Homework (series 8)

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Problem 1

$$u_x = \varepsilon x - \gamma y$$
$$u_y = \gamma x - \varepsilon y$$

Incompressibility $\iff$ $\nabla \cdot \vec{u} = 0$

$$\nabla \cdot \vec{u} = \partial_x u_x + \partial_y u_y = \varepsilon - \varepsilon = 0 \checkmark$$

By definition of the stream function:

$$\vec{u} = \nabla \times (\Psi \hat{e_z})$$

$$\Rightarrow \begin{cases} u_x = \frac{\partial \Psi}{\partial y} \\ u_y = -\frac{\partial \Psi}{\partial x} \end{cases}$$

$$\frac{\partial \psi}{\partial y} = \varepsilon x - \gamma y$$

$$\Rightarrow \quad \psi = \varepsilon x y - \frac{1}{2}\gamma y^2 + C(x)$$

$$\frac{\partial \psi}{\partial x} = \varepsilon y + C'(x)$$

$$\frac{\partial \psi}{\partial x} = -u_y \quad \Rightarrow \quad \cancel{\varepsilon y} + C'(x) = -\gamma x + \cancel{\varepsilon y}$$

$$\Rightarrow \quad C(x) = -\frac{1}{2}\gamma x^2 + \overset{\text{const}}{C}$$

$$\Rightarrow \quad \psi = \varepsilon x y - \frac{1}{2}\gamma(x^2 + y^2) + \psi_0$$

$$\psi\Big|_{\text{streamlines}} = \text{const}$$

So, along streamlines:

$$\varepsilon x y - \frac{1}{2}\gamma(x^2 + y^2) = \text{const}$$

$$2\varepsilon xy - \gamma(x^2+y^2) = \text{const}$$

$$\varepsilon\left((x+y)^2 - x^2 - y^2\right) - \gamma\left((x+y)^2 - 2xy\right) = \text{const}$$

$$(\varepsilon-\gamma)(x^2+y^2) - \varepsilon(x^2+y^2) + 2\gamma xy = \text{const}$$

$$(\varepsilon-\gamma)(x^2+y^2) - \varepsilon\left((x-y)^2 + 2xy\right) - \gamma\left((x-y)^2 - x^2 - y^2\right) = \text{const}$$

$$(\varepsilon-\gamma)(x+y)^2 - (\varepsilon+\gamma)(x-y)^2 - 2\varepsilon xy + \gamma(x^2+y^2) = \text{const}$$

