# Problem Set 10 – Microhydrodynamics & fluctuations

**Problem 1 (warm-up): Sedimenting spheres**

Consider two identical spheres sedimenting in a viscous fluid. Do they sediment faster when their line of centres is parallel or perpendicular to gravity? Use the point particle approximation.

**Problem 2: Rotne-Prager form of the mobility correction**

Reconsider the problem of two identical spheres moving under the action of forces $\boldsymbol{F}_{1,2}$. To formulate the Rotne-Prager approximation to the mobility matrix, follow the consecutive steps:

(a) Consider sphere 1 in isolation, at the origin. The velocity field produced by a point force $\boldsymbol{F}$ at a distance $\boldsymbol{r}$ is $\boldsymbol{v}_0(\boldsymbol{r}) = \boldsymbol{G}(\boldsymbol{r}) \cdot \boldsymbol{F}$. Show that the velocity field around the sphere can be written as

$$\boldsymbol{v}_1(\boldsymbol{r}) = \left(1 + \frac{a^2}{6}\nabla^2\right) \boldsymbol{v}_0(\boldsymbol{r}).$$

This can be interpreted as the sphere being in the ambient flow $\boldsymbol{v}_0(\boldsymbol{r})$.

(b) Now consider the ambient field generated by sphere 1, incoming at sphere 2 centred at a distance $\boldsymbol{r} = \boldsymbol{R}$. What is the velocity of sphere 2 in this external flow?

(c) Defining the Rotne-Prager mobility matrix element $\boldsymbol{\mu}_{21}$ by the equation

$$\boldsymbol{U}_2 = \boldsymbol{\mu}_{21} \cdot \boldsymbol{F}_1,$$

deduce the final form

$$\boldsymbol{\mu}_{21} = \left(1 + \frac{a^2}{3}\nabla^2\right) \boldsymbol{v}_0(\boldsymbol{R}).$$

What is the form of $\boldsymbol{\mu}_{12}$? Evaluate the expression above explicitly as a function of $\boldsymbol{R}$.

(d) How would the expression for $\boldsymbol{\mu}_{12}$ change if the spheres had different radii $a_{1,2}$?

(*Note*) $\boldsymbol{U}_2$ defined in this way is only approximate and not equal to the true velocity of sphere 2 in a two-sphere system, because the stick boundary conditions are not satisfied at sphere 1.

**Problem 3: Torque-induced hydrodynamic interaction**

Sphere 1 placed at the origin is acted on by a torque $\boldsymbol{T}_1$. Far from sphere 1, the disturbance flow is represented by the rotlet

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{8\pi\eta} \frac{\boldsymbol{T}_1 \times \boldsymbol{x}}{|\boldsymbol{x}|^3}.$$

(a) Compute the vorticity at the centre of sphere 2 and use the rotational Faxén law for a force-free sphere to find the induced angular velocity of sphere 2.

(b) Interpret your result as a far-field rotational pair-mobility block $\boldsymbol{\mu}_{21}^{rr}$.

(c) Compare its distance dependence with that of the translational pair coupling from Problem 1.

**Problem 4: Force-free pair of spheres in a shear flow**

Two identical spheres of radius $a$ are suspended in the ambient simple-shear flow

$$\boldsymbol{u}^\infty(\boldsymbol{x}) = \dot{\gamma} y \, \boldsymbol{e}_x.$$

Let their centers be at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and define $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$. Assume the particles are neutrally buoyant and that no external or interparticle forces act on them, so that each sphere is force-free and torque-free.

(a) Decompose the ambient flow gradient as

$$\nabla \boldsymbol{u}^\infty = \boldsymbol{E}^\infty + \boldsymbol{\Omega}^\infty,$$

where $\boldsymbol{E}^\infty$ is symmetric and traceless, and $\boldsymbol{\Omega}^\infty$ is antisymmetric. Determine $\boldsymbol{E}^\infty$ and the corresponding angular velocity of the undisturbed flow, $\boldsymbol{\omega}^\infty = \nabla \times \boldsymbol{u}^\infty/2$.

(b) Using the Faxén laws for an isolated freely suspended sphere, show that in the limit of very large separation,

$$\boldsymbol{U}_1^\infty = \boldsymbol{u}^\infty(\boldsymbol{x}_1), \qquad \boldsymbol{U}_2^\infty = \boldsymbol{u}^\infty(\boldsymbol{x}_2),$$

and

$$\boldsymbol{\omega}_1^\infty = \boldsymbol{\omega}^\infty(\boldsymbol{x}_1), \qquad \boldsymbol{\omega}_2^\infty = \boldsymbol{\omega}^\infty(\boldsymbol{x}_2).$$

Deduce that the far-field relative velocity is

$$\boldsymbol{U}_{\text{rel}}^\infty = \boldsymbol{U}_2^\infty - \boldsymbol{U}_1^\infty = (\nabla \boldsymbol{u}^\infty) \cdot \boldsymbol{r}.$$

(c) Explain why the leading hydrodynamic interaction between the spheres is not Stokeslet-driven, but stresslet-driven. Hence argue that the disturbance velocity

$$\boldsymbol{U}_i' = \boldsymbol{U}_i - \boldsymbol{u}^\infty(\boldsymbol{x}_i)$$

scales as

$$\boldsymbol{U}_i' \sim r^{-2}.$$

(d) For equal spheres, use symmetry to argue that

$$\boldsymbol{U}_2' = -\boldsymbol{U}_1'.$$

What does this imply about the interaction-induced correction to the center-of-mass velocity of the pair?

(e) Discuss qualitatively how the pair motion differs from the sedimenting-pair problem. In particular, explain why in shear flow the particles are driven by the ambient flow, yet still deviate from the isolated-particle motion because of hydrodynamic interactions.