Problem Set 6 – Microhydrodynamics & fluctuations

Problem 1: Sphere in Poiseuille flow (warm-up)

A rigid sphere of radius a is freely suspended in the plane Poiseuille flow

$$\boldsymbol{v}^{\infty}(y) = U_0 \left(1 - \frac{y^2}{H^2}\right) \boldsymbol{e}_x.$$

Its centre is at $y = y_0$. Show that the translational velocity of the sphere is

$$\boldsymbol{U} = U_0 \left(1 - \frac{y_0^2}{H^2} - \frac{a^2}{3H^2}\right) \boldsymbol{e}_x.$$

Compare this with the velocity of a point tracer at the same position $y = y_0$. Why is the sphere faster or slower?

Problem 2: Symmetry forms of the resistance tensor for an axisymmetric particle

Let a rigid axisymmetric particle have orientation $\boldsymbol{p}$, with $|\boldsymbol{p}| = 1$. Suppose the hydrodynamic force and torque are written in the linear resistance form

$$\begin{pmatrix} \boldsymbol{F}^H \\ \boldsymbol{L}^H \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\zeta}^{tt} & \boldsymbol{\zeta}^{tr} \\ \boldsymbol{\zeta}^{rt} & \boldsymbol{\zeta}^{rr} \end{pmatrix} \begin{pmatrix} \boldsymbol{U} - \boldsymbol{v}^{\infty}(\boldsymbol{x}_0) \\ \boldsymbol{\omega} - \boldsymbol{\Omega}^{\infty}(\boldsymbol{x}_0) \end{pmatrix} - \begin{pmatrix} \boldsymbol{\zeta}^{td} \\ \boldsymbol{\zeta}^{rd} \end{pmatrix} : \boldsymbol{E}^{\infty},$$

where $\boldsymbol{E}^{\infty}$ is the local rate-of-strain tensor.

(a) Using axisymmetry, show that

$$\boldsymbol{\zeta}^{tt} = \zeta_{\perp}^{t}(\boldsymbol{I} - \boldsymbol{pp}) + \zeta_{\parallel}^{t}\, \boldsymbol{pp}, \qquad \boldsymbol{\zeta}^{rr} = \zeta_{\perp}^{r}(\boldsymbol{I} - \boldsymbol{pp}) + \zeta_{\parallel}^{r}\, \boldsymbol{pp}.$$

(b) Show that the most general forms of the translation-rotation couplings are

$$\zeta_{ij}^{tr} = \alpha\, \epsilon_{ijk} p_k, \qquad \zeta_{ij}^{rt} = \beta\, \epsilon_{ijk} p_k.$$

Use Lorentz reciprocity (symmetry of $\boldsymbol{\zeta}$) to relate α and β.

(c) Now assume the particle is fore-aft symmetric ($\boldsymbol{p}$ and $-\boldsymbol{p}$ are physically equivalent). Show that this symmetry forces

$$\boldsymbol{\zeta}^{tr} = \boldsymbol{\zeta}^{rt} = 0.$$

(d) Show also that the most general strain-induced force has the form

$$\boldsymbol{F}^{(E)} = -a\, \boldsymbol{p}\, (\boldsymbol{p} \cdot \boldsymbol{E}^{\infty} \cdot \boldsymbol{p}) - b\, \boldsymbol{E}^{\infty} \cdot \boldsymbol{p},$$

and hence deduce that fore-aft symmetry implies $\boldsymbol{\zeta}^{td} = 0$.

(e) Show that the only possible strain-induced torque on such a particle has the form

$$\boldsymbol{L}^{(E)} = -\chi\, \boldsymbol{p} \times (\boldsymbol{E}^{\infty} \cdot \boldsymbol{p}),$$

for some shape-dependent scalar χ.

Problem 3: Sedimentation of an axisymmetric particle

An axisymmetric body falls slowly under gravity through viscous incompressible fluid. The shape is such that the motion is one of pure translation (i.e. $\boldsymbol{\zeta}^{tr} = 0$). In one orientation the velocity $\boldsymbol{V}$ of the body makes an angle $\theta = \theta_1$, with the symmetry axis, and an angle $\alpha = \alpha_1$

with the downward vertical; in another the body falls at a different velocity $\boldsymbol{U}$, with $\theta = \alpha = \phi$, say. By considering the resistance matrix, prove that

$$\tan^2\phi = 1 - 2\tan\theta_1 \cot(\theta_1 + \alpha_1),$$

and that

$$|\boldsymbol{U}| = |\boldsymbol{V}| \left[\frac{2\sin\theta_1 \cos\phi}{\sin(\theta_1 + \alpha_1)}\right].$$

Problem 4: Jeffery's equation and Jeffery orbits

Consider a freely suspended fore-aft symmetric axisymmetric particle with orientation $\boldsymbol{p}$. Assume the torque law from Problem 2 has the form

$$\boldsymbol{L}^H = -\zeta_\perp^r \left[\boldsymbol{\omega} - \boldsymbol{\Omega}^\infty\right]_\perp - \zeta_\parallel^r \left[\boldsymbol{\omega} - \boldsymbol{\Omega}^\infty\right]_\parallel + \chi\, \boldsymbol{p} \times (\boldsymbol{E}^\infty \cdot \boldsymbol{p}),$$

and that $\dot{\boldsymbol{p}} = \boldsymbol{\omega} \times \boldsymbol{p}$.

(a) For a torque-free particle, show that

$$\boldsymbol{\omega} = \boldsymbol{\Omega}^\infty + \lambda\, \boldsymbol{p} \times (\boldsymbol{E}^\infty \cdot \boldsymbol{p}), \qquad \lambda = \frac{\chi}{\zeta_\perp^r}.$$

What is the physical meaning of the two terms in this expression?

(b) Hence derive Jeffery's equation

$$\dot{\boldsymbol{p}} = \boldsymbol{\Omega}^\infty \times \boldsymbol{p} + \lambda \Big[\boldsymbol{E}^\infty \cdot \boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{E}^\infty \cdot \boldsymbol{p})\boldsymbol{p}\Big].$$

Explain why the second term is always perpendicular to $\boldsymbol{p}$, and hence why $|\boldsymbol{p}| = 1$ is preserved.

(c) Consider simple shear $\boldsymbol{v}^\infty = (\dot\gamma y, 0, 0)$. Write the component equations for $\boldsymbol{p} = (p_1, p_2, p_3)$. Identify which part of the motion comes from rigid-body rotation of the ambient flow and which part comes from strain.

(d) For a spheroid of aspect ratio r, one has

$$\lambda = \frac{r^2 - 1}{r^2 + 1}.$$

Show that

$$\frac{r\, p_3}{\sqrt{p_1^2 + r^2 p_2^2}}$$

is constant along trajectories. What does this imply geometrically about the orientational motion?

(e) By introducing the variable

$$\tan\psi = \frac{r p_2}{p_1},$$

show that

$$\dot\psi = -\frac{\dot\gamma r}{r^2 + 1},$$

and hence deduce the Jeffery period

$$T = \frac{2\pi}{\dot\gamma}\left(r + \frac{1}{r}\right).$$

How does the period behave in the limits $r \to 1$ and $r \gg 1$? Give a physical interpretation.

(f) Describe qualitatively the motion for the following special cases:

(i) $p_3 = 0$ initially,

(ii) $p_1 = 0$ initially,

(iii) $r = 1$.