Integrable Quantum Field Theories

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Abstract

These lecture notes cover the fundamentals of Integrable Quantum Field Theories with a focus on applications to condensed matter systems.

Contents

Literature

The lectures are mainly based on

- Statistical Field Theory, G. Mussardo, Oxford University Press (2009)

Additionally, you might want to consult different lecture notes on IQFT and integrability available online

- Exact S-matrices, P. Dorey,
  `https://arxiv.org/abs/hep-th/9810026`
- Introduction in integrable quantum field theory, B. Doyon,
  `https://benjamindoyon.weebly.com/uploads/3/8/5/8/38586121/notesiqft08.pdf`
- Introduction to integrability, N. Beisert,
  `http://people.phys.ethz.ch/~nbeisert/lectures/IntHS16-Notes.pdf`

To explore the world of the bosonization, try one of these,

- Quantum Physics in One Dimension, T. Giamarchi, Clarendon Press (2003)
- Bosonizing one-dimensional cold atomic gases, M. A. Cazalilla,
  `https://arxiv.org/abs/cond-mat/0307033`.

For applications of IQFT in cond-mat

- Applications of Massive Integrable Quantum Field Theories to Problems in Condensed Matter Physics, F. Essler, R. Konik,
  `https://arxiv.org/abs/cond-mat/0412421`

For many-body exactly solvable models of quantum mechanics

- An introduction to integrable techniques for one-dimensional quantum systems, F. Franchini, https://arxiv.org/abs/1609.02100

Chapter 1

Basics of QFT's in (1+1)d

1.1 Introduction

What is a Quantum Field Theory? One way of formulating quantum field theory is through path integrals. Having a classical action $S[\phi]$

$$S[\phi] = \int \mathrm{d}x\mathrm{d}t\, \mathcal{L}(\phi, \partial_\mu \phi), \tag{1.1}$$

which depends on (for example) a scalar field $\phi(x,t)$ (and its derivatives) we get a quantum theory but weighting each configuration by a factor

$$\exp(iS[\phi]/\hbar). \tag{1.2}$$

Here the configuration means a profile of function $\phi(x,t)$ at every x and t, so the field configuration. We can thus ask a question what is an expectation value of $\phi(x,t)$ at a certain position and time (x,t). This is then given by a weighted sum over all possible configurations

$$\langle \phi(x,t) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \exp(iS[\phi]/\hbar)\phi(x,t). \tag{1.3}$$

Factor $\mathcal{Z}$, the partition function, ensures the normalization

$$\mathcal{Z} = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \exp(iS[\phi]\hbar). \tag{1.4}$$

What we are very often interested in are the correlation functions. An example of a two-point correlation function is

$$\langle \phi(x_2,t_2)\phi(x_1,t_1) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \exp(iS[\phi]/\hbar)\phi(x_2,t_2)\phi(x_1,t_2), \tag{1.5}$$

and an example of a three-point correlation function is

$$\langle \phi(x_3,t_3)\phi(x_2,t_2)\phi(x_1,t_1) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \exp(iS[\phi]/\hbar)\phi(x_3,t_3)\phi(x_2,t_2)\phi(x_1,t_1), \tag{1.6}$$

Quantum field theories appear in physics and mathematics in numerous places. In the high energy physics, as a basic language to describe fundamental particles and interactions between them, in condensed-matter as an effective theory to describe the low-energy dynamics. They

also appear in mathematics, where attempts to formalise the path integrals led to discovering applications of QFT to topology and a classification of manifolds.

An alternative way of defining QFT is through promoting fields to field operators in a Hilbert space. We will consider this approach in future lectures. For now, let us see how we can recover classical physics from path integrals.

In the classical limit, $\hbar \to 0$, the weighting factor localizes to field configurations for which the action is extremal. These are the saddle points (stationary phase) configurations which follow the Lagrange-Euler equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0. \tag{1.7}$$

Therefore in the classical limit physics is dominated by field configurations obeying the Lagrange-Euler equations. The Lagrange-Euler equations are equations of motions for these dominant contributions.

Remark: Relativistic notation

Throughout the lecture notes we adopt the standard relativistic notation. We will work only in the Minkowski space with the metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.8}$$

with the metric tensor used to rise and lower the indices of a vector or a tensor

$$x^\mu = (t, x), \qquad x_\mu = \eta_{\mu\nu} x^\nu = (t, -x). \tag{1.9}$$

For the derivative we have

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \partial_x). \tag{1.10}$$

The scalar product of two vectors $x_\mu = (x_0, x_1)$ and $y_\mu = (y_0, y_1)$ is

$$x \cdot y = x_\mu y^\mu = x^\mu y_\mu = x_0 y_0 - x_1 y_1. \tag{1.11}$$

For example

$$\partial_\mu \partial^\mu = \partial_t^2 - \partial_x^2. \tag{1.12}$$

Here is an exercise in this notation

Exercise 1: Equations of motion

Find the classical equations of motion for the following Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - U(\phi) = \frac{1}{2} \left((\partial_t \phi)^2 - (\partial_x \phi)^2 \right) - U(\phi) \tag{1.13}$$

Derivation 1: Euler-Lagrange equations

How to find the most important contributions to the path integral in the classical limit? Let $\bar{\phi}$ be such a configuration. The condition for the action to be extremal is that under

a slight perturbation of $\bar{\phi}$ there is no linear change to the action. This corresponds to the usual condition for the extremum of a function:

$$f(\bar{x}+\delta x) = f(\bar{x}) + \delta x \frac{\partial f(x)}{\partial x} + \mathcal{O}(\delta x^2) \longrightarrow \frac{\partial f(x)}{\partial x} = 0. \tag{1.14}$$

For the action we have

$$S[\bar{\phi}+\delta\phi] = S[\bar{\phi}] + \int \mathrm{d}x\mathrm{d}t \, \frac{\delta S[\bar{\phi}]}{\delta \bar{\phi}} \delta\phi + \mathcal{O}(\delta\phi^2) \longrightarrow \frac{\delta S[\bar{\phi}]}{\delta \bar{\phi}} = 0. \tag{1.15}$$

How to compute $\frac{\delta S[\phi]}{\delta \phi}$? We perturb $\bar{\phi}$ by $\delta\phi$ and look for the linear (in $\delta\phi$) contribution. We have

$$S[\bar{\phi}+\delta\phi] - S[\phi] = \int \mathrm{d}x\mathrm{d}t \left(\mathcal{L}(\phi+\delta\phi, \partial_\mu\phi + \partial_\mu\delta\phi) - (\mathcal{L}(\phi, \partial_\mu\phi)\right). \tag{1.16}$$

Expanding the Lagrangian to the zeroth and first order in $\delta\phi$ we get

$$S[\bar{\phi}+\delta\phi] - S[\phi] = \int \mathrm{d}x\mathrm{d}t \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi\right) \tag{1.17}$$

Integrating the second term by parts, and neglecting the boundary term, we get

$$S[\bar{\phi}+\delta\phi] - S[\phi] = \int \mathrm{d}x\mathrm{d}t \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi. \tag{1.18}$$

This gives

$$\frac{\delta S[\phi]}{\delta\phi} = \frac{\partial \mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial \mathcal{L}}{\partial\phi} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x\phi)}. \tag{1.19}$$

Therefore in the classical limit $\hbar \to 0$ the dominating field configurations are indeed given by solutions to the Euler-Lagrange equations.

1.2 QFT in cond-mat

What we are after is a QFT description of condensed matter physics. We will consider a specific example leading to the Sine-Gordon model. Before, let us make few remarks of a general nature. The QFT can be thought of as consisting of 3 objects: field, action, correlation functions. In the high-energy context the relation between them and the physical reality is the following. Fields correspond to fundamental particles, action describes interaction between them, QFT provides us with the scattering matrix describing effects of collisions between the elementary particles.

In the cond-mat setting however the situation is a bit different. Now, fields describe effective low energy degrees of freedom, when appropriate called quasi-particles, action encode, again effective, interactions between the quasi-particles. Finally, the outcome of the QFT which interest us the most are the correlation functions which reflect the effective nature of the quasi-particles and interactions between them. Correlation functions are also directly related to the outcomes of the typical experimental techniques.

Let us see how this works in a concrete example. We consider a 1d quantum magnet, a

	hep	cond-mat
fields	particles	quasi-particles
action	standard model	effective action
aim	scattering data	corr. func.

chain of $\frac{1}{2}$-spins in an uniform magnetic field,

$$H_{XXZ} = -J\sum_{j=1}^{L}\left(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + \Delta s_j^z s_{j+1}^z\right) - h\sum_{j=1}^{L} s_j^z. \tag{1.20}$$

Here s_j^α are spin operators located at the j-th lattice site. There is a coupling between the neighbouring spins parameterized by J, and for $\Delta \neq 1$ the coupling is anisotropic in the z-direction. Moreover there is an external, uniform, magnetic field h also in the z-direction. This model is called XXZ spin chain.

Recall that the spin-operators on a single site form an $su(2)$ algebra

$$[s^x, s^y] = is^z, \tag{1.21}$$

and commute with each other when on different sites. When $J > 0$ then spins tend to be oriented opposite to their neighbours, the chain is antiferromagnetic. For $J > 0$, the chain exhibit tendency for ferromagnetic ordering. The XXZ spin chains can be found in real crystals.

