Problem Set 8 – Microhydrodynamics & fluctuations

Problem 1: Electrophoresis of a sphere in the laboratory frame

Consider a charged sphere of radius a with a constant zeta potential ζ in equilibrium. We consider a thin electric double layer where $\kappa a \gg 1$, with κ^{-1} being the Debye screening length. We are interested in the electrophoretic velocity $\mathbf{U}$ when we apply an external electric field $\mathbf{E}_0$. You may assume that $\mathbf{E}_0$ is sufficiently small such that the electrophoretic mobility does not depend on $\mathbf{E}_0$. We describe the electric double layer by a boundary layer located at $a < r < a^*$.

(a) Determine the electrostatic potential outside the electric double layer. What is the local electric field on the outer boundary of the electric double layer $\mathbf{E}^*(\mathbf{r}) = \mathbf{E}(\mathbf{r})|_{r=a^*}$? Make a sketch of $\mathbf{E}^*$ and its magnitude.

(b) We now consider the fluid flow inside the electric double layer. Determine the so-called slip velocity at $r = a^*$ defined by $\mathbf{U}^* = \mathbf{v}(\mathbf{r})|_{r=a^*} - \mathbf{U}$ by a suitable analysis of the electrokinetic flow in the boundary layer.

(c) Is there a normal component of the electric field within the electric double layer? Explain your answer and what this results entails for the shape of the electric double layer.

(d) Now we consider the flow for $r > a^*$ and we impose the slip condition $\mathbf{v}(\mathbf{r})|_{r=a^*} = \mathbf{U} + \mathbf{U}^*$. What are the conditions that this flow needs to satisfy?

(e) Write $\mathbf{v}(\mathbf{r})$ for $r > a^*$ in terms of a singularity representation. How does the singularity representation differ from an uncharged translating sphere with stick boundary conditions?

(f) Evaluate the singularity representation for $\mathbf{v}(\mathbf{r})$ explicitly for $r > a^*$. Show that this solution is a case of potential flow $\mathbf{v}(\mathbf{r}) = \nabla\Phi(\mathbf{r})$ for some scalar function Φ, and determine Φ. In particular, demonstrate that all the conditions you found in (d) are satisfied.

Problem 2: Electrophoretic mobility from the Lorentz reciprocal theorem

Consider a conducting particle of arbitrary shape S_p with uniform surface potential ζ in equilibrium. Under the application of an external electric field $\mathbf{E}_0$ the particle can translate and rotate $\mathbf{v}(\mathbf{r})|_{\mathbf{r}\in S_\mathrm{p}} = \mathbf{U} + \boldsymbol{\Omega} \times \mathbf{r}$. We are interested in steady rigid-body motion, in the linear electrophoresis regime, where $\mathbf{U} = \boldsymbol{\mu}^{\mathrm{tE}} \cdot \mathbf{E}_0$ and $\boldsymbol{\Omega} = \boldsymbol{\mu}^{\mathrm{rE}} \cdot \mathbf{E}_0$, respectively. In this case the electrophoretic mobility tensor $\boldsymbol{\mu}^{\mathrm{tE}}$ and electrorotation tensor $\boldsymbol{\mu}^{\mathrm{rE}}$ are independent of $\mathbf{E}_0$.

(a) Write down the Lorentz reciprocal theorem where the flow of interest $\mathbf{v}$ and auxiliary flow $\mathbf{v}^{(0)}$ are both subjected to a body force density acting on the fluid.

(b) Take for the auxiliary problem a suspended particle that carries no charge, subjected to the boundary condition $\mathbf{v}^{(0)}(\mathbf{r})|_{\mathbf{r}\in S_\mathrm{p}} = \mathbf{U}^{(0)} + \boldsymbol{\Omega}^{(0)} \times \mathbf{r}$. Use the force- and torque-free condition on the particle to derive that

$$\boldsymbol{\zeta}^{\mathrm{tt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{tr}} \cdot \boldsymbol{\Omega} = \int_{\mathcal{V}_\mathrm{f}} \mathrm{d}V\, \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \cdot [\mathsf{V}(\mathbf{r}) - \mathsf{I}],$$

$$\boldsymbol{\zeta}^{\mathrm{rt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{rr}} \cdot \boldsymbol{\Omega} = \int_{\mathcal{V}_\mathrm{f}} \mathrm{d}V\, \rho(\mathbf{r})\mathbf{E}(\mathbf{r}) \cdot [\mathsf{W}(\mathbf{r}) - \mathbf{r} \cdot \boldsymbol{\epsilon}].$$

