Vector differential operators

Cylindrical Coordinates (r, φ, z).

- Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

- Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \qquad (\nabla f)_\varphi = \frac{1}{r}\frac{\partial f}{\partial \varphi}; \qquad (\nabla f)_z = \frac{\partial f}{\partial z}$$

- Curl

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right]\hat{e}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{e}_\varphi + \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi) - \frac{1}{r}\frac{\partial A_r}{\partial \varphi}\right]\hat{e}_z$$

- Scalar Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplacian of a vector

$$\left(\nabla^2 \mathbf{A}\right)_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial \varphi} - \frac{A_r}{r^2}$$

$$\left(\nabla^2 \mathbf{A}\right)_\varphi = \nabla^2 A_\varphi + \frac{2}{r^2}\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r^2}$$

$$\left(\nabla^2 \mathbf{A}\right)_z = \nabla^2 A_z$$

- Gradient of a vector

$$\mathrm{Grad}\mathbf{A} = \begin{bmatrix} \frac{\partial A_r}{\partial r} & \frac{1}{r}\frac{\partial A_r}{\partial \varphi} - \frac{A_\varphi}{r} & \frac{\partial A_r}{\partial z} \\ \frac{\partial A_\varphi}{\partial r} & \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{A_r}{r} & \frac{\partial A_\varphi}{\partial z} \\ \frac{\partial A_z}{\partial r} & \frac{1}{r}\frac{\partial A_z}{\partial \varphi} & \frac{\partial A_z}{\partial z} \end{bmatrix}$$

- Components of $(\mathbf{A} \cdot \nabla)\mathbf{B}$

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}]_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\varphi}{r}\frac{\partial B_r}{\partial \varphi} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\varphi B_\varphi}{r}$$

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}]_\varphi = A_r\frac{\partial B_\varphi}{\partial r} + \frac{A_\varphi}{r}\frac{\partial B_\varphi}{\partial \varphi} + A_z\frac{\partial B_\varphi}{\partial z} + \frac{A_\varphi B_r}{r}$$

$$[(\mathbf{A} \cdot \nabla)\mathbf{B}]_z = A_r\frac{\partial B_z}{\partial r} + \frac{A_\varphi}{r}\frac{\partial B_z}{\partial \varphi} + A_z\frac{\partial B_z}{\partial z}$$

- Divergence of a tensor

$$\left(\nabla \cdot \hat{\mathbf{T}}\right)_r = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial}{\partial \varphi}(T_{\varphi r}) + \frac{\partial}{\partial z}(T_{zr}) - \frac{1}{r}T_{\varphi\varphi}$$

$$\left(\nabla \cdot \hat{\mathbf{T}}\right)_\varphi = \frac{1}{r}\frac{\partial}{\partial r}(rT_{r\varphi}) + \frac{1}{r}\frac{\partial}{\partial \varphi}(T_{\varphi\varphi}) + \frac{\partial}{\partial z}(T_{z\varphi}) + \frac{1}{r}T_{\varphi r}$$

$$\left(\nabla \cdot \hat{\mathbf{T}}\right)_z = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial}{\partial \varphi}(T_{\varphi z}) + \frac{\partial}{\partial z}(T_{zz})$$

Spherical Coordinates (r, ϑ, φ).

- Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(A_\vartheta \sin\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$$

- Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \qquad (\nabla f)_\varphi = \frac{1}{r}\frac{\partial f}{\partial\vartheta}; \qquad (\nabla f)_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial f}{\partial\varphi}$$

- Curl

$$\nabla\times\mathbf{A} = \left[\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(A_\varphi\sin\vartheta\right) - \frac{1}{r\sin\vartheta}\frac{\partial A_\vartheta}{\partial\varphi}\right]\hat{e}_r + \left[\frac{1}{r\sin}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right)\right]\hat{e}_\vartheta + \left[\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\vartheta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\vartheta}\right]\hat{e}_\varphi$$