Exercise 2: Symmetries of the XXZ chain

Show first that the following transformation respects the spin commutation relations

$$s_j^x \to (-1)^j s_j^x, \tag{1.22}$$
$$s_j^y \to (-1)^j s_j^y, \tag{1.23}$$
$$s_j^z \to s_j^z. \tag{1.24}$$

Then, observe the effect of this transformation on the Hamiltonian. Which sets of J and Δ are related to each other?

Let us analyze now the structure of the ground state of the XXZ Hamiltonian. From the previous exercise we know that models at (J, Δ, h) and $(-J, -\Delta, h)$ are equivalent. Therefore we set $J > 0$ and consider the ground state structure for various values of Δ and h. To this end it is convenient to first rewrite it in the terms of spin raining and lowering operators

$$s_j^\pm = s_j^x \pm i s_j^y. \tag{1.25}$$

They obey the following commutation relations

$$[s_j^+, s_j^-] = 2s_j^z, \qquad [s_j^z, s_j^\pm] = \pm s_j^\pm. \tag{1.26}$$

Exercise 3: Spin raising and lowering operators

Show the commutation relations among s_j^α operators with $\alpha = +, -, z$.

In terms of the raising and lowering operators the Hamiltonian reads

$$H_{XXZ} = -J \sum_{j=1}^{L} \left(\frac{s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+}{2} + \Delta s_j^z s_{j+1}^z \right) - h \sum_{j=1}^{L} s_j^z. \tag{1.27}$$

We consider now various ranges of Δ.

- $\Delta > 1$: The term $s_j^z s_{j+1}$ dominates and leads to the ferromagnetic ordering in the ground state. The external magnetic field supports such ordering and chooses polarization of the ground-state: up or down. In the absence of the magnetic field, the ground state is doubly degenerate.
- $\Delta = 1$: the symmetry is restored
- $|\Delta| < 1$ the "kinetic" part of the Hamiltonian dominates over the "ordering" part. The magnetic field tends to polarize the spins and clashes with the "kinetic" part. For small h we expect that the kinetic part wins and the ground state is an eigenstate of the kinetic energy operator: it's a wave and the state is paramagnetic. As we crank up the magnetic field, at large values we obtain again ferromagnetic state.
- $\Delta = -1$: the symmetry is again restored, this time with the anti-ferromagnetic flavour.
- $\Delta < -1$. The ordering is now of antiferromagnetic nature and dominates over the "kinetic" part. Still it might clash with the external magnetic field which supports ferromagnetic order. For small h we expect anti-ferromagnet and for large h ferromagnet. There is yet another phase between, the paramagnetic one, connected with the $|\Delta < 1$ phase.

This quantitative picture can be actually made exact by exact solvability of the XXZ spin chain. The results are summarized in the phase diagram

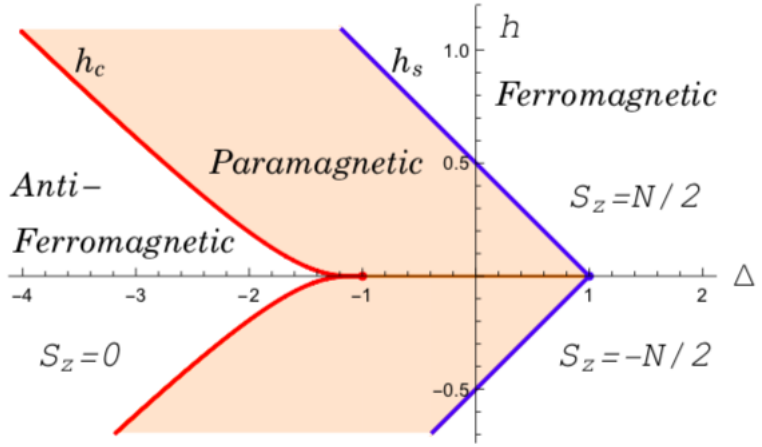


Figure 1.1: The phase diagram of the XXZ spin chain. Picture taken from "An introduction to integrable techniques for one-dimensional quantum systems" by F. Franchini.

The shaded region on the phase diagram corresponds to a massless theory, excitations over the ground state are like waves. Outside of this region the excitations over the ground state are gapped, their are massive, and particle-like. The thick lines are quantum phase transitions. The QFT picture in cond-mat is best suited for gapless phases or in the vicinity of the phase transitions.

1.2.1 Jordan-Wigner transformation

The spin raising and lowering operators on a single site obey anticommutation relations and commute when on different sites. They can be turned into a fully fermionic operators by introducing the Jordan-Wigner string. We define the following transformation

$$s_j^+ = c_j^+ \exp\left(i\pi \sum_{k<j} c_k^+ c_k\right), \tag{1.28}$$

$$s_j^z = c_j^+ c_j - \frac{1}{2}. \tag{1.29}$$

Here c_j, c_j^+ are fermionic operator obeying

$$\{c_j, c_k^+\} = \delta_{jk}, \tag{1.30}$$

$$\{c_j, c_k\} = \{c_j^+, c_k^+\} = 0. \tag{1.31}$$

The operator

$$\exp\left(i\pi \sum_{k<j} c_k^+ c_k\right), \tag{1.32}$$

simply introduces a sign ± 1 depending on the number of the particles to the left of the j-th particle. This is enough to turn the anticommutation between fermionic operators on different lattice sites to a commutation between spin operators on different sites. In the same time the Jordan-Wigner string does not modify the on-site commutation relations.

The XXZ Hamiltonian becomes

$$H_{XXZ} = -\frac{J}{2}\sum_{j=1}^{L}\left[c_{j+1}^+ c_j + c_j^+ c_{j+1} + \Delta\left(c_{j+1}^+ c_{j+1} - \frac{1}{2}\right)\left(c_j^+ c_j - \frac{1}{2}\right)\right] - h\sum_{j=1}^{L}\left(c_j^+ c_j - \frac{1}{2}\right), \tag{1.33}$$

which is a model of spinless fermions on $1d$ lattice with the nearest neighbour interactions. We can write it in a standard form

$$H_{\text{fermions}} = -t\sum_{j=1}^{L}\left(c_{j+1}^+ c_j + c_j^+ c_{j+1}\right) + V\sum_{j=1}^{L} c_{j+1}^+ c_{j+1} c_j^+ c_j - \mu\sum_{j=1}^{L}\left(c_j^+ c_j - \frac{1}{2}\right) + \text{const}, \tag{1.34}$$

with the identification

$$t = \frac{J}{2}, \qquad V = -\frac{J}{2}\Delta, \qquad \mu = h + 2\Delta. \tag{1.35}$$

Exercise 4: Jordan-Wigner transformation

Fill in the gaps in applying the Jordan-Wigner transformation to the XXZ spin chain. Specifically:

- check that the spin operators defined in terms of the fermionic operators obey the correct commutation and anti-commutation relations,
- express the spin chain Hamiltonian with fermionic operators.

1.2.2 Continuum limit

We consider now a continuum limit of the theory. That is we want to look at the dynamics at the scale much larger than the lattice spacing. Let us denote a the lattice spacing. We introduce

$$c_j \sim \sqrt{a}\psi(x), \qquad x = aj. \tag{1.36}$$

In such limit the sums over lattice points turn into integrals

$$a\sum \to \int \mathrm{d}x. \tag{1.37}$$

For example, for the kinetic part we get

$$\sum_j \left(c^+_{j+1}c_j + \text{h.c.}\right) \to \int \mathrm{d}x \left(\psi^\dagger(x+a)\psi(x) + \text{h.c.}\right). \tag{1.38}$$

Expanding the field $\psi^\dagger(x+a)$ in the Taylor series and integrating by parts we find

$$\sum_j \left(c^+_{j+1}c_j + \text{h.c.}\right) \to 2\int \mathrm{d}x\, \psi^\dagger(x)\psi(x) + a^2 \int \mathrm{d}x\, \psi^\dagger(x)\partial_x^2\psi(x) + \mathcal{O}(a^4). \tag{1.39}$$

The first term just shifts the chemical potential. The second term is the kinetic energy. For the interaction term it is convenient to first generalize it a bit allowing also for interaction between spins that are further apart

$$\sum_j c^\dagger_{j+1}c_{j+1}c^\dagger_j c_j \to \frac{1}{2}\sum_{k,j} V_{|k-j|}c^\dagger_k c_k c^\dagger_j c_j. \tag{1.40}$$

In the continuum limit we then find

$$\frac{1}{2}\sum_{k,j} V_{|k-j|}c^\dagger_k c_k c^\dagger_j c_j \to \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\, V(x-x')\rho(x')\rho(x) + \mathcal{O}(a), \tag{1.41}$$

where we introduced

$$\rho(x) = \psi^\dagger(x)\psi(x), \tag{1.42}$$

and $V(x) = V_j$. The Hamiltonian thus becomes

$$H = -m\int \mathrm{d}x\, \psi^\dagger(x)\partial_x^2\psi(x) + \frac{1}{2}\int \mathrm{d}x\mathrm{d}x'\, V(x-x')\rho(x')\rho(x) - \mu\int \mathrm{d}x\rho(x), \tag{1.43}$$

with $m = ta^2$.

Remark: When QFT applies to cond-mat

In taking the continuum limit we were treating operators like functions. Specifically, we have assumed that these functions vary smoothly from one lattice side to another. Can we justify this assumption? To do so we have to restrict the space of all possible state in a spin chain to a subset of configuration on which the expectation values of the operators vary smoothly. For that restricted Hilbert space we can make that the continuum limit at the level of operators. Is this assumption physical?