Here the fluid domain $\mathcal{V}_\mathrm{f}$ contains a net ionic charge density $\rho(\mathbf{r})$ for the flow of interest and $\mathbf{E}(\mathbf{r})$ is the corresponding local electric field. Furthermore $\boldsymbol{\zeta}^{ab}$, $a, b \in \mathrm{t}, \mathrm{r}$ are the friction tensors for the auxiliary problem and we defined $\mathbf{v}^{(0)}(\mathbf{r}) = \mathsf{V}(\mathbf{r}) \cdot \mathbf{U}^{(0)} + \mathsf{W}(\mathbf{r}) \cdot \boldsymbol{\Omega}^{(0)}$.

(c) Since $\mathbf{E}_0$ is sufficiently small, we can expand $\psi(\mathbf{r}) = \psi^{\mathrm{eq}}(\mathbf{r}) + \boldsymbol{\varphi}(\mathbf{r}) \cdot \mathbf{E}_0$. Here, ψ^{eq} is the equilibrium electrostatic potential. For $\kappa a \gg 1$, we have that $\nabla^2 \boldsymbol{\varphi} = 0$ and $\partial_n \boldsymbol{\varphi}|_{S_{\mathrm{p}}} = 0$ with ∂_n denoting the normal derivative. What happens to $\boldsymbol{\varphi}(\mathbf{r})$ for $r \to \infty$? Show that the result in (b) reduces to

$$\boldsymbol{\zeta}^{\mathrm{tt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{tr}} \cdot \boldsymbol{\Omega} = \left\{ \varepsilon \int_{\mathcal{V}_{\mathrm{f}}} \mathrm{d}V \, \nabla^2 \psi^{\mathrm{eq}}(\mathbf{r}) \cdot [\mathsf{V}^\top(\mathbf{r}) - \mathsf{I}] \cdot \nabla \boldsymbol{\varphi}(\mathbf{r}) \right\} \cdot \mathbf{E}_0,$$

$$\boldsymbol{\zeta}^{\mathrm{rt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{rr}} \cdot \boldsymbol{\Omega} = \left\{ \varepsilon \int_{\mathcal{V}_{\mathrm{f}}} \mathrm{d}V \, \nabla^2 \psi^{\mathrm{eq}}(\mathbf{r}) \cdot [\mathsf{W}^\top(\mathbf{r}) + \mathbf{r} \cdot \boldsymbol{\epsilon}] \cdot \nabla \boldsymbol{\varphi}(\mathbf{r}) \right\} \cdot \mathbf{E}_0.$$

Here, $\top$ denotes the tensor transpose.

(d) Perform two partial integrations to show that

$$\boldsymbol{\zeta}^{\mathrm{tt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{tr}} \cdot \boldsymbol{\Omega} = \left\{ \varepsilon \zeta \int_{S_{\mathrm{p}}} \mathrm{d}S \, \left[\partial_n \mathsf{V}^\top(\mathbf{r}) \right] \cdot \nabla \boldsymbol{\varphi}(\mathbf{r}) \right\} \cdot \mathbf{E}_0,$$

$$\boldsymbol{\zeta}^{\mathrm{rt}} \cdot \mathbf{U} + \boldsymbol{\zeta}^{\mathrm{rr}} \cdot \boldsymbol{\Omega} = \left\{ \varepsilon \zeta \int_{S_{\mathrm{p}}} \mathrm{d}S \, \partial_n [\mathsf{W}^\top(\mathbf{r}) + \mathbf{r} \cdot \boldsymbol{\epsilon}] \cdot \nabla \boldsymbol{\varphi}(\mathbf{r}) \right\} \cdot \mathbf{E}_0,$$

within the Smoluchowski limit of a thin electric double layer.

(e) Consider for S_{p} a sphere of radius a. Determine $\boldsymbol{\mu}^{\mathrm{tE}}$ and $\boldsymbol{\mu}^{\mathrm{rE}}$ explicitly for $\kappa a \gg 1$ using the result in (d). What can you say about electrorotation?