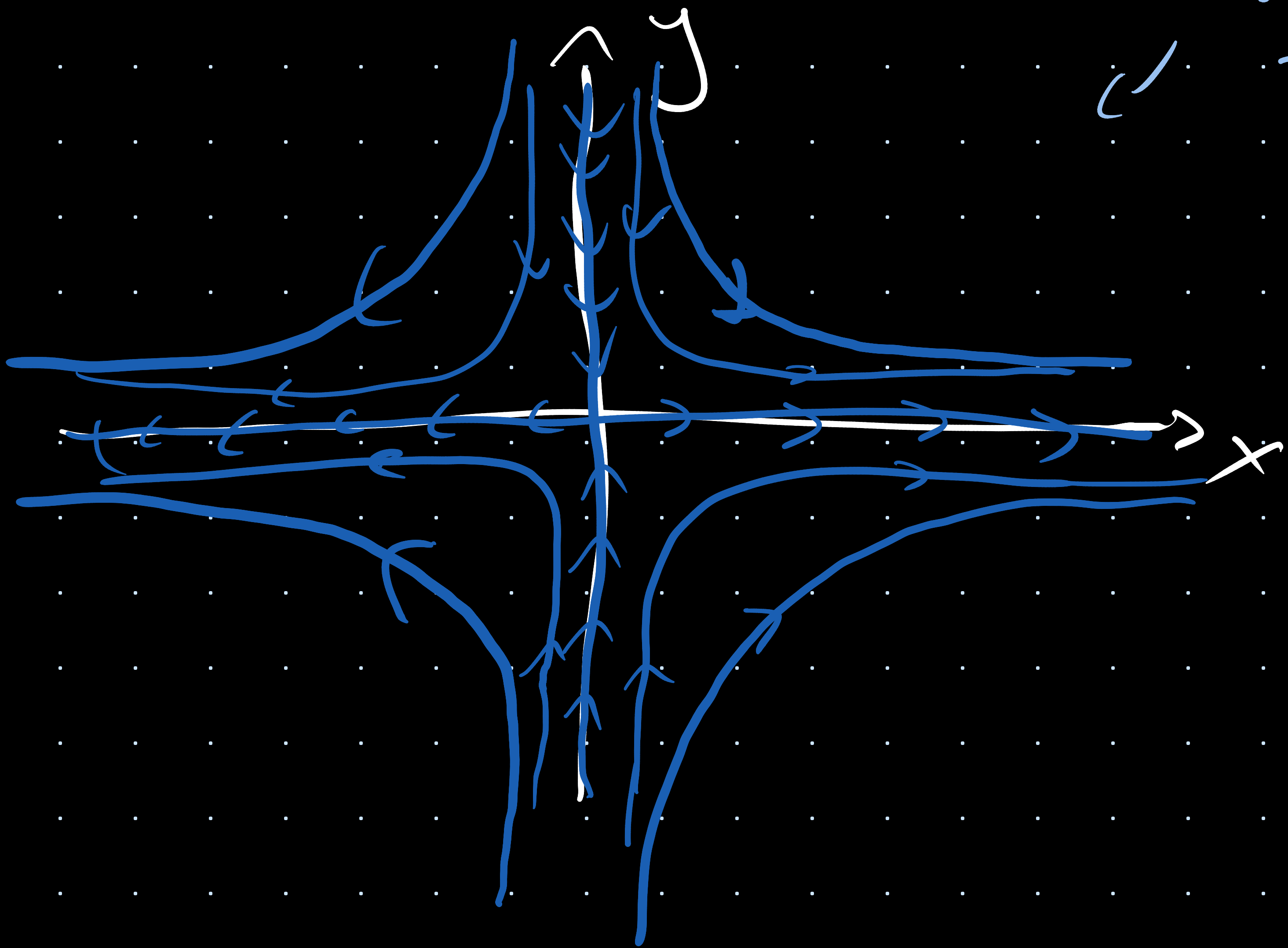
$$(\varepsilon-\gamma)(x+y)^2 - (\varepsilon+\gamma)(x-y)^2 - 2\Big(\overbrace{\varepsilon xy - \tfrac{1}{2}\gamma(x^2+y^2)}^{\text{const}}\Big) = \text{const}$$