To quantify this assumption we look at a 2-point function, for example, a Green's function $\langle c^\dagger_j c_1\rangle \sim \langle\psi^\dagger(x)\psi(0)\rangle$ and analyze it's dependence on x. For physical systems this function decays with a distance and we can distinguish two situations:

- exponential decay, $\langle\psi^\dagger(x)\psi(0)\rangle \sim \exp(-|x|/\xi)$, with a characteristic length scale ξ,
- algebraic decay, $\langle\psi^\dagger(x)\psi(0)\rangle \sim 1/|x|^\nu$, with a power law ν.

Note that the algebraic decay can be viewed as a limiting case of the exponential decay when the correlation length ξ is very large.

In taking the continuum limit we disregard correlations at the length scale of lattice sites. This is justified if the correlation length $\xi \gg a$. This condition is fulfilled in the gapless (massless) phases and in the vicinity of the quantum phase transition. Therefore in our considerations we focus on the XXZ model with $|\Delta| < 1$ and $|h| < h_s = (1-\Delta)/2$.

1.2.3 Bosonization

We would like to understand now the low-energy physics. There is a universal, systematic way of obtaining the low energy QFT of 1d models, which goes under the name of bosonization. The name reflects the fact that the low-energy degrees of freedom will be bosons. Here we will merely sketch the procedure, for a more rigorous treatment we refer to the literature mentioned in the beginning of the notes.

We consider a 1d system. Particles move along a line and there is a sense of ordering. A particle can be said to be left or right of the another particle. Particles can be labelled following this ordering. We can define a function $\phi_l(x)$ such that at position x_j of the j-th particle it takes value $\phi_l(x_j) = 2\pi j$. We will use this function to describe fluctuations in particles positions.

The density operator $\rho(x)$ can be written as

$$\rho(x) = \sum_j \delta(x - x_j), \tag{1.44}$$

where x_i is the position operator of the i-th particle. From the properties of the Dirac δ function we find

$$\rho(x) = \sum_j \delta(x - x_j) = |\partial_x \phi_l(x)| \sum_j \delta(\phi_l(x) - 2\pi j). \tag{1.45}$$

The counting function $\phi_l(x)$ is monotonically increasing and we can drop the absolute values. We rewrite the sum over δ-functions with the help of the Poisson summation formula

$$\rho(x) = \frac{\partial_x \phi_l(x)}{2\pi} \sum_p e^{ip\phi_l(x)}. \tag{1.46}$$

We are after a low energy dynamics, therefore we can assume that the particles are not too far from the perfect crystalline order. In the case of the perfect order the position of the j-th particle would be $x_j = \rho_0^{-1} j$, where ρ_0 is the 1d density (a ratio of the number of particles to the length of the system). We define a field describing the difference between the actual density of particles and the crystalline case

$$\phi_l(x) = 2\pi\rho_0 x - 2\phi(x). \tag{1.47}$$

For the density operator we then find

$$\rho(x) = \left(\rho_0 - \frac{1}{\pi}\partial_x \phi(x)\right) \sum_{p\in\mathcal{Z}} e^{i2p(\pi\rho_0 x - \phi(x))}. \tag{1.48}$$

Note that in general, if we were interested in small fluctuations of the density around some constant value we would write

$$\rho(x) = \rho_0 + \delta\rho(x). \tag{1.49}$$

The expression (1.48) that we found can be thought of a refinement that takes into account a peculiarity of one spatial dimension.

We have now an effective representation of the density operator to describe the low-energy physics. What about the particle operator? Let us start with a bosonic theory. We can write

$$\psi_B^\dagger(x) = [\rho(x)]^{1/2}\, e^{-i\theta(x)}, \tag{1.50}$$

where $\theta(x)$ is some new, hermitian, field. It can be shown that the commutation relation between particle operators leads to the following relation

$$[\frac{1}{\pi}\partial_x\phi(x), \theta(x')] = -i\delta(x - x'). \tag{1.51}$$

This implies that fields θ and $\partial_x\phi$ are canonically conjugated. Integrating this relation by parts we can also view fields ϕ and $\partial_x\theta$ as canonically conjugated. The field operator can be written as

$$\psi_B^\dagger(x) = \left(\rho_0 - \frac{1}{\pi}\partial_x\phi(x)\right)^{1/2} \sum_{p\in\mathcal{Z}} e^{2ip(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}, \tag{1.52}$$

where we used that the square root of the delta function is proportional to the delta function. The proportionality constant depends on the microscopic of the model and we do not fix it here. Therefore the relation between the two expressions is specified up to this non-universal constant.

Exercise 5: Commutation relations

The aim of this exercise is to derive the commutation relation between the phase field $\theta(x)$ and density field $\rho(x)$. Deriving the commutation with the full density operator is a bit involved, therefore let us simplify the problem by neglecting the harmonics, that is assume that the density operator is just

$$\rho(x) \approx \rho_0 - \frac{1}{\pi}\partial_x\phi(x). \tag{1.53}$$

To get an expression for fermionic particle operator we need to respect the anti-commutation relations. To this end we take the bosonic operator and supplement it with the Jordan-Wigner string. The Jordan-Wigner string is easily expressible through the $\phi_l(x)$ operator, i.e. it is given by $e^{i\phi_l(x)/2}$. The fermionic particle operator is

$$\psi_F^\dagger(x) = \psi_B^\dagger(x) e^{i\phi_l(x)/2} = \left(\rho_0 - \frac{1}{\pi}\partial_x\phi(x)\right)^{1/2} \sum_{p} e^{i(2p+1)(\pi\rho_0 x - \phi(x))} e^{-i\theta(x)}. \tag{1.54}$$

What is the Hamiltonian in terms of the new operators? We have two fluctuating fields and we look for the Hamiltonian describing their low-energy dynamics. The guess, which can be supported by the renormalization group analysis, is

$$H_{LL} = \frac{v_s}{2\pi}\int \mathrm{d}x \left(K\left(\partial_x\theta(x)\right)^2 + \frac{1}{K}\left(\partial_x\phi(x)\right)^2\right). \tag{1.55}$$

Here v_s is the sound velocity, and dimensionless K (the Luttinger parameter) describes the stiffness. For $K > 1$ the phase fluctuations dominate, while for $K < 1$ the density fluctuations dominate.

As a final step let us write the Lagrangian of the theory. Because fields ϕ and $\partial_x\theta$ are canonically conjugated, field $\Pi = \partial_x\theta$ is the conjugated momentum to ϕ. The Lagrangian density is then

$$\mathcal{L} = \frac{1}{2\pi K}\left(-\frac{1}{u}\left(\partial_t\phi(x)\right)^2 + u\left(\partial_x\phi(x)\right)^2\right), \tag{1.56}$$

and describes a massless bosonic field.

We can now get back to the XXZ spin model. Say we are interested in the longitudinal spin-spin correlation function. As a first step we need to express the spin s_j^z operator in terms of the bosonized fields. For the spin s_j^z operator we find

$$a^{-1}s_j^z \to \rho(x) - \frac{1}{2a} = \rho_0 - \frac{1}{2} - \frac{1}{\pi}\partial_x\phi(x) + 2(\rho_0 - \frac{1}{\pi}\partial_x\phi(x))\sum_{p>0}\cos(2p(\pi\rho_0 x - \phi(x)). \tag{1.57}$$

For half-filling (no magnetic field), $\rho_0 = 1/(2a)$ and

$$a^{-1}s_j^z \to -\frac{1}{\pi}\partial_x\phi(x) + (1 - 2\frac{1}{\pi}\partial_x\phi(x))\sum_{p>0}\cos(2p(\pi\rho_0 x - \phi(x)). \tag{1.58}$$

Finally looking at the leading expression in fields we get

$$a^{-1}s_j^z \to -\frac{1}{\pi}\partial_x\phi(x) + \cos(\pi j - 2\pi\phi(x)) = -\frac{1}{\pi}\partial_x\phi(x) + (-1)^x\cos(2\phi(x)). \tag{1.59}$$

In a similar way, albeit somehow more complicated, an expression for the spin raising and lowering operators can be found

$$a^{-1/2}s_j^+ \to e^{-i\theta(x)}((-1)^x + \cos(2\phi(x)). \tag{1.60}$$

To compute the correlation function we then need to average the product of such operators with the Luttinger liquid action. The action is Gaussian (a free bosonic theory) and all the correlation functions can be computed. For example

$$\langle s_{j+1}^z s_1^z\rangle \to \frac{a^2}{\pi^2}\langle\partial_x\phi(x)\partial_x\phi(0)\rangle + a^2\langle\cos(\pi x/a_0 - 2\pi\phi(x))\cos(2\pi\phi(0)\rangle. \tag{1.61}$$

If you know QFT, you can compute the r.h.s.

comment on the possible rescalling of the operators

1.2.4 Massive perturbations

In writing the XXZ Hamiltonian

$$H_{XXZ,h} = J\sum_{j=1}^{L}\left(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + \Delta s_j^z s_{j+1}^z\right) - h\sum_{j=1}^{L}s_j^z. \tag{1.62}$$

we assumed that it takes somehow an idealized form. In real materials they might be important effects that modify it.

