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Problem 3

We are given an inviscid and incompressible fluid, which occupies the region $x \geq 0$ bounded by rigid wall at $x = 0$. At a point $(d, 0)$ there is a linear vortex of circulation Γ. ✓

(a) The wall gives us the impenetrable condition. It requires no normal flow so:

$v_n = 0$ on $x = 0$. In the 2D flow (which is incompressible and potential) this is equivalent to the fact: wall is a streamline $\rightsquigarrow$ $\Psi = \text{const.}$ on $x = 0$ ✓

We will use complex variable $z = x + iy$ and complex potential (the same as during the lecture and tutorials). So $w = \Phi + i\Psi$

Method of images: We place an imagine vortex across the wall (mirror is in the wall in $x = 0$) at $x = -d$. (so $(-d, 0)$). To make a $x = 0$ a streamline, the imagine must have the opposite circulation $-\Gamma$.

So the total complex potential is the superposition of:

- a real vortex at $z = d$: $w_1(z) = -\frac{i\Gamma}{2\pi} \ln(z - d)$ (we take the formula for the complex potential from the sheet)
- an imagine vortex at $z = -d$ with circulation $-\Gamma$, so:

$$w_2(z) = -\frac{i(-\Gamma)}{2\pi} \ln(z + d) = \frac{i\Gamma}{2\pi} \ln(z + d)$$

Therefore: $w(z) = -\frac{i\Gamma}{2\pi} \ln(z - d) + \frac{i\Gamma}{2\pi} \ln(z + d) = -\frac{i\Gamma}{2\pi} \ln\left(\frac{z - d}{z + d}\right)$ ✓

Let's make quick check with boundary (at $x=0$):

On the wall $z = iy$, thus:

$\left| \frac{iy-d}{iy+d} \right| = \frac{\sqrt{d^2+y^2}}{\sqrt{d^2+y^2}} = 1$ so $\ln\left(\frac{iy-d}{iy+d}\right)$ is purely imaginary, hence w is

purely real and $\Psi = \mathrm{Im}(w) = 0$ on $\underset{x=0}{\checkmark}$ so the wall is indeed the streamline ✓

Let's find the streamfunction. Writing: $z-d = (x-d)+iy$ and

$z+d = (x+d)+iy$ and define distances to the real vortex and an imagine vortex:

$r_1 = \sqrt{(x-d)^2+y^2}$ and $r_2 = \sqrt{(x+d)^2+y^2}$ and let their polar angles be

(their arguments): $\theta_1 = \arg(z-d)$ and $\theta_2 = \arg(z+d)$

Use that $\ln(z-d) = \ln r_1 + i\theta_1$ and $\ln(z+d) = \ln r_2 + i\theta_2$. Then:

$$w = -\frac{i\Gamma}{2\pi}\left[(\ln r_1 + i\theta_1) - (\ln r_2 + i\theta_2)\right] = -\frac{i\Gamma}{2\pi}\left[\ln\left(\frac{r_1}{r_2}\right) + i(\theta_1-\theta_2)\right] =$$

$$= \frac{\Gamma}{2\pi}(\theta_1-\theta_2) + i\left(-\frac{\Gamma}{2\pi}\ln\frac{r_1}{r_2}\right) \quad \text{so:}$$

$$\Psi(x,y) = -\frac{\Gamma}{2\pi}\ln\left(\frac{r_1}{r_2}\right) = -\frac{\Gamma}{2\pi}\ln\left(\frac{\sqrt{(x-d)^2+y^2}}{\sqrt{(x+d)^2+y^2}}\right)$$ ✓

and again we obtain on $x=0$: $r_1 = r_2 \implies \Psi = 0$ ✓

Streamlines are $\Psi = \Psi_0 = \text{const}$. From the formula above we have:

$$-\frac{\Gamma}{2\pi}\ln\left(\frac{r_1}{r_2}\right) = \Psi_0 \implies \frac{r_1}{r_2} = k \quad \text{where } k = \exp\left(-\frac{2\pi}{\Gamma}\Psi_0\right) > 0$$

Midterm Pointdale

Problem 3 (continued)

So streamlines satisfy: $\dfrac{\sqrt{(x-d)^2+y^2}}{\sqrt{(x+d)^2+y^2}} = k$

$\longrightarrow$ this means the ratio of distances to the two vortexes is constant: these are circles of constant ratio of distances of two points (whole family)

We can rearrange and square above formula: $(x-d)^2+y^2 = k^2\left((x+d)^2+y^2\right)$

it finally becomes:

$$\left(x - d\,\frac{1+k^2}{1-k^2}\right)^2 + y^2 = \left(\frac{2dk}{1-k^2}\right)^2 \longrightarrow \text{circles with centers on the } x\text{-axis} \checkmark$$

Special case: $k=1$ gives $r_1 = r_2 \Longrightarrow x = 0$ so the wall itself. ✓

(b) A point vortex moves with local ambient flow at its center, excluding its own singular velocity. Hence the ambient flow at the real vortex $(d,0)$ is produced only by imagine vortex at $(-d,0)$ with circulation Γ.

Imagine vortex potential is: $w_{im}(z) = \dfrac{i\Gamma}{2\pi}\ln(z+d)$ "$w_2(z)$"

$\dfrac{dw_{im}(z)}{dz} = \dfrac{i\Gamma}{2\pi}\cdot\dfrac{1}{z+d}$ ✓ now we need to evaluate it at real the real vortex

position $z_0 = d$: $\left.\dfrac{dw_{im}(z)}{dz}\right|_{z=d} = \dfrac{i\Gamma}{2\pi}\cdot\dfrac{1}{2d} = \dfrac{i\Gamma}{4\pi d}$

Convention from the lecture (or tutorials): $\dfrac{dw}{dz} = u - iv \Longrightarrow$

$\Longrightarrow u = \mathrm{Re}\left(\dfrac{dw}{dz}\right) = 0$ and $v = -\mathrm{Im}\left(\dfrac{dw}{dz}\right) = -\dfrac{\Gamma}{4\pi d}$

Therefore the vortex moves parallel to the wall with constant speed:

$$\bar{V} = u\,\bar{e}_x + v\,\bar{e}_y = -\frac{\Gamma}{4\pi d}\,\bar{e}_y \qquad \left(\text{direction depends of sign of } \Gamma\right)$$

Let's derive the trajectory:

Since $u = 0$, the distance from the wall stays constant: $x(t) = d$.

$$\frac{dy}{dt} = -\frac{\Gamma}{4\pi d} \implies y(t) = y(0) - \frac{\Gamma}{4\pi d}\,t$$

If we take the initial position $(d, 0) \implies y(0) = 0$

Thus: $x(t) = d$; $y(t) = -\dfrac{\Gamma}{4\pi d}\,t$

So the trajectory is a straight line parallel to the wall. ✓

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