$$(\varepsilon-\gamma)(x+y)^2 - (\varepsilon+\gamma)(x-y)^2 = \text{const}$$

i) $\quad xy = \text{const} \Rightarrow y = \frac{A}{x}$

If $A=0$: $x=0 \vee y=0$,

Otherwise hiperbola

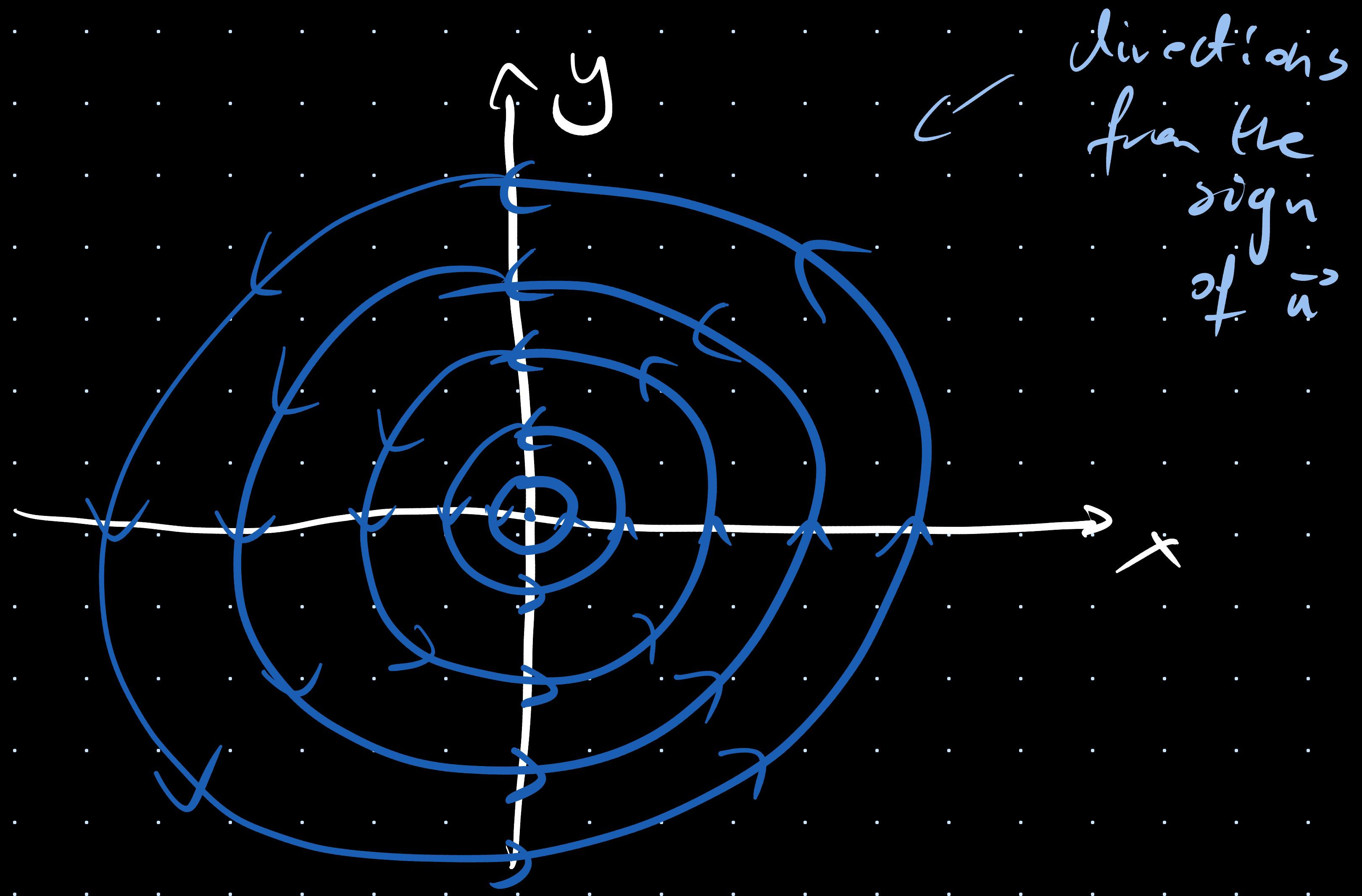


directions from the sign of $\vec{u}$

The fluid is flowing into the origin on the y-axis and away from it on the x-axis. The streamlines away from the axes are hiperbolae. (The origin is a stagnation point).

ii) $x^2+y^2=\text{const}$

equation of a circle



The fluid is flowing in circles counter-clockwise. (stagnation point in the middle).