Consider modifying the XXZ Hamiltonian by

$$+ h \sum_j (-1)^j s_j^x. \tag{1.63}$$

Such terms arise in a case of Dzyaloshinskii-Moriya interactions or when there are two spins per cell and we take into account the tensorial nature of coupling between the external magnetic field and spins. In any case, mapping this expression to fermions and using the bosonization identities we find the leading term to be

$$+ h \sum_j (-1)^j s_j^x \to \rho_0 a^{-1/2} \int \mathrm{d}x \cos(\theta(x)). \tag{1.64}$$

Another possibility is to make the anisotropy and magnetic field in different directions

$$H = J \sum_{j=1}^{L} \left(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + s_j^z s_{j+1}^z \right) - h \sum_{j=1}^{L} s_j^z + (\Delta - 1) \sum_{j=1}^{L} s_j^y s_{j+1}^y. \tag{1.65}$$

The first part of the Hamiltonian is just the XXX model ($\Delta = 1$) and leads to the Luttinger liquid Hamiltonian. The last part becomes

$$(\Delta - 1) \sum_{j=1}^{L} s_j^y s_{j+1}^y \to \mu(\Delta) \int \mathrm{d}x \cos(\theta(x)). \tag{1.66}$$

We see that different perturbations of the XXZ spin chain lead to the same field theory. It's Lagrangian is

$$\mathcal{L}_{SG} = \frac{1}{2\pi K} \left(-\frac{1}{u} \left(\partial_t \theta(x) \right)^2 + u \left(\partial_x \theta(x) \right)^2 \right) - \mu \cos(\beta\theta). \tag{1.67}$$

By rescalling the fields, length and time, and changing θ into ϕ we can write it in a canonical form

$$\mathcal{L}_{SG} = \frac{1}{2} (\partial_t \phi(x,t))^2 - \frac{1}{2} (\partial_x \phi(x,t))^2 + \frac{\mu^2}{g^2} \cos g\phi(x,t). \tag{1.68}$$

This is the Sine-Gordon theory, which will be one of our main examples of an Integrable QFT. We see that it arises as an effective, low-energy, description of certain spin chains.

There is a closely related model, the Sinh-Gordon theory, which we obtain by replacing the interaction parameter g by ig.

$$\mathcal{L}_{SG} = \frac{1}{2} (\partial_t \phi(x,t))^2 - \frac{1}{2} (\partial_x \phi(x,t))^2 - \frac{\mu^2}{g^2} \cosh g\phi(x,t). \tag{1.69}$$

Exercise 6

Show, using the bosonization expression for the spin operators, that indeed the $\cos(\theta(\phi))$ terms arises for both modifications of the XXZ Hamiltonian.

1.3 Free massive bosons in $(1+1)$d

We will recall now the solution to free massive bosonic QFT in $1+1$ dimensions. By solving we mean computing correlation functions. We provide the solution with the path integral approach and with the canonical quantization.

1.3.1 Euclidian Formalism

To avoid problems with ill-defined mathematical expressions we adopt the Euclidian formalism. That is we switch from Minkowski space-time to an Euclidean space-time by considering imaginary time $t = -i\tau (\tau \in \mathbf{R})$. The action then becomes

$$S_E[\phi; x, \tau] = -iS[\phi; x, t \to -i\tau]. \tag{1.70}$$

The Euclidian action is positive defined (strictly speaking nonnegative) and in the path integral approach the weighting factor becomes $\exp(-S_E[\phi])/\hbar$. This change to imaginary time is called Wick rotation. The Euclidean Lagrangean is

$$\mathcal{L}_E(\phi, x, \tau) = -\mathcal{L}(\phi; x, t \to -i\tau) \tag{1.71}$$

The effect of the Wick rotation is that the previously oscillatory character that suppressed trajectories with large action is replaced by an exponential damping. We can always get back to the Minkowski space-time by analytic continuation of the final results. We will exemplify this later in a concrete example.

The Euclidean formalism is also directly related to the path integral approach to classical statistical physics in $2d$. We will comment on this relation in future.

1.3.2 Free bosons - path integrals

We start with the following Lagrangean (still in the Minkowski space-time)

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi(x,t))^2 - \frac{1}{2}(\partial_x \phi(x,t))^2 - \frac{1}{2}\mu^2 \phi^2(x,t). \tag{1.72}$$

Note that it corresponds to weakly interacting Sine-Gordon model when the interaction parameter g is small. Applying the Wick rotation we find

$$\mathcal{L}_E = \frac{1}{2}(\partial_\tau \phi(x,\tau))^2 + \frac{1}{2}(\partial_x \phi(x,\tau))^2 + \frac{1}{2}\mu^2 \phi^2(x,\tau). \tag{1.73}$$

From now on we work in Euclidean formalism and to simplify the notation we drop the subindex E. Moreover, now both x and τ can be treated on equal footing and we write $\mathbf{x} = (x, \tau)$.

We are after correlation functions. The main role is played by the Green's function, the two-point correlation function,

$$\langle \phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}\phi \exp(-S[\phi]/\hbar)\phi(\mathbf{x}_2)\phi(\mathbf{x}_1), \tag{1.74}$$

To compute such correlation function we need to understand how to perform sum (integral) over all possible configurations of field $\phi(\mathbf{x})$ weighted with the action $S[\phi]$. The way to go is by decomposing the field in Fourier modes

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{x}} \phi(\mathbf{k}), \tag{1.75}$$

where $\mathbf{k} = (k, \omega)$ and $\mathbf{k} \cdot \mathbf{x} = kx + \omega\tau$. Because field $\phi(\mathbf{x})$ is real, Fourier modes must obey $\phi(-\mathbf{k}) = \phi^*(\mathbf{k})$. The action in terms of the Fourier modes becomes

$$S[\phi] = \frac{1}{2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} (\mathbf{k}^2 + \mu^2) |\phi(\mathbf{k})|^2. \tag{1.76}$$

Exercise 7

Derive eq. (1.76) for the action in terms of the Fourier modes.

The action is now a sum of independent contributions from each $\mathbf{k}$ mode. Therefore it makes sense to divide the integration measure $\mathcal{D}\phi$ also in small blocks, each corresponding to a different mode. Field $\phi(\mathbf{k})$ is complex-valued and values for $\mathbf{k}$ and $-\mathbf{k}$ are complex conjugations of each other. Any field configuration $\phi(\mathbf{x})$ can be specified by either:

- values of $\phi(\mathbf{k})$ for each $\mathbf{k}$
- values of $\phi(\mathbf{k})$ and $\phi^*(\mathbf{k})$ for half of $\mathbf{k}$'s [1]

We choose the latter and write

$$\mathcal{D}\phi = \prod_{\mathbf{k}_+} \mathrm{d}^2\phi(\mathbf{k}). \tag{1.77}$$

Here the product is a product over half of the Fourier modes and we denoted $\mathrm{d}^2\phi(\mathbf{k}) = \mathrm{d}\phi(\mathbf{k})\mathrm{d}\phi^*(\mathbf{k})$. Finally, it is easier to start with the propagator in the momentum space. To this end we write

$$\langle\phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}_2}{(2\pi)^2} e^{i\mathbf{k}_1\cdot\mathbf{x}_1 + i\mathbf{k}_2\cdot\mathbf{x}_2}\langle\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\rangle, \tag{1.78}$$

where

$$\langle\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{D}\phi \exp(-S[\phi]/\hbar)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1), \tag{1.79}$$

We want now to evaluate $\langle\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\rangle$. The action couples modes $\phi(\mathbf{k})$ and $\phi(-\mathbf{k})$. Otherwise the modes are independent. Therefore for $\mathbf{k}_1 \neq -\mathbf{k}_2$ we have

$$\langle\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\rangle = \langle\phi(\mathbf{k}_2)\rangle\langle\phi(\mathbf{k}_1)\rangle \tag{1.80}$$

A single operator expectation value is 0. Therefore the propagator is not zero only if $\mathbf{k}_2 = -\mathbf{k}_1$. We then have

$$\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{D}\phi \exp(-S[\phi]/\hbar)|\phi(\mathbf{k})|^2, \tag{1.81}$$

where we used the observed before property $\phi(-\mathbf{k}) = \phi^*(\mathbf{k})$.

Exercise 8

Show that the one point expectation value $\langle\phi(\mathbf{k})\rangle = 0$.

