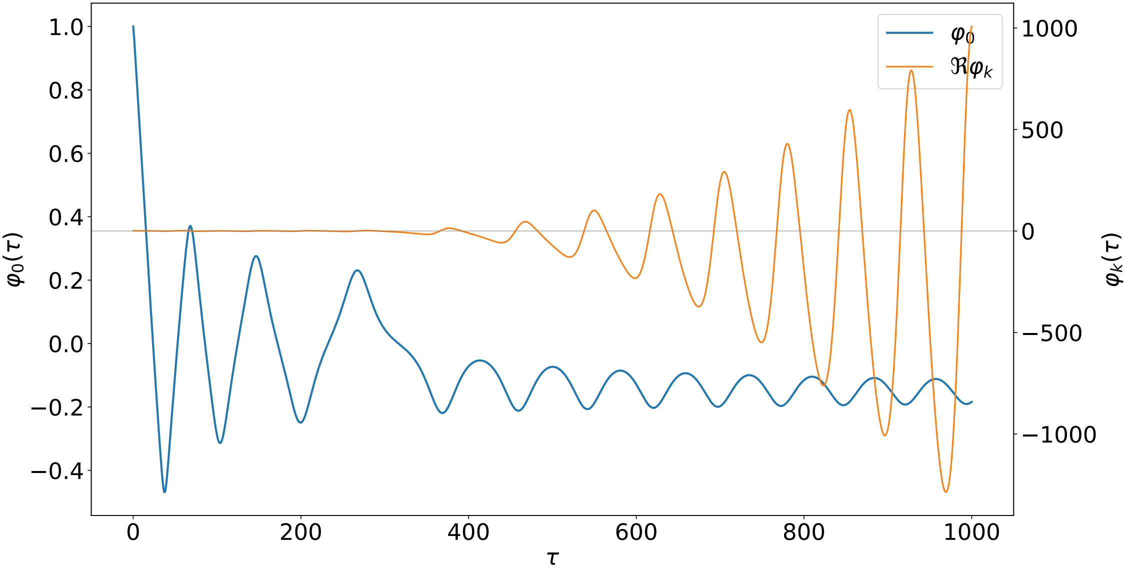
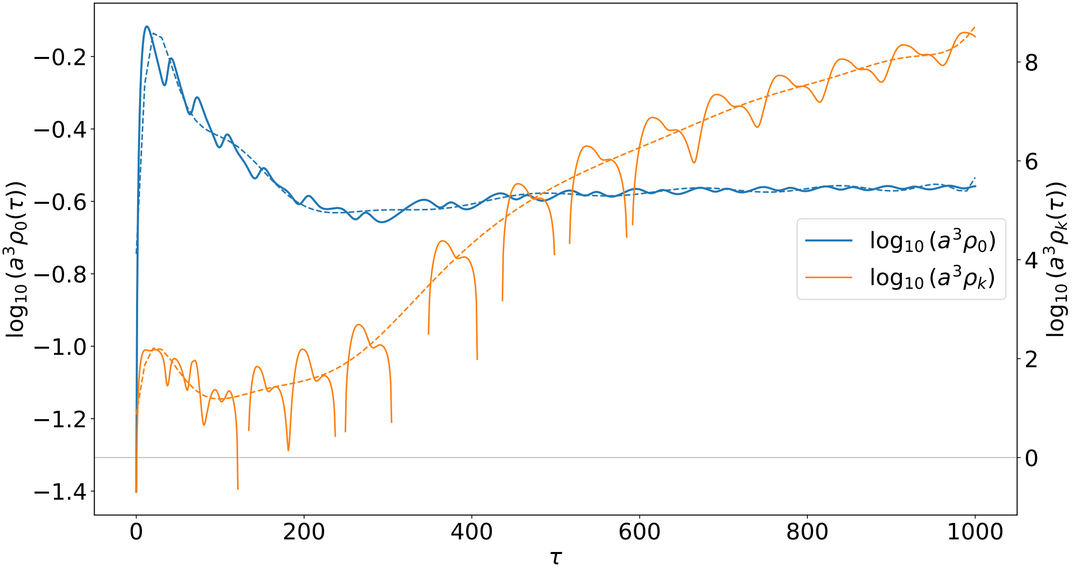


Sample field mode evolution just after Inflation, $\mu^2 = -0.0041$, $\eta = 0.21$, $\kappa = 1.0$.



