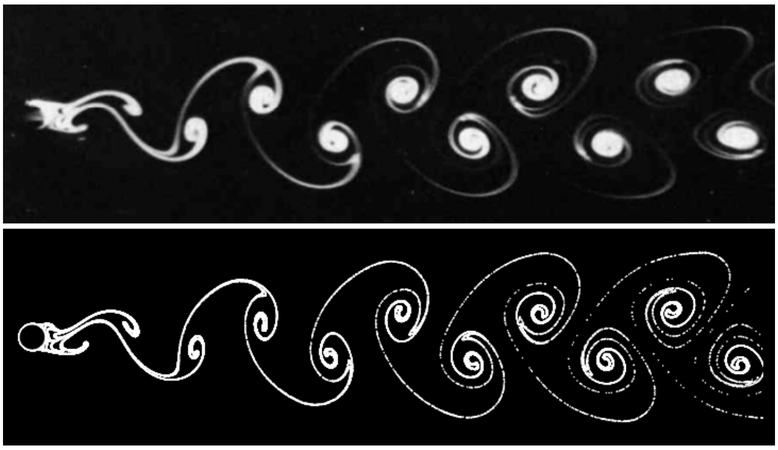
Hydrodynamics and Elasticity 2025/2026

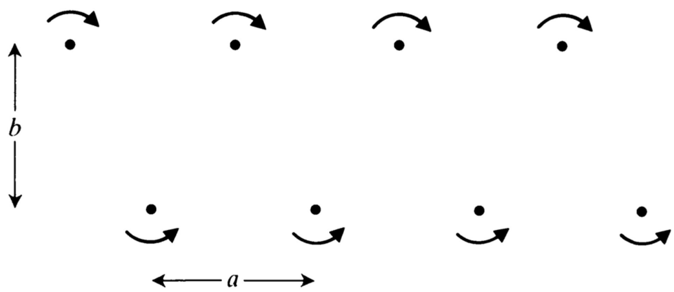
Christmas Problem Sheet

Here are a few problems for you for Christmas to keep your mind occupied during lengthy family dinners. The problems are not obligatory, but you can make yourself a Christmas present by getting two extra points for the solutions.

❄ **Problem 1** A von Karman vortex street is the repeating pattern in parallel rows of vortices that form in the wake of an obstruction in flowing fluid.

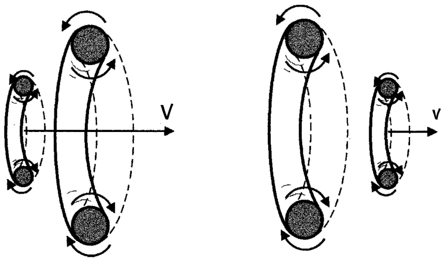


A mathematical model of such a vortex street, introduced by von Karman himself, represents it as two infinite rows of vortices: one set of line vortices of strength Γ at $z = na$, and another set of strength $-\Gamma$ at $z = (n + \frac{1}{2})a + ib$, with $n = 0, \pm 1, \pm 2, \ldots$. Find the speed at which such a pattern of vortices moves, assuming that each line vortex moves at the local flow velocity due to everything other than itself.



☆ **Problem 2** A smoke ring in still air travels slowly in the direction perpendicular to the plane of the ring (see the figure). In such a ring the smoke particles rotate around the hollow toroidal axis of the doughnut (toroid) in the directions indicated with the arrows. What makes smoke rings travel through the air? Which way will the smoke ring in the diagram travel?

Two smoke rings can chase one another, the trailing ring accelerating and shrinking while the leading ring slows down and expands. The smaller ring catches up with the larger one and passes through. Then the roles are reversed and the process is repeated! A fascinating show, but how do we explain it?



More smoke ring tricks can be found here. Disclaimer: smoking is bad etc etc.

✺ **Problem 3** An infinite cylinder of radius a rotates in a viscous, incompressible fluid with angular velocity Ω. Show that the vortex of a following form:

$$\mathbf{u} = \frac{\Omega a^2}{r}\mathbf{e}_\theta \qquad r \geq a \tag{1}$$

is in this case an exact solution of the Navier-Stokes equation satisfying the boundary conditions. Show that there is a nonzero torque exterted on such a cylinder by a fluid and find the value of this torque. Next, find a mistake in the following reasoning:
"Navier-Stokes eqution for viscous, incompressible fluid

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} + \eta\nabla^2\mathbf{u}$$

can be transformed, using a formula

$$\nabla^2\mathbf{u} = \text{grad div}\mathbf{u} - \text{rot rot}\mathbf{u},$$

into the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g} - \eta \text{ rot rot}\mathbf{u}.$$

However the flow (1) is irrotational (check!), thus the viscous term in the Navier-Stokes equations vanishes; therefore the viscous forces are zero; and so the torque on the cylinder is zero."

✺✺**Problem 4** The velocity field

$$u_r = \frac{Q}{2\pi r}, \quad u_\theta = 0$$

where Q is a constant, is called a line source flow if $Q > 0$ and a line sink if $Q < O$. Show that it is irrotational and that it satisfies $\nabla \cdot \mathbf{u} = 0$, save at $r = 0$, where it is not defined. Find the complex potential for such a flow.
Next, consider a mapping

$$Z = f(z)$$

where f is an analytic function of z. Provided that $f'(z_0) \neq 0$, points in the neighbourhood of $z = z_0$ are mapped by $Z = f(z)$, according to Taylor's theorem, in such a way that

$$Z - Z_0 = f'(z_0)(z - z_0) + O(z - z_0)^2$$

where $Z_0 = f(z_0)$. Use this to show that a line source of strength Q at $z = z_0$ is mapped into a line source of strength Q at $Z = Z_0$, provided that $f'(z_0) \neq 0$.
Next, consider fluid which occupies the region between two plane rigid boundaries at $y = \pm b$, and there is a line source of strength Q at $z = 0$. Find the complex potential $w(z)$ for the flow

1. by the method of images,
2. by using the mapping $Z = e^{\alpha z}$ with a suitably chosen $\alpha > 0$

Please bring the solution of not more than one problem to the first tutorial in the New Year

Merry Christmas and a Happy New Year!

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