The functional integration in (1.81) splits in a product of integrals over all modes. For all modes but the $\mathbf{k}$, the contribution from the numerator and denominator cancel. Therefore we have

$$\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle = \frac{\int \mathrm{d}^2\phi(\mathbf{k})\exp(-A_{\mathbf{k}}|\phi(\mathbf{k})|^2)|\phi(\mathbf{k})|^2}{\int \mathrm{d}^2\phi(\mathbf{k})\exp(-A_{\mathbf{k}}|\phi(\mathbf{k})|^2)}, \qquad A_{\mathbf{k}} = \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}(\mathbf{k}^2+\mu^2). \tag{1.82}$$

[1] For every $\mathbf{k}$ and $\mathbf{p}$ such that $\mathbf{k} = -\mathbf{p}$ we keep only one in the integration meassure

Note that the integration here is over values of $\phi(\mathbf{k})$. So this is just a regular 2-dimensional ($\phi(\mathbf{k})$ is a complex number) Gaussian integral. For $A > 0$, we have the following identities

$$\int \mathrm{d}z\mathrm{d}\bar{z}\exp(-A|z|^2) = \frac{\pi}{A},$$
$$\int \mathrm{d}z\mathrm{d}\bar{z}\exp(-A|z|^2)|z|^2 = \frac{\pi}{A^2}, \tag{1.83}$$

where the integration is over the whole complex plane. Therefore

$$\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle = \frac{(2\pi)^2}{\mathrm{d}\mathbf{k}}\frac{1}{\mathbf{k}^2+\mu^2}, \tag{1.84}$$

where $1/\mathrm{d}\mathbf{k}$ can be understood as weight of the Dirac δ-function at zero argument. That is, we can write

$$\langle\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\rangle = \frac{(2\pi)^2\delta(\mathbf{k}_2+\mathbf{k}_1)}{\mathbf{k}_1^2+\mu^2}. \tag{1.85}$$

The real space two-point function (1.78) is then

$$\langle\phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}}{\mathbf{k}^2+\mu^2}, \tag{1.86}$$

and depends only on the distance $x = |\mathbf{x}_2 - \mathbf{x}_1|$. For further convenience we will now redefine the two-point function in the momentum space writing

$$\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle = \frac{1}{\mathbf{k}^2+\mu^2}, \tag{1.87}$$

such that

$$\langle\phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)}\langle\phi(\mathbf{k})\phi(-\mathbf{k})\rangle. \tag{1.88}$$

We parametrize $\mathbf{k}$ in polar coordinates as (k, θ) where k is the length of $\mathbf{k}$ and θ is an angle between $\mathbf{k}$ and $(\mathbf{x}_2 - \mathbf{x}_1)$. We get

$$\langle\phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \int \frac{\mathrm{d}k\mathrm{d}\theta}{(2\pi)^2}\frac{ke^{ikx\cos\theta}}{k^2+\mu^2}. \tag{1.89}$$

The answer can be expressed with the help of special functions. We use two identities

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}e^{-kx\cos\theta} = J_0(|kx|), \tag{1.90}$$
$$\int_0^{\infty}\mathrm{d}k\,\frac{k}{k^2+\mu^2}J_0(|kx|) = K_0(\mu x). \tag{1.91}$$

to get

$$\langle\phi(\mathbf{x}_2)\phi(\mathbf{x}_1)\rangle = \frac{1}{2\pi}K_0(\mu|\mathbf{x}_2-\mathbf{x}_1|), \tag{1.92}$$

where $J_0(x)$ is the Bessel function of the first kind and $K_0(x)$ is the modified Bessel function of the second kind. This is our final result for the two-point function in the theory of free massive bosons. The only parameter of our theory is the mass μ.

Consider now a short and long distance behaviour of the two-point function. The scale of the distance is set by μ^{-1}. For $x \gg \mu^{-1}$, we look at the asymptote of the Bessel function,

which yields $e^{-\mu x}$. We observe an exponential decay with a correlation length $\xi = 1/\mu$. This is a typical behaviour of a two-point function in a massive theory. The larger the mass the shorter the correlation length. In the opposite limit, $x \ll \mu^{-1}$, the Bessel function diverges. This is an ultraviolet divergence, caused by the fact that we allow for modes with arbitrary large momentum k. Indeed, if we introduce a momentum cutoff Λ, the two point function for small x is finite

$$\langle \phi(\mathbf{x})\phi(\mathbf{x})\rangle = \int_0^\Lambda \frac{\mathrm{d}k}{2\pi}\frac{k}{k^2+\mu^2} = \frac{1}{4\pi}\log\left(1+\left(\frac{\Lambda}{\mu}\right)^2\right) \approx \frac{1}{2\pi}\log\frac{\Lambda}{\mu}. \tag{1.93}$$

The momentum cutoff Λ corresponds to the inverse of the lattice spacing a of the underlying microscopic model.

Derivation 2

We will derive here eq. (1.82) expressing the two-point function through Gaussian integrals. Our starting point is eq. (1.81). We think about $\mathbf{k}$ as a discrete set of variables such that the integral in the action can approximated by a sum

$$S[\phi] = \frac{1}{2}\sum_{\mathbf{k}} A_{\mathbf{k}}|\phi(\mathbf{k})|^2, \qquad A_{\mathbf{k}} = \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}(\mathbf{k}^2+\mu^2). \tag{1.94}$$

Consider now evaluation of the partition function $\mathcal{Z}$. It is defined as

$$\mathcal{Z} = \int \mathrm{D}\phi \exp(-S[\phi]). \tag{1.95}$$

Functional integration over functions ϕ is discretised according to (1.77). Therefore

$$\mathcal{Z} = \int_{-\infty}^{\infty}\prod_{\mathbf{q}_+} d^2\phi(\mathbf{q}) \exp\left(-\frac{1}{2}\sum_{\mathbf{k}} A_{\mathbf{k}}|\phi(\mathbf{k})|^2\right), \tag{1.96}$$

which can be written as

$$\mathcal{Z} = \int_{-\infty}^{\infty}\prod_{\mathbf{q}_+} d^2\phi(\mathbf{q}) \prod_{\mathbf{k}_+} \exp\left(-A_{\mathbf{k}}|\phi(\mathbf{k})|^2\right). \tag{1.97}$$

This expression is a multiple integral over a function that is a product of functions each depending on a single variable. Schematically it is of the following form

$$\int \mathrm{d}x\mathrm{d}y f(x)f(y) = \left(\int \mathrm{d}x f(x)\right)\times\left(\int \mathrm{d}y f(y)\right). \tag{1.98}$$

Therefore

$$\mathcal{Z} = \prod_{\mathbf{q}_+}\int_{-\infty}^{\infty} d^2\phi(\mathbf{q}) \exp\left(-A_{\mathbf{q}}(\mathbf{q}^2+\mu^2)|\phi(\mathbf{q})|^2\right). \tag{1.99}$$

Following in an analogous way with the numerator of the two-point function we find

$$\int \mathrm{D}\phi \exp(-S[\phi])|\phi(\mathbf{p})|^2 = \mathcal{Z} \times \frac{\int_{-\infty}^{\infty} d^2\phi(\mathbf{p}) \exp\left(-A_{\mathbf{p}}|\phi(\mathbf{p})|^2\right)|\phi(\mathbf{p})|^2}{\int_{-\infty}^{\infty} d^2\phi(\mathbf{p}) \exp\left(-A_{\mathbf{p}}|\phi(\mathbf{p})|^2\right)}. \tag{1.100}$$

When computing the two-point function we divide this expression by the partition function $\mathcal{Z}$ which leads to the ratio presented in (1.82).

Analytic continuation

We want now to get back to the Minkowski space-time by analytically continuing the two-point functions. We have seen that the correlation function depends on $|\mathbf{x}_2 - \mathbf{x}_1|$. Analytic continuation changes this to

$$|\mathbf{x}_2 - \mathbf{x}_1| = \sqrt{(x_2 - x_1)^2 + (\tau_2 - \tau_1)^2} \to \sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2}. \tag{1.101}$$

As long as $(x_2 - x_1)^2 > (t_2 - t_1)^2$ than qualitatively the propagator behaves in the same way. The novelty comes when the opposite is true. Consider equal point correlation function, $x_2 = x_1$. Then the argument of the Bessel function is a purely imaginary number $i|t_2 - t_1|$. The two-point function is now a complex functions describing the quantum mechanical amplitude, instead of a classical probability.

1.3.3 Free bosons - canonical quantization

Let us now turn to another method of computing the correlation functions, that is through the canonical quantization. Let us see how this works. We get back to the Minkowski space. This approach is based on quantization of the classical Lagrangian. Quantization means promotion of field $\phi(x,t)$ to the status of an operator acting in the Hilbert space. Let us recall the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{1}{2}\mu^2\phi^2(x,t). \tag{1.102}$$

Now $\phi(x,t)$ is an operator acting at each point (x,t) of space-time. In the canonical formalism the field operator should fulfill the operatorial Euler-Lagrange equations, the equations of motion,

$$\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right) - \partial_x\left(\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{1.103}$$

In the case of the free massive bosons the equations are

$$(\partial_t^2 - \partial_x^2 + \mu^2)\phi(x,t) = 0. \tag{1.104}$$

In a classical field theory, when $\phi(x,t)$ is an ordinary function, a general solution to these equations is just a superposition of plane waves

$$\phi(x,t) = \int \frac{dk}{4\pi E_k}\left(A_k e^{ikx - iE_k t} + A_k^* e^{-ikx + iE_k t}\right). \tag{1.105}$$

where

$$E_k = \sqrt{k^2 + \mu^2}. \tag{1.106}$$

The factor $2E_k$ assures that the expression for $\phi(x,t)$ is relativistically invariant, as it should be, cause $\phi(x,t)$ is a scalar function.

When field configuration $\phi(x,t)$ obeys the equation of motion we call it *on-shell*. Otherwise it is *off-shell*.