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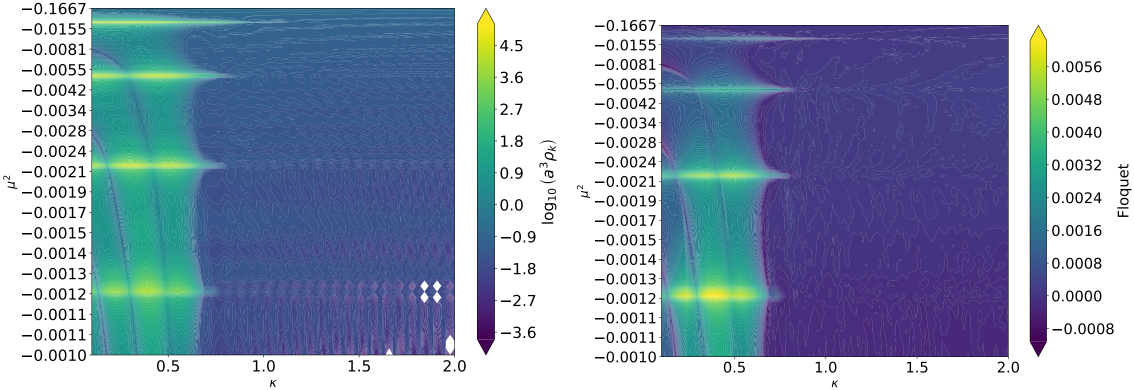
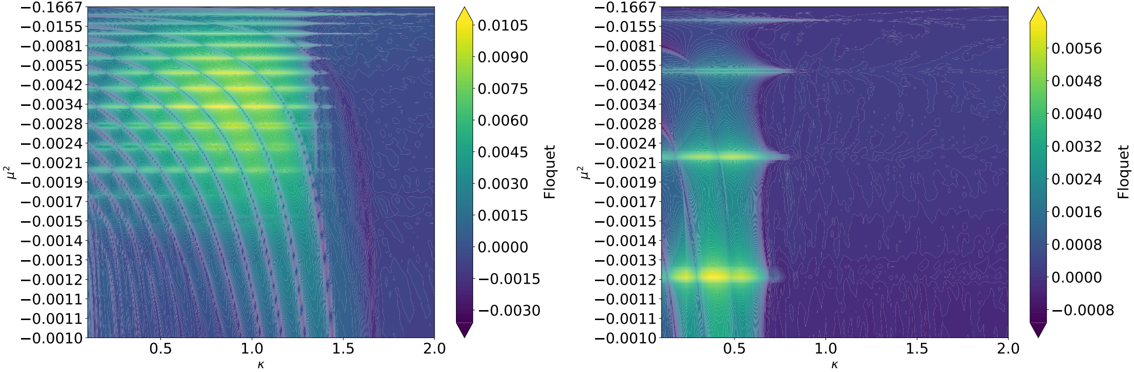
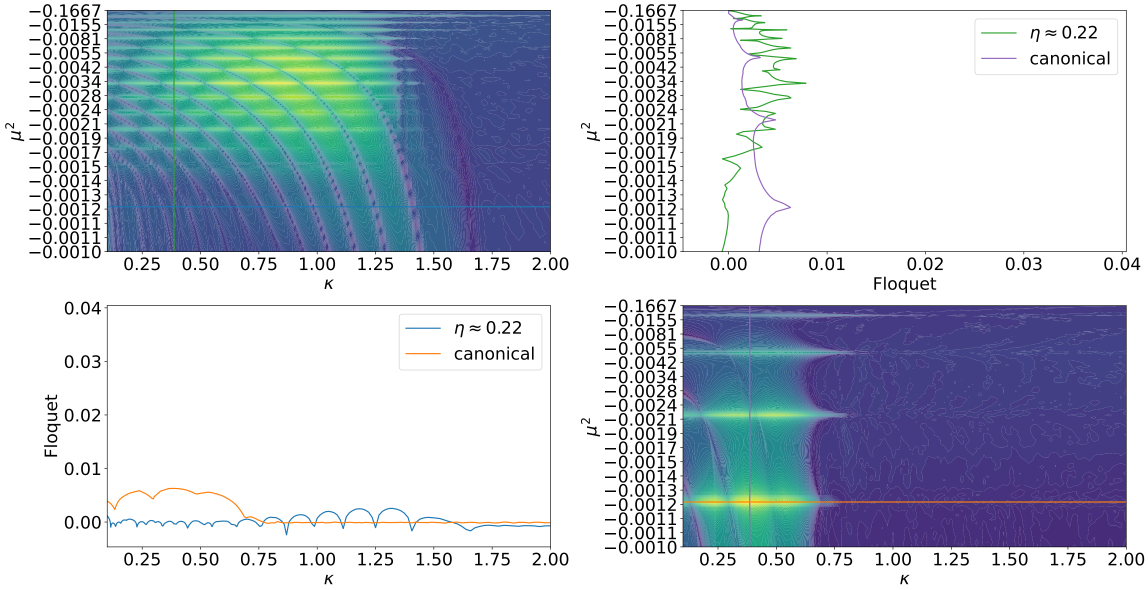