- Scalar Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}$$

- Laplacian of a vector

$$\left(\nabla^2\mathbf{A}\right)_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\vartheta}{\partial\vartheta} - \frac{2A_\vartheta\cot\vartheta}{r^2} - \frac{2}{r^2\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\left(\nabla^2\mathbf{A}\right)_\vartheta = \nabla^2 A_\varphi + \frac{2}{r^2}\frac{\partial A_r}{\partial\vartheta} - \frac{A_\vartheta}{r^2\sin^2\vartheta} - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\left(\nabla^2\mathbf{A}\right)_\varphi = \nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\vartheta} + \frac{2}{r^2\sin\vartheta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial\varphi}$$

- Gradient of a vector

$$\mathrm{Grad}\mathbf{A} = \begin{bmatrix} \frac{\partial A_r}{\partial r} & \frac{1}{r}\frac{\partial A_r}{\partial\vartheta} - \frac{A_\vartheta}{r} & \frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial\varphi} - \frac{A_\varphi}{r} \\ \frac{\partial A_\vartheta}{\partial r} & \frac{1}{r}\frac{\partial A_\vartheta}{\partial\vartheta} + \frac{A_r}{r} & \frac{1}{r\sin\vartheta}\frac{\partial A_\vartheta}{\partial\varphi} - \frac{A_\varphi\cot\vartheta}{r} \\ \frac{\partial A_\varphi}{\partial r} & \frac{1}{r}\frac{\partial A_\varphi}{\partial\vartheta} & \frac{1}{r\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi} + \frac{A_r}{r} + \frac{A_\vartheta\cot\vartheta}{r} \end{bmatrix}$$

- Components of $(\mathbf{A}\cdot\nabla)\,\mathbf{B}$

$$\left[(\mathbf{A}\cdot\nabla)\,\mathbf{B}\right]_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\vartheta}{r}\frac{\partial B_r}{\partial\vartheta} + \frac{A_\varphi}{r\sin\vartheta}\frac{\partial B_r}{\partial\varphi} - \frac{A_\vartheta B_\vartheta + A_\varphi B_\varphi}{r}$$

$$\left[(\mathbf{A}\cdot\nabla)\,\mathbf{B}\right]_\vartheta = A_r\frac{\partial B_\vartheta}{\partial r} + \frac{A_\vartheta}{r}\frac{\partial B_\vartheta}{\partial\vartheta} + \frac{A_\varphi}{r\sin\vartheta}\frac{\partial B_\vartheta}{\partial\varphi} + \frac{A_\vartheta B_r}{r} - \frac{A_\varphi B_\varphi\cot\vartheta}{r}$$

$$\left[(\mathbf{A}\cdot\nabla)\,\mathbf{B}\right]_\varphi = A_r\frac{\partial B_\varphi}{\partial r} + \frac{A_\vartheta}{r}\frac{\partial B_\varphi}{\partial\vartheta} + \frac{A_\varphi}{r\sin\vartheta}\frac{\partial B_\varphi}{\partial\varphi} + \frac{A_\varphi B_r}{r} + \frac{A_\varphi B_\vartheta\cot\vartheta}{r}$$

- Divergence of a tensor

$$\left(\nabla\cdot\hat{\mathbf{T}}\right)_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{rr}\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(T_{\vartheta r}\sin\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\varphi r}}{\partial\varphi} - \frac{T_{\vartheta\vartheta} + T_{\varphi\varphi}}{r}$$

$$\left(\nabla\cdot\hat{\mathbf{T}}\right)_\vartheta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{r\vartheta}\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(T_{\vartheta\vartheta}\sin\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\varphi\vartheta}}{\partial\varphi} + \frac{T_{\vartheta r}}{r} - \frac{\cot\vartheta}{r}T_{\varphi\varphi}$$

$$\left(\nabla\cdot\hat{\mathbf{T}}\right)_\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 T_{r\varphi}\right) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(T_{\vartheta\varphi}\sin\vartheta\right) + \frac{1}{r\sin\vartheta}\frac{\partial T_{\varphi\varphi}}{\partial\varphi} + \frac{T_{\varphi r}}{r} + \frac{\cot\vartheta}{r}T_{\varphi\vartheta}$$