We can now formally promote the field $\phi(x,t)$ to be an operator. To make the sense of (1.105) we shall promote the Fourier coefficients A_k and A_k^* to be operators as well. We could write

$$\hat{\phi}(x,t) = \int \frac{dk}{4\pi E_k} \left(\hat{A}_k e^{ikx - iE_k t} + \hat{A}_k^\dagger e^{-ikx + iE_k t} \right), \tag{1.107}$$

but we will abuse the notation and do not put hat's over the operators. Instead we keep on writing

$$\phi(x,t) = \int \frac{dk}{4\pi E_k} \left(A_k e^{ikx - iE_k t} + A_k^\dagger e^{-ikx + iE_k t} \right). \tag{1.108}$$

To proceed further with the quantization we define the conjugate momentum operator

$$\pi(x,t) \equiv \frac{\delta \mathcal{L}}{\delta \partial_t \phi(x,t)} = \partial_t \phi(x,t). \tag{1.109}$$

From the definition we find

$$\pi(x,t) = -i \int \frac{\mathrm{dk}}{4\pi E_k} E_k \left(A_k e^{ikx - iE_k t} - A_k^\dagger e^{-ikx + iE_k t} \right). \tag{1.110}$$

Classically fields $\phi(x,t)$ and $\pi(x,t)$ obey the Poisson bracket relation. Now, that they are operators, the relations are promoted to the canonical (equal-time) commutation relations

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y), \tag{1.111}$$
$$[\phi(x,t), \phi(y,t]] = 0, \tag{1.112}$$
$$[\pi(x,t), \pi(y,t)] = 0. \tag{1.113}$$

These commutation relations imply the following commutation between the A_k and $A_k^\dagger$ operators

$$[A_k, A_p^\dagger] = 4\pi E_k \delta(k-p), \tag{1.114}$$
$$[A_k, A_p] = [A_k^\dagger, A_p^\dagger] = 0. \tag{1.115}$$

The central role in the quantum theory is played by the Hamiltonian. We construct it, just like in the classical formalism through Legendre's transformation. The Hamiltonian density is

$$\mathcal{H} = \pi(x,t)\partial_t \phi(x,t) - \mathcal{L} = \frac{1}{2}(\pi(x,t))^2 + \frac{1}{2}(\partial_x \phi(x,t))^2 + \mu^2 \phi^2(x,t). \tag{1.116}$$

The Hamiltonian and momentum are

$$H = \int \mathrm{d}x\, \mathcal{H}, \tag{1.117}$$
$$P = -\int \mathrm{d}x\, \pi(x,t)\partial_x \phi(x,t). \tag{1.118}$$

We can express both operators in terms of the A_k and $A_k^\dagger$ as

$$H = \frac{1}{2} \int \frac{\mathrm{dk}}{4\pi E_k} E_k \left(A_k^\dagger A_k + A_k A_k^\dagger \right) = \int \frac{\mathrm{dk}}{4\pi E_k} E_k \left(A_k^\dagger A_k + 2\pi E_k \delta(0) \right), \tag{1.119}$$
$$P = -i \int \frac{\mathrm{dk}}{4\pi E_k} k A_k^\dagger A_k. \tag{1.120}$$

Observe that, in both expressions, the combination $A_k^\dagger A_k$ appears. We denote it

$$N_k = A_k^\dagger A_k. \tag{1.121}$$

The commutation relations between the N_k and the other two operators are[2]

$$[N_k, N_p] = 0 \tag{1.122}$$

$$[N_k, A_p] = -4\pi\delta(k-p)E_p A_p, \qquad [N_k, A_p^\dagger] = 4\pi\delta(k-p)E_p A_p^\dagger. \tag{1.123}$$

We want to diagonalize now the Hamiltonian. Note that the Hamiltonian commutes with the N_k operator, $[H, N_k] = 0$, and therefore it is enough to construct eigenstates of the N_k operator. This is our next aim.

Exercise 9

In this exercise you will fill in the gaps in the formulating of canonical quantization. Namely

1. Show the commutation relations for the operators A_k and $A_k^\dagger$.
2. Confirm the equation (1.116) for the Hamiltonian density and find expression for it in terms of the A_k and $A_k^\dagger$ operators
3. Show commutation relations of N_k with A_k and $A_k^\dagger$.

Fock space

Remark 1

To simplify the notation we define new operators a_k and $a_k^\dagger$ which are an rescaled version of A_k and $A_k^\dagger$,

$$a_k = A_k/\sqrt{2E_k}, \qquad a_k^\dagger = A_k^\dagger/\sqrt{2E_k}, \qquad \hat{n}_k = a_k^\dagger a_k = N_k/(2E_k). \tag{1.124}$$

This simplifies the algebra to

$$[a_k, a_p^\dagger] = 2\pi\delta(k-p), \qquad [\hat{n}_k, a_p] = -2\pi\delta(k-p)a_p, \qquad [n_k, a_p^\dagger] = 2\pi\delta(k-p)a_p^\dagger. \tag{1.125}$$

We need to understand the structure of the Hilbert space from the point of view of the three operator a_k, $a_k^\dagger$ and $\hat{n}_k$. One way of proceeding is by assuming the existence of a special state, the vacuum $|0\rangle$, that is annihilated by all the A_k operators

$$a_k|0\rangle = 0. \tag{1.126}$$

From the definition of $\hat{n}_k$ it follows that also $\hat{n}_k|0\rangle = 0$. As a consequence $H|0\rangle = 0$. What is $a_k^\dagger|0\rangle$? We can characterize it by acting with other operators on it. We get

$$a_p a_k^\dagger|0\rangle = 2\pi\delta(p-k)|0\rangle, \tag{1.127}$$

$$\hat{n}_p a_k^\dagger|0\rangle = 2\pi\delta(p-k)a_k^\dagger|0\rangle. \tag{1.128}$$

[2]Recall the identity: $[AB, C] = A[B, C] + [A, C]B$.

We see that action of a_k somehow undoes the action of $a_k^\dagger$. We also see that the state $a_k^\dagger|0\rangle$ is an eigenstate of the $\hat{n}$ operator. Moreover acting with Hamiltonian we would get $Ha_k^\dagger|0\rangle \sim E_k a_k^\dagger|0\rangle$, which shows that it is a state with energy E_k. This motivates us to call $a_k^\dagger$ a creation operator, that creates a particle of energy E_k. Since a_k undoes the action of $a_k^\dagger$ we call it the annihilation operator. What is the $\hat{n}_k$. Since it's eigenvalue is zero on the vacuum and one on a single-particle state it seems that it's counting the particles with momentum k. To establish this we need to first take care of normalization of states. To this end we define the dual vacuum $|0\rangle$ by the following condition

$$\langle 0|0\rangle = 1. \tag{1.129}$$

We can now compute a norm of a single particle state. We find

$$\|a_k^\dagger|0\rangle\|^2 = \langle 0|a_k a_k^\dagger|0\rangle = 2\pi\delta(0). \tag{1.130}$$

Therefore the single particle state is normalized to the Dirac δ-function in the momentum space. If this is unsettling, recall that the $\delta(0)$ in the momentum space is the volume (in our case, the length) of the system in the real space[3]. This is a problem also familiar from a quantum mechanics, where plane waves in an infinite volume are also normalized to the δ-function.

Consider now a two particle state, $(a_k^\dagger)^2|0\rangle$. Its norm is

$$\|(a_k^\dagger)^2|0\rangle\|^2 = \langle 0|a_k a_k a_k^\dagger a_k^\dagger|0\rangle = 2\left(2\pi\delta(0)\right)^2. \tag{1.131}$$

The factor 2 is of a combinatorial nature and arises because the bosonic creation operators commute with each other. Properly normalized (with respect to the δ-function) states are

$$|n_{k_1},\ldots,n_{k_m}\rangle = \frac{1}{\sqrt{n_{k_1}!\cdots n_{k_m}!}}\left(a_{k_1}^\dagger\right)^{n_{k_1}}\cdots\left(a_{k_m}^\dagger\right)^{n_{k_m}}|0\rangle. \tag{1.132}$$

Exercise 10

Verify that $\hat{n}_{k_1}|n_{k_1},\ldots,n_{k_m}\rangle = 2\pi\delta(0)\times n_{k_1}|n_{k_1},\ldots,n_{k_m}\rangle$.

Two-point function

Consider now two-point correlation function of fields

$$\langle\phi(x_2,t_2)\phi(x_1,t_1)\rangle = \langle 0|T\left(\phi(x_2,t_2)\phi(x_1,t_1)\right)|0\rangle \tag{1.133}$$

Here T denotes the time ordering of the operators

$$T\left(\phi(x_2,t_2)\phi(x_1,t_1)\right) = \begin{cases}\phi(x_2,t_2)\phi(x_1,t_1), & t_2 > t_1,\\ \phi(x_1,t_1)\phi(x_2,t_2), & t_1 > t_2.\end{cases} \tag{1.134}$$

That both expressions for the correlation function are equal can be shown by constructing the path integral representation from the operatorial one. We refer to any book on QFT for details. The correlation function becomes

$$\langle\phi(x_2,t_2)\phi(x_1,t_1)\rangle = \langle 0|\phi(x_2,t_2)\phi(x_1,t_1)|0\rangle\theta(t_2-t_1) + \langle 0|\phi(x_1,t_1)\phi(x_2,t_2)|0\rangle\theta(t_1-t_2) \tag{1.135}$$

[3]To see this observe that $2\pi\delta(k) = \int_{-\infty}^{\infty} dx e^{ikx}$. Going to finite volume L, means restricting the integral to interval $[-L/2, L/2]$. For $k=0$ we have that $2\pi\delta(0)$ becomes L.