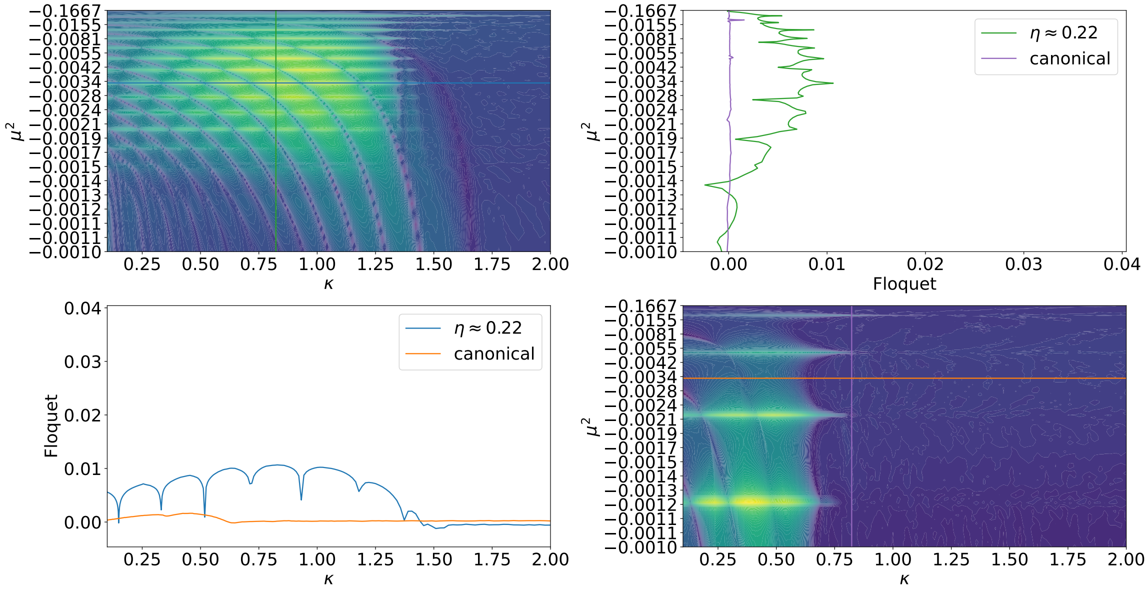
Plots of contribution to commoving energy density and "Floquet exponents" for different κ and μ^2 in the canonical limit ($\eta \to \infty$). κ given at the start of simulation (soon after the end of Inflation).



