Problem Set 4 – Microhydrodynamics & fluctuations

Problem 1: stress field of the Oseen tensor

Consider the stress field, denoted by $\boldsymbol{F} \cdot \boldsymbol{\Sigma}$, of the Green's function $\boldsymbol{\mathcal{G}}$. The stress field of the Oseen tensor is then the triadic $8\pi\eta_s\boldsymbol{\sigma}$. Show that

$$8\pi\eta_s\Sigma_{ijk} = -6\eta_s\frac{r_i r_j r_k}{r^5}.$$

With the singular point being placed at $\boldsymbol{\xi}$, show the following statements:

$$8\pi\eta_s\frac{\partial}{\partial x_k}\Sigma_{ijk}(\boldsymbol{x}-\boldsymbol{\xi}) = -\frac{\partial}{\partial x_i}\mathcal{P}_j(\boldsymbol{x}-\boldsymbol{\xi}) + \eta_s\nabla^2\mathcal{G}_{ij} = -8\pi\eta_s\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{\xi}),$$

$$\frac{\partial}{\partial x_i}\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{\xi}) = 0.$$

Problem 2: Representation of flow outside a rigid particle

Consider an ambient Stokes flow (in the absence of particles), $\boldsymbol{v}^{\infty}$. A rigid particle immersed in this flow, and moving with a translational velocity $\boldsymbol{U}$ and rotational velocity Ω, creates a disturbance flow, such that the velocity field in the presence of the particle, $\boldsymbol{v}$ matches the particle (rigid body) motion, $\boldsymbol{v}^{RBM}$ on its surface, and the ambient flow far away. Define the disturbance field $\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^{\infty}$.

(a) Write the integral representation for $\boldsymbol{v}^D$.

(b) Now apply the integral representation theorem to the field $\boldsymbol{U}+\boldsymbol{\Omega}\times\boldsymbol{r}-\boldsymbol{v}^{\infty}$ inside the particle. Mind the orientation of the normal vector!

(c) Add the representations (for $\boldsymbol{v}^D$ and $\boldsymbol{v}^{RBM} - \boldsymbol{v}^{\infty}$), and obtain the result that the flow past a rigid particle can be represented with just single layer potentials.

Hints: for a rigid-body motion, show that the surface traction $\boldsymbol{\sigma}^{RBM} \cdot \boldsymbol{n} = -p_0\boldsymbol{n}$ and that since $\boldsymbol{\mathcal{G}}$ is solenoidal, the integral of $\boldsymbol{n} \cdot \boldsymbol{\mathcal{G}}$ over the surface of the particle is identically zero.

Problem 3: Constant single layer density on a circular disk Consider a constant single layer density ψ distributed over a circular disk of radius R. Show by direct integration in cylindrical coordinates that the velocity field generated by this distribution is continuous passing through the disk. Are the tractions also continuous?

Problem 4: Components of the double layer potential

Consider the decomposition of the double layer kernel into the Oseen pressure field and the rate-of-strain field of the Oseen tensor. Show that

$$8\pi\eta_s(\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi})\cdot\boldsymbol{n})\cdot\boldsymbol{\varphi} = -(\boldsymbol{\varphi}\cdot\boldsymbol{n})\mathcal{P}(\boldsymbol{x}-\boldsymbol{\xi}) + \eta_s(\boldsymbol{\varphi}\boldsymbol{n}+\boldsymbol{n}\boldsymbol{\varphi}) : \nabla\boldsymbol{\mathcal{G}}(\boldsymbol{x}-\boldsymbol{\xi}),$$

where : denotes double contraction.

The first term corresponds to sources and sinks, depending on the sign of the density, $\boldsymbol{\varphi}$, while the second group corresponds to a true bilayer of Stokeslets. Argue that that the source/sink term gives a jump in the normal, but continuous tangential velocities, while for the bilayer the opposite occurs. Verify this by considering a local analysis using a constant density on a plane. In that situation, show also that only the tangential component of double layer density, $\boldsymbol{\varphi}^{\perp} = \boldsymbol{\varphi} - \boldsymbol{\varphi}\cdot\boldsymbol{n}\boldsymbol{n}$, contributes to the velocity field of the bilayer distribution.

Hint: The double contraction of two tensors is $(A : B)_{i_1\cdots i_m j_1\cdots j_n} = A_{i_1\cdots i_m pq}B_{pqj_1\cdots j_n}$.