Problem 1. Consider a model of a universe which is a generalization of the model analysed in class. In addition to ordinary matter with density ρ, there is also dark energy with density ρ_0. We assume that the dark energy density is constant in time and homogeneous in space.

In this case, the formula for the energy E, derived in class, takes the form

$$E = \frac{1}{2}\dot{a}^2 - G\frac{M + M_0}{a} = \frac{4\pi G}{3}a^2\big[\rho_c - (\rho + \rho_0)\big].$$

Here, $M = \frac{4}{3}\pi\rho a^3$ is the mass of ordinary matter (which obeys the continuity equation), and we define the effective dark energy mass as $M_0 = \frac{4}{3}\pi\rho_0 a^3$. Finally, ρ_c is the critical density, as defined in class.

(a) Find the equation describing the time evolution of the energy, $\dot{E}$, remembering that ρ_c is not constant. Next show that the condition of constancy of energy leads to the following dynamic equation

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho - 2\rho_0)$$

(b) Can such a universe be stationary (with time-independent scale factor)? Under which conditions this can happen? How would you interpret the situation in which $2\rho_0 > \rho$?

(c) Next, consider a situation in which $\rho_0 < \rho_c/3$. Show that if, at a certain moment of time, $2\rho_0 < \rho < \rho_c - \rho_0$, then initially the expansion of the universe will decelerate with time ($\ddot{a} < 0$), but at a certain moment the expansion speed will begin to increase. Recent observations seem to indicate that the cosmic expansion is in fact accelerating and that it may have been decelerating in the past.

Solution: We started with $\mathbf{v}(\mathbf{r}, t) = H(t)\mathbf{r}$. The cosmic scaling factor was introduced as $\dot{a} = Ha$. The critical density is $\rho_c = 3H^2/8\pi G$. Continuity equation leads to $\dot{\rho} = -3H\rho$.

(a) We differentiate the energy with respect to time, so that

$$\begin{aligned}
\frac{3}{4\pi G}\dot{E} &= 2a\dot{a}\big[\rho_c - (\rho + \rho_0)\big] + a^2\left[\dot{\rho}_c - \dot{\rho}\right] \\
&= 2a^2H\left[\frac{3H^2}{8\pi G} - (\rho + \rho_0)\right] + a^2\left[\frac{3H\dot{H}}{4\pi G} + 3H\rho\right] \\
&= a^2H\left(\frac{3H^2}{4\pi G} - 2(\rho + \rho_0) + \frac{3\dot{H}}{4\pi G} + 3\rho\right) \\
&= a^2H\left(\frac{3}{4\pi G}\left[\dot{H} + H^2\right] + \rho - 2\rho_0\right)
\end{aligned}$$

from which we get the equation describing the time evolution of energy

$$\dot{E} = a^2H\left[\dot{H} + H^2 + \frac{4\pi G}{3}(\rho - 2\rho_0)\right].$$

Requiring energy to be constant in time ($\dot{E} = 0$) leads to the desired dynamic equation

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho - 2\rho_0).$$

(b) Rewriting the last equation in terms of the scaling factor $a(t)$ yields

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2} = -\frac{4\pi G}{3}(\rho - 2\rho_0),$$

which can be rewritten as

$$\ddot{a} = -\frac{G\tilde{M}}{a^2}, \quad \tilde{M} = \frac{4\pi}{3}(\rho - 2\rho_0)a^3,$$

which is equivalent to a falling object in Newtonian gravity. Such a universe can be stationary only if $\tilde{M} = 0$, which means that $\rho = 2\rho_0$.

When $2\rho_0 > \rho$, then $\ddot{a} > 0$, which means that the expansion of the universe is accelerating.

(c) Since initially $\rho > 2\rho_0$, then $\ddot{a} < 0$. Because $\rho < \rho_c - \rho_0$, it follows that the energy of the universe is positive $E > 0$. Since $E > 0$ corresponds to $a(t)$ increasing indefinitely, the universe will expand and ρ will decrease. Since ρ_0 is constant, at some time τ we will reach a transition point $\rho(\tau) = 2\rho_0$, and the universe will start accelerating ($\ddot{a} > 0$).