Caparison of the band structure of a non-canonical DBI model (here $\eta \approx 0.22$) (left panel) with a canonical model (right panel).



Sections through the band structures of DBI ($\eta \approx 0.22$) and canonical model.



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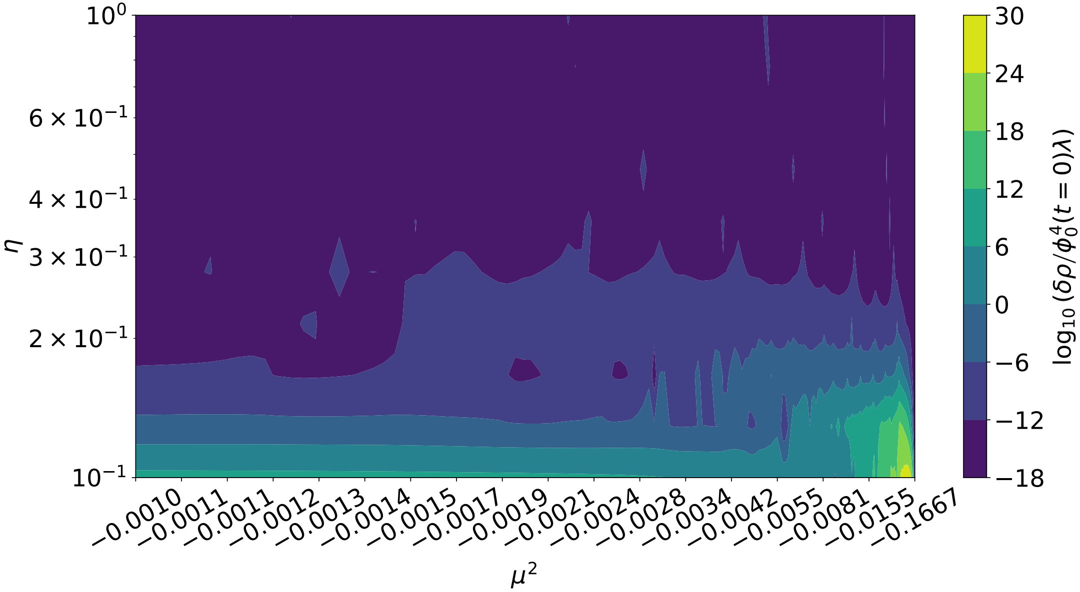

Contributions from different modes integrated to get $\delta\rho$.

Conclusions

- Field theories with properly constructed non-canonical kinetic terms are consistent with Lorentz covariance and gauge invariance of the action;
- They are rarely investigated due to nonlinear equations of motion and associated difficulty in quantizing them;
- The DBI scalar has a modified particle production band structure compared to the canonical model. In some regions it is more efficient.
- A better quantization scheme might be needed.

Thank you for your attention!

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Extra: Polyakov like action[6]

$$\tilde{S} = \int \mathrm{d}^4x\sqrt{-g}\left(\Lambda^4\left[-\frac{e}{2}\left(e^{-2}\left(\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}-1\right)-1\right)-1\right]-V(\phi)\right) \tag{28}$$

$$\delta_e\tilde{S} = \int \sqrt{-g}\Lambda^4\left[-\frac{\delta e}{2}\left(e^{-2}\left(\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}-1\right)-1\right)-\frac{e}{2}\left(-2e^{-3}\delta e\left(\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}-1\right)\right)\right] \tag{29}$$

$$\delta_e\tilde{S} = 0 \qquad \Rightarrow \qquad e^2 = \left(1-\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}\right) \tag{30}$$

$$\tilde{S}\Big|_{\delta_e\tilde{S}=0} = \int \mathrm{d}^4x\sqrt{-g}\left(-\Lambda^4\left[\sqrt{1-\frac{\partial_\mu\phi\partial^\mu\phi}{\Lambda^4}}-1\right]-V(\phi)\right) \tag{31}$$

[6] Blumenhagen, Lüst, and Theisen 2013.

