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$$v_\theta = \begin{cases} \dfrac{\kappa r}{a^2} & r<a \\ \dfrac{\kappa}{r} & r>a \end{cases} \qquad v_r = 0$$

a) Vorticity

$$\omega = \mathrm{rot}\, v = \frac{1}{r}\begin{vmatrix} e_r & r e_\varphi & e_z \\ \partial_r & \partial_\theta & \partial_z \\ v_r & r v_\theta & v_z \end{vmatrix} = e_z \frac{1}{r}\partial_r r v_\theta - \overset{0}{e_r \partial_z r v_\theta}$$

only nonzero

$$\omega_z = \frac{1}{r}\partial_r r v_\theta = \frac{1}{r}\partial_r \begin{Bmatrix} \dfrac{\kappa r^2}{a^2} \\ \kappa \end{Bmatrix} = \frac{1}{r}\begin{Bmatrix} 2\dfrac{\kappa r}{a^2} \\ 0 \end{Bmatrix} = \begin{cases} \dfrac{2\kappa}{a^2} & r<a \\ 0 & r>a \end{cases}$$

$$\Gamma(a) = \int \omega \cdot dS = \int \omega_z\, r\, d\varphi\, d\theta = \underbrace{\int_{r<a} \frac{2\kappa}{a^2} r\, d\varphi\, d\theta}_{\text{this is area of the disk of radius } a} + \overset{\nearrow 0}{\int_{r>0} 0\, d\varphi\, d\theta} =$$

$$\Gamma(a) = \frac{2\kappa}{a^2}\cdot \pi a^2 = 2\kappa\pi$$

b) Let's use Lamb equation:

$$\underbrace{\frac{\partial v}{\partial t}}_{\substack{0 \\ \text{for steady}}} + \underbrace{(\nabla \times v)}_{\omega} \times v = -\nabla\Big(\frac{v^2}{2} + \frac{p}{\rho} + \underbrace{\phi}_{\substack{0 - \text{no potential.}}}\Big)$$

$$\omega \times v = -\nabla\Big(\frac{v^2}{2} + \frac{p}{\rho}\Big)$$

$$\omega = \omega e_z \qquad v = v_\theta e_\theta$$

$$\omega \times v = -\omega v_\theta e_r$$

$$\omega v_\theta e_r = \nabla\Big(\frac{v^2}{2} + \frac{p}{\rho}\Big)$$

So it is one diff. equation

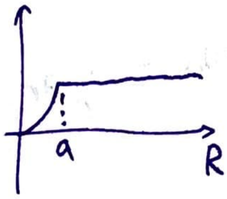
$$\omega v_\theta = \frac{\partial}{\partial r}\left(\frac{v^2}{2} + \frac{p}{\rho}\right)$$

$$\downarrow$$

$$\begin{cases} \frac{2\kappa}{a^2}\cdot\frac{\kappa r}{a^2} & r<a \\ 0 & r>a \end{cases}$$

$$\int_0^R \omega v_\theta \, dr = \cancel{2}\frac{\kappa^2}{a^4}\cdot\frac{1}{\cancel{2}}R^2 \leftarrow \text{for } R \le a$$

$$\frac{\kappa^2}{a^2} \quad \leftarrow \text{for } R \ge a$$



$$\frac{\kappa^2}{a^4}\min(\cancel{R^2}\, r^2, a^2) = \frac{v^2}{2} + \frac{p(r)}{\rho} + C$$

at $r \to \infty$ we have $v \to 0$

$$\frac{\kappa^2}{a^2} = 0 + \frac{p_\infty}{\rho} + C \qquad C = \frac{\kappa^2}{a^2} - \frac{p_0}{\rho}$$

$$\frac{\kappa^2}{a^4}\min(r^2, a^2) = \frac{1}{2}\begin{Bmatrix} \frac{\kappa^2 r^2}{a^4} & r<a \\ \frac{\kappa^2}{r^2} & r>0 \end{Bmatrix} + \frac{p(r)}{\rho} + \frac{\kappa^2}{a^2} - \frac{p_0}{\rho}$$

Let's split it in two regions:

1) $r<a$

$$\frac{\kappa^2 r^2}{a^4} = \frac{1}{2}\frac{\kappa^2 r^2}{a^4} + \frac{p(r)-p_0}{\rho} + \frac{\kappa^2}{a^2}$$

$$\rho\left(\frac{1}{2}\frac{\kappa^2 r^2}{a^4} - \frac{\kappa^2}{a^2}\right) = p(r) - p_0$$

$$\boxed{p(r) = p_0 + \rho\frac{\kappa^2}{a^4}\left(\frac{1}{2}r^2 - a^2\right)}$$

2) $r \ge a$

$$\cancel{\frac{\kappa^2}{a^2}} = \frac{1}{2}\frac{\kappa^2}{r^2} + \frac{p(r)}{\rho} + \cancel{\frac{\kappa^2}{a^2}} - \frac{p_0}{\rho}$$

$$-\rho\frac{1}{2}\frac{\kappa^2}{r^2} = p(r) - p_0$$

$$\boxed{p(r) = p_0 - \frac{\rho}{2}\frac{\kappa^2}{r^2}}$$

In the middle

$$p(0) = p_0 + \rho\frac{\kappa^2}{a^4}(0 - a^2)$$

$$p(0) = p_0 - \rho\frac{\kappa^2}{a^2}$$

$$\Delta p = -\rho\frac{\kappa^2}{a^2}$$