Plugging the operatorial expression for fields and using properties of the creation and annihilation operators we find for $t_2 > t_1$

$$\langle 0|\phi(x_2,t_2)\phi(x_1,t_1)|0\rangle = \int \frac{\mathrm{dk}}{4\pi E_k} e^{ik(x_2-x_1)-iE_k(t_2-t_1)}, \tag{1.136}$$

while for $t_1 > t_2$

$$\langle 0|\phi(x_1,t_1)\phi(x_2,t_2)|0\rangle = \int \frac{\mathrm{dk}}{4\pi E_k} e^{-ik(x_2-x_1)+iE_k(t_2-t_1)}. \tag{1.137}$$

We can put both expressions into a single one by introducing integration over ω

$$\langle \phi(x_2,t_2)\phi(x_1,t_1)\rangle = -i \int \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\omega}{2\pi} \frac{e^{ik(x_2-x_1)-i\omega(t_2-t_1)}}{\omega^2-k^2-\mu^2}. \tag{1.138}$$

This follows from the following identity

$$-i\int_{\mathcal{F}} \frac{\mathrm{d}\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2-k^2-\mu^2} = \frac{e^{-iE_k t}}{2E_k}\theta(t) + \frac{e^{iE_k t}}{2E_k}\theta(-t). \tag{1.139}$$

Exercise 11

Prove this identity. The integral can be evaluated with the help of the residue technique. As the integration contour choose the Feynman contour $\mathcal{F}$, see fig. 1.2. To employ the residue technique the contour has to be closed with an infinite semi-circle in either upper-half (for $t < 0$) or lower-half (for $t > 0$) of the complex plane.

Figure 1.2: Feynman contour

We can read off the correlation function in the momentum space

$$\langle \phi(k,\omega)\phi(-k,-\omega)\rangle = \frac{-i}{\omega^2-k^2-\mu^2}. \tag{1.140}$$

This expression has a pole at the mass shell $\mathbf{k}^2 = \mu^2$ which is regularised in the equivalent ways using the Feynman contour or introducing $i\epsilon$. In the latter case, we write the propagator as

$$\langle \phi(k,\omega)\phi(-k,-\omega)\rangle = \frac{-i}{\omega^2-k^2-\mu^2+i\epsilon}, \tag{1.141}$$

The propagator has now poles at $\omega = \pm\sqrt{k^2+\mu^2} \mp i\epsilon/(2\sqrt{k^2+\mu^2})$. We see that the pole for $\omega > 0$ is now shifted slightly below the real line. Shifting the position of the pole by $i\epsilon$ is equivalent to slightly deforming the contour as per Feynman contour which is deformed to encircle the pole from above for $\omega > 0$. Thus both ways of regularising the propagator are equivalent.

Going to the Euclidian space means $t \to -i\tau$. This corresponds to $\omega \to -i\omega$ [4]. We perform the analytic continuation at the level of eq. (1.138) to find

$$\langle \phi(x_2,\tau_2)\phi(x_1,\tau_1)\rangle = \int \frac{\mathrm{d}k}{2\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{e^{ik(x_2-x_1)+i\omega(\tau_2-\tau_1)}}{\omega^2+k^2+\mu^2}. \tag{1.142}$$

with the two point correlation function in the momentum space

$$\langle \phi(k,\omega)\phi(-k,-\omega)\rangle = \frac{1}{\omega^2+k^2+\mu^2}. \tag{1.143}$$

We obtained the same formula from the path integral approach, see (1.87).

[4]The transformation of ω can be determined from the requirement that temporal exponent in the Fourier expansion $-i\omega t$ turns into the spatial exponent $i\omega\tau$.

1.4 Perturbation theory and Sinh-Gordon model

We will study now the Sinh-Gordon theory using the perturbation theory. Recall the Lagrangian

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{\mu^2}{g^2}\cosh g\phi(x,t). \tag{1.144}$$

Assume that we can expand the interaction term $\cosh g\phi$ in a Taylor series, the leading term is a constant that we can drop, the next term is $\frac{1}{2}\mu^2\phi^2(x,t)$. This is a mass term. The next term is $\frac{1}{4!}g^2\mu^2\phi^4(x,t)$. We cannot solve the QFT exactly with such a term. However we can use a perturbation theory. So we consider free massive bosons with an extra ϕ^4 term

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{1}{2}\mu^2\phi^2(x,t) - \frac{1}{4!}\mu^2g^2\phi^4(x,t). \tag{1.145}$$

Let us say that we want to compute the two-point function. Recall the expression in the path-integral formalism

$$\langle\phi(x_2,t_2)\phi(x_1,t_1)\rangle = \frac{\int D\phi\, e^{iS[\phi]}\phi(x_2,t_2)\phi(x_1,t_1)}{\int D\phi\, e^{iS[\phi]}}. \tag{1.146}$$

The action consists now of two pieces

$$S[\phi] = S_0[\phi] + S_{\text{int}}[\phi], \tag{1.147}$$

where

$$S_0[\phi] = \int \mathrm{d}x\mathrm{d}t\left(\frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{1}{2}\mu^2\phi^2(x,t)\right), \tag{1.148}$$

$$\begin{aligned} S_{int}[\phi] &= -\frac{1}{4!}\mu^2g^2\int \mathrm{d}x\mathrm{d}t\,\phi^4(x,t) \\ &= -\frac{1}{4!}\mu^2g^2\int\left(\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\right)^4\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(\mathbf{k}_4)(2\pi)^2\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4). \end{aligned} \tag{1.149}$$

Exercise 12: Interaction in the momentum space

Verify eq. (1.149) with

$$\phi(x,t) = \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k}), \tag{1.150}$$

where $\mathbf{k} = (k,\omega)$ and $\mathbf{k}\cdot\mathbf{x} = kx - \omega t$.

In the spirit of the perturbation theory we expand the weighting factor in powers of the interaction term

$$e^{iS[\phi]} = e^{iS_0[\phi]}\left(1 + iS_{int}[\phi] - \frac{1}{2}S_{int}[\phi]^2 + \dots\right). \tag{1.151}$$

For the partition function we find

$$Z = \int D\phi e^{iS[\phi]} = \int D\phi e^{iS_0[\phi]}\left(1 + iS_{int}[\phi] + \dots\right), \tag{1.152}$$

with $S_{int}[\phi]$ involving, as per eq. (1.149), 4 fields. In a similar way we can compute the correction to the numerator

$$\int D\phi\, e^{iS[\phi]}\phi(x_2,t_2)\phi(x_1,t_1) = \int D\phi\, e^{iS_0[\phi]}(1+iS_{int}[\phi])\phi(x_2,t_2)\phi(x_1,t_1), \tag{1.153}$$

which involves now computing a 6-point function. Therefore to develop a perturbation theory we need an effective way of computing higher order correlation functions in a free theory. Fortunately there is an almost automatic way to get them with the help of the generating function which we will now sketch.

1.4.1 Generating function and Wick's theorem

We go the Euclidean formalism. We define the generating function as

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(-S_0[\phi] - \int \mathrm{d}\mathbf{x}\, J(\mathbf{x})\phi(\mathbf{x})\right). \tag{1.154}$$

It is convenient to move to the Fourier space in which

$$\int \mathrm{d}\mathbf{x}\, J(\mathbf{x})\phi(\mathbf{x}) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} J(-\mathbf{k})\phi(\mathbf{k}), \tag{1.155}$$

with $J(-\mathbf{k}) = J^*(\mathbf{k})$ for real field $J(\mathbf{x})$. The weighting factor is now

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\left[-\frac{1}{2}\left(\mathbf{k}^2+\mu^2\right)|\phi(\mathbf{k})|^2 - J(-\mathbf{k})\phi(\mathbf{k})\right]. \tag{1.156}$$

The momentum space correlation functions can be now computed by functional derivative of the generating function $Z_0[J]$

$$\langle\phi(\mathbf{k}_n)\ldots\phi(\mathbf{k}_1)\rangle = (-1)^n\frac{(2\pi)^{2n}}{Z[J]}\left.\frac{\delta^n Z[J]}{\delta J(-\mathbf{k}_n)\ldots\delta J(-\mathbf{k}_1)}\right|_{J=0}. \tag{1.157}$$

Remark 2: Functional derivative

We will think about a functional derivative as a generalization of partial derivatives. Let $f(x_1,\ldots,x_n)$ be a function of n variables. You are familiar with a partial derivative defined as

$$\frac{\partial f(x_1,\ldots,x_n)}{\partial x_j} = \lim_{\epsilon\to 0}\frac{f(x_1,\ldots,x_{j-1},x_j+\epsilon,x_{j+1},\ldots,x_n)-f(x_1,\ldots,x_n)}{\epsilon}. \tag{1.158}$$

We can make the notation more efficient by denoting $\mathbf{x} = \{x_1,\ldots,x_n\}$ and then writing

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = \lim_{\epsilon\to 0}\frac{f(\mathbf{x}+\epsilon_j)}{\epsilon}, \tag{1.159}$$

where we understand that $\mathbf{x}+\epsilon_j$ means adding ϵ to the j-th element of the set $\mathbf{x}$.