iii)

$$xy - \frac{1}{2}(x^2+y^2) = \text{const}$$

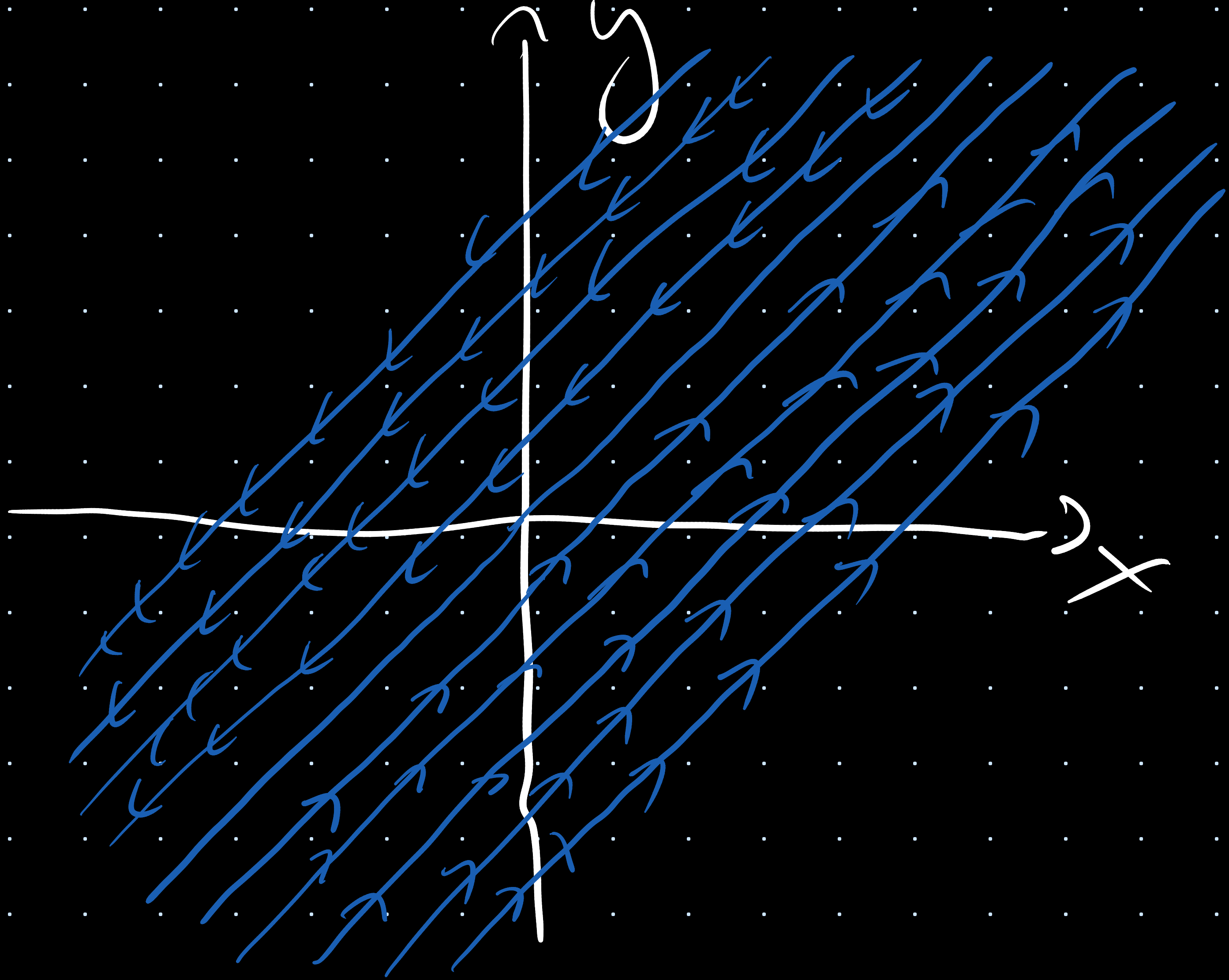
$$(x-y)^2 = \text{const}$$

$x - y = A$

$\Rightarrow y = x - A$

$u_x = u_y = A$

„lowered" lines will have both positive components



The fluid flows in straight diagonal lines, but the direction of the flow changes when we cross the line going through the origin (stagnation line).