Extra: Expressions for energy density and pressure in DBI

$$\rho_0 = \frac{\dot{\phi}_0^2}{\sqrt{1 - \frac{\dot{\phi}_0^2}{\Lambda^4}}} + \Lambda^4 \left(\sqrt{1 - \frac{\dot{\phi}_0^2}{\Lambda^4}} - 1 \right) + V(\phi_0)\,. \tag{32}$$

$$p_0 = -\Lambda^4 \left(\sqrt{1 - \frac{\dot{\phi}_0^2}{\Lambda^4}} - 1 \right) - V(\phi_0)\,. \tag{33}$$

Notice that when $\dot{\phi}_0$ is small, $p_0 = -\rho_0 + \mathcal{O}\left(\dot{\phi}_0^2\right)$.

Extra: Expressions for energy density and pressure in DBI

$$\delta\rho = \frac{1}{(aH)^3}\int \mathrm{d}k \frac{k^2}{2\pi^2}\left[\left(1-\frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{-\frac{5}{2}}\left(\frac{1}{2}+\frac{\dot{\phi}_0^2}{\Lambda^4}\right)|\dot{\phi}_k|^2 + \left\{\frac{1}{2}V''(\phi_0) + \right.\right.$$

$$\left.\left. -\left(1-\frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{-\frac{3}{2}}\left(\frac{1}{2}-\frac{\dot{\phi}_0^2}{\Lambda^4}\right)\frac{k^2}{a^2}\right\}|\phi_k|^2\right] \equiv \frac{1}{(aH)^3}\int \mathrm{d}k \frac{k^2}{2\pi^2}\rho_k \, . \quad (34)$$

Similarly:

$$p_k \equiv \left(1-\frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{-\frac{3}{2}}\frac{1}{2}|\dot{\phi}_k|^2 + \left[-\frac{1}{2}V''(\phi_0) + \left(1-\frac{\dot{\phi}_0^2}{\Lambda^4}\right)^{-\frac{1}{2}}\frac{1}{6}\frac{k^2}{a^2}\right]|\phi_k|^2 \, . \quad (35)$$

Extra: Initial conditions for ϕ_k

Should be given by Bunch–Davies vacuum... But evolving from Inflation to post–Inflation requires choosing a specific Inflation model.
Instead, to stay Inflation model agnostic, I follow the approach of[7]:
construct late time particle count:

$$N = \int \mathrm{d}k \frac{k^2}{2\pi^2} n_k \,. \tag{36}$$

Assume initial vacuum $N = 0 \Rightarrow n_k = 0$.
Get initial condition for ϕ_k from that.

[7] Greene et al. 1997; Felder, Kofman, and Linde 2001; Kamada and Sakurai 2026.

Extra: Initial conditions for ϕ_k

$$\hat{\phi}_k = u_k(t)\hat{a}_k + u_k^*(t)\hat{a}_k^\dagger \tag{37}$$

Assuming $\phi_0 = \mathrm{VeV}$ at late times and a changes slowly gives:

$$u_k(t) = \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k t}\,, \tag{38}$$

with $\omega_k^2 = V''(\mathrm{VeV}) + k^2/a^2$, therefore

$$n_k = \frac{1}{2\omega_k}|\dot{u}_k|^2 + \frac{1}{2}\omega_k|u_k|^2 - \frac{1}{2}\,. \tag{39}$$