

Problem Set 7 – Microhydrodynamics & fluctuations

Problem 1: Transient Stokeslet and pressure vector for an oscillating force source

Consider the instationary Stokes equations for an incompressible fluid,

$$\rho \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) = -\nabla p(\mathbf{r}, t) + \eta_s \nabla^2 \mathbf{v}(\mathbf{r}, t) + \mathbf{f}(\mathbf{r}, t), \quad \nabla \cdot \mathbf{v}(\mathbf{r}, t) = 0,$$

with $\mathbf{v}(\mathbf{r}, t)$ the fluid flow velocity field, $p(\mathbf{r}, t)$ the absolute pressure, η_s the dynamic shear viscosity, and $\mathbf{f}(\mathbf{r}, t)$ a body force density acting on the fluid. Furthermore, ρ is the (constant) mass density of the fluid.

- (a) Perform a Fourier transform with respect time for all the fields and show that the Fourier components satisfy the relation

$$-\nabla p_\omega(\mathbf{r}) + \eta_s \nabla^2 \mathbf{v}_\omega(\mathbf{r}) - \eta_s \alpha^2 \mathbf{v}_\omega(\mathbf{r}) = -\mathbf{f}_\omega(\mathbf{r}), \quad \nabla \cdot \mathbf{v}_\omega(\mathbf{r}) = 0, \quad (1)$$

and determine an expression for α^2 .

- (b) Set $\mathbf{f}_\omega(\mathbf{r}) = \mathbf{F}_\omega \delta(\mathbf{r})$ and write $\mathbf{v}_\omega(\mathbf{r}) = \mathbf{G}_\omega(\mathbf{r}) \cdot \mathbf{F}_\omega$ and $p_\omega(\mathbf{r}) = \mathbf{P}_\omega(\mathbf{r}) \cdot \mathbf{F}_\omega$. Determine the differential equations for $\mathbf{G}_\omega(\mathbf{r})$ and $\mathbf{P}_\omega(\mathbf{r})$.
- (c) As a warm-up, determine the Green's function $G_0(\mathbf{r})$ of the Laplace equation, satisfying $\nabla^2 G_0(\mathbf{r}) = -\delta(\mathbf{r})$. Use the incompressibility condition and $G_0(\mathbf{r})$ to determine $\mathbf{P}_\omega(\mathbf{r})$.
- (d) For the determination of $\mathbf{G}_\omega(\mathbf{r})$, we make the ansatz

$$\mathbf{G}_\omega(\mathbf{r}) = (\nabla \nabla - \mathbf{I} \nabla^2) \psi(\mathbf{r}).$$

Explain why this ansatz makes sense, and derive a differential equation for the scalar potential $\psi(\mathbf{r})$.

- (e) By explicitly determining $\psi(\mathbf{r})$, show that

$$8\pi\eta_s \mathbf{G}_\omega(\mathbf{r}) = \frac{4\alpha}{(\alpha r)^3} [1 - (1 + \alpha r)e^{-\alpha r}] \hat{\mathbf{r}} \hat{\mathbf{r}} + \frac{2\alpha}{(\alpha r)^3} \{ [1 + \alpha r + (\alpha r)^2] e^{-\alpha r} - 1 \} (\mathbf{I} - \hat{\mathbf{r}} \hat{\mathbf{r}}).$$

- (f) Show that $\mathbf{G}_\omega(\mathbf{r})$ behaves as the steady Oseen tensor for small r .
- (g) Consider now the far-field. Explain that the far field behaves like a source dipole. Furthermore, show that this result coincides with inviscid flow where the fluid's inertia is balanced by pressure. Discuss the range of the transient Stokeslet compared to the steady Stokeslet.

Problem 2: Stokes drag on an oscillating sphere

We now consider an oscillating sphere of radius a suspended in a fluid described by the unsteady Stokes equations. For this we consider Eq. (1) with $\mathbf{f}_\omega = 0$ supplemented by the boundary conditions $\mathbf{v}_\omega(\mathbf{r})|_{r=a} = \mathbf{U}_\omega$, $\lim_{r \rightarrow \infty} \mathbf{v}_\omega(\mathbf{r}) = 0$, and $\lim_{r \rightarrow \infty} p_\omega(\mathbf{r}) = p_0$, with p_0 being a constant.

- (a) Using our experience from steady Stokes flow, we make the ansatz

$$\mathbf{v}_\omega(\mathbf{r}) = 6\pi\eta_s \mathbf{U}_\omega \cdot (B_0 + B_2 a^2 \nabla^2) \mathbf{G}_\omega(\mathbf{r}). \quad (2)$$

Explain why this ansatz solves Eq. (1) and determine B_0 and B_2 in terms of $\lambda = \alpha a$.

- (b) Determine $p_\omega(\mathbf{r})$ using a similar ansatz.
- (c) Let $\hat{\mathbf{n}}$ be the outward pointing unit normal on the particle and denote the Fourier components of the stress tensor by $\boldsymbol{\sigma}_\omega(\mathbf{r})$. Show that the traction vector on the sphere is

$$\boldsymbol{\sigma}_\omega(\mathbf{r}) \cdot \hat{\mathbf{n}}|_{r=a} = -\frac{3\eta_s}{2a} \mathbf{U}_\omega \cdot \left[(1 + \lambda) \mathbf{I} + \frac{\lambda^2}{3} \hat{\mathbf{n}} \hat{\mathbf{n}} \right]$$

- (d) Using the result from (c) compute the Fourier components of the force \mathbf{F}_ω that is exerted on the sphere by the fluid, and show that

$$\mathbf{F}_\omega = -6\pi\eta_s a \left(1 + \lambda + \frac{\lambda^2}{9} \right) \mathbf{U}_\omega.$$

The two last terms are out of phase with \mathbf{U}_ω . What are the phase differences?

- (e) The result in (d) can be obtained without explicit calculation of the tractions. Hereby, we can take the sources in the centre of the sphere and determine a fictitious force density $\tilde{\mathbf{f}}_\omega(\mathbf{r})$ centered at the origin. Write down $\tilde{\mathbf{f}}_\omega(\mathbf{r})$ using the singularity representation Eq. (2) and compute \mathbf{F}_ω by using Gauss' law to relate the integral to $\tilde{\mathbf{f}}_\omega(\mathbf{r})$.
- (f) Consider a stationary sphere in an oscillating ambient flow field, i.e. $\mathbf{v}_\omega(\mathbf{r})|_{r=a} = 0$, $\lim_{r \rightarrow \infty} \mathbf{v}_\omega(\mathbf{r}) = \mathbf{U}_\omega^\infty$. What is now the expression for \mathbf{F}_ω ? What happens with $p_\omega(\mathbf{r})$ for $r \rightarrow \infty$?