Consider now a functional $F[f]$. This is an operation that assigns to a function f a number. An example of a functional is integration:

$$F_g[f] = \int_{-\infty}^{\infty}\mathrm{d}x f(x)g(x), \tag{1.160}$$

where $g(x)$ is some other function. What's the result of taking a functional derivative of $F_q[f]$ with respect to $f(x)$. The answer is $g(x)$. How to understand this? Let's recast the integral as a sum and write

$$F_g[\{f(x_i)\}] = \lim_{\Delta x_i \to 0} \sum_i \Delta x_i f(x_i) g(x_i), \tag{1.161}$$

where we assume some discretisation of the integration interval. This turns the functional into a function of $\{f(x_i)\}$. We can ask now how the answer varies if we modify one of the arguments, say $\{f(x_j)\}$. This can be computed as a partial derivative

$$\frac{\partial F_g[\{f(x_i)\}]}{\partial f(x_j)} = \Delta x_j g(x_j). \tag{1.162}$$

We can take now the continuum limit $\Delta x_i \to 0$ defining

$$\frac{\delta F_g[f]}{\delta f(x)} = \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_j} \frac{\partial F_g[\{f(x_i)\}]}{\partial f(x_j)} \tag{1.163}$$

Exercise 13: Exercise in functional derivatives

Consider the following functional

$$F_g[f] = \exp\left(\int \mathrm{d}x f(x) g(x)\right). \tag{1.164}$$

Compute $\delta F_g[f]/\delta f(x)$. Having this result, show eq. (1.157).

Eq. (1.157) shows that all we need now is to compute $Z_0[J]$. This can be easily achieved. To this end we shift $\phi(\mathbf{k})$ by a $J(\mathbf{k})$ term to complete the square in (1.156). We write

$$\tilde{\phi}(\mathbf{k}) = \phi(\mathbf{k}) - \frac{J(\mathbf{k})}{\mathbf{k}^2 + \mu^2}, \tag{1.165}$$

which leads to the action equal to

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \left[-\frac{1}{2}\left(\mathbf{k}^2 + \mu^2\right) |\tilde{\phi}(\mathbf{k})|^2 + \frac{|J(\mathbf{k})|^2}{2(\mathbf{k}^2 + \mu^2)}\right]. \tag{1.166}$$

The shift of the field by a constant does not influence the integration measure $\mathcal{D}\phi$. Therefore, the partition function with the external current can be expressed as

$$Z_0[J] = Z_0[0] \exp\left(\frac{1}{2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \frac{|J(\mathbf{k})|^2}{\mathbf{k}^2 + \mu^2}\right). \tag{1.167}$$

The contribution from the external current simply factorizes. For example the two point function is

$$\Delta(\mathbf{k}) = \langle \phi(\mathbf{k}) \phi(-\mathbf{k}) \rangle = \frac{1}{\mathbf{k}^2 + \mu^2}. \tag{1.168}$$

The 4-point function is

$$\begin{aligned} \langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(\mathbf{k}_4) \rangle =& \Delta(\mathbf{k}_1, \mathbf{k}_2)\Delta(\mathbf{k}_3, \mathbf{k}_4) + \Delta(\mathbf{k}_1, \mathbf{k}_3)\Delta(\mathbf{k}_2, \mathbf{k}_4) \\ &+ \Delta(\mathbf{k}_1, \mathbf{k}_4)\Delta(\mathbf{k}_2, \mathbf{k}_3). \end{aligned} \tag{1.169}$$

For the 6-point function we have

$$\langle\phi(\mathbf{k}_1)\dots\phi(\mathbf{k}_6)\rangle = \sum_\sigma \Delta(\mathbf{k}_{\sigma_1},\mathbf{k}_{\sigma_2})\Delta(\mathbf{k}_{\sigma_3},\mathbf{k}_{\sigma_4})\Delta(\mathbf{k}_{\sigma_5},\mathbf{k}_{\sigma_6}), \tag{1.170}$$

where the summation goes over all possible pairings of $1,2,\dots,6$ and

$$\Delta(\mathbf{k}_1,\mathbf{k}_2) = (2\pi)^2\delta(\mathbf{k}_1+\mathbf{k}_2)\Delta(\mathbf{k}_1). \tag{1.171}$$

This result can be easily generalized to any $2n$-point correlation function of the field $\phi(\mathbf{k})$.

Exercise 14: On Wick's theorem

Confirm expressions for the 4-points and 6-points function by the functional derivative of the generating function.

In general, the $(2n)$-point function is a sum over all possible pairwise contractions with the Green's function and δ-function appearing for every contraction. The correlation functions with odd number of fields are 0. This is the Wick's theorem.

1.4.2 Back to the perturbation theory and Feynmann diagrams

Using the result of the previous section, the correction to the partition function is

$$\begin{aligned}\langle S_{int}[\phi]\rangle &= -\frac{1}{4!}\mu^2 g^2\int\left(\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\right)^2\langle\phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(\mathbf{k}_4)\rangle(2\pi)^2\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\\ &= -\frac{3}{4!}\mu^2 g^2\mathcal{V}\left(\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\Delta(\mathbf{k})\right)^2.\end{aligned} \tag{1.172}$$

where $\mathcal{V} = (2\pi)^2\delta^2(0)$ comes as regularization of the 2d Dirac δ-function at 0. The correction to the numerator of the two-point function involves 6-point function. Let us start with understanding its structure. 4 momenta are constrained by the momentum conservation coming from the interaction term. So when we consider pairing among the 6 momenta with 4 of them constrained, we can divide pairings in two groups. In the first group we pair 4 constrained momenta with each other, in the second group two constrained momenta are paired with 2 unconstrained (external) momenta. We find

$$\begin{aligned}\langle\phi(\mathbf{p})\phi(\mathbf{q})S_{int}\rangle =& -\frac{3}{4!}\mu^2 g^2\Delta(\mathbf{p},\mathbf{q})\mathcal{V}\left(\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\Delta(\mathbf{k})\right)^2\\ &-\frac{12}{4!}\mu^2 g^2\Delta(\mathbf{p})\Delta(\mathbf{q})(2\pi)^2\delta(\mathbf{p}+\mathbf{q})\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\Delta(\mathbf{k}).\end{aligned} \tag{1.173}$$

The two-point function, including the first order correction, is

$$\langle\phi(\mathbf{p})\phi(\mathbf{q})\rangle = \frac{\Delta(\mathbf{p},\mathbf{q}) - \langle S_{int}[\phi]\phi(\mathbf{p})\phi(\mathbf{q})\rangle}{1-\langle S_{int}[\phi]\rangle}. \tag{1.174}$$

Therefore, with the same accuracy, we find

$$\begin{aligned}\langle\phi(\mathbf{p})\phi(\mathbf{q})\rangle &= \left(\Delta(\mathbf{p},\mathbf{q}) - \langle S_{int}[\phi]\phi(\mathbf{p})\phi(\mathbf{q})\rangle\right)\left(1+\langle S_{int}[\phi]\rangle\right)\\ &= \Delta(\mathbf{p},\mathbf{q}) + \Delta(\mathbf{p},\mathbf{q})\langle S_{int}[\phi]\rangle - \langle S_{int}[\phi]\phi(\mathbf{p})\phi(\mathbf{q})\rangle.\end{aligned} \tag{1.175}$$

After some rewriting

$$\langle\phi(\mathbf{p})\phi(\mathbf{q})\rangle = (2\pi)^2\delta(\mathbf{p}+\mathbf{q})\Delta(\mathbf{p})\left(1-\frac{1}{2}\mu^2 g^2\Delta(\mathbf{p})\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\Delta(\mathbf{k})\right). \tag{1.176}$$

This two factors can be understood in terms of the Feynman diagrams

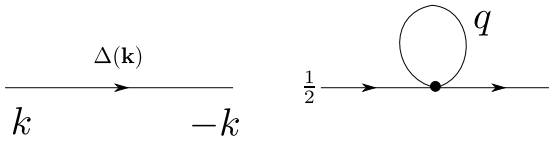


Figure 1.3: Propagator and the first order correction to it together with the combinatorial factor.

Connected contributions

From the computations of the first-order corrections to the 2-point function we see that there was a cancellation between a contribution from the partition function and from the numerator. Such cancellations occur also at higher orders of perturbation theory. Therefore it would be nice if we could somehow automatically get only contributions to the correlation functions without having to independently consider numerator, denominator and then execute the cancellations. Fortunately there is such a procedure and it is directly connected with the structure of the Feynmann diagrams. It can be shown, for a proof we refer to any decent QFT book, that only connected diagrams contribute to perturbative expansion of the correlation functions.

Feynmann rules for the ϕ^4 theory [in Euclidean formalism]

For a given diagram we have to perform the following steps

1. assign to each vertex a factor $(-\mu^2 g^2)$
2. assign to each line a momentum $\mathbf{k}$ and propagator $\Delta(\mathbf{k})$
3. impose momentum conservation at each vertex
4. integrate over the remaining internal momenta
5. work out the combinatorial factors (there are rules for that as well)

The propagator is given by

$$\Delta(\mathbf{k}) = \frac{1}{\mathbf{k}^2 + \mu^2}, \tag{1.177}$$

with $\mathbf{k}^2 = \omega^2 + k^2$.

Consider the second order correction to the two-point function. There are 3 kinds of connected diagrams, see fig. 1.4. The contribution to the two-point function is

$$\mu^4 g^4 \Delta(\mathbf{k})^2 \left(\frac{1}{4}\Delta(\mathbf{k}) \left(\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\Delta(\mathbf{q}) \right)^2 + \frac{1}{4}\left(\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^2}\Delta(\mathbf{q}) \right)^3 + \frac{1}{6}\int \left(\frac{\mathrm{d}\mathbf{q}}{(2\pi)^2} \right)^2 \Delta(\mathbf{q}_1)\Delta(\mathbf{q}_2)\Delta(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2) \right). \tag{1.178}$$

To switch to the Minkowski space, we need to replace $(-\mu^2 g^2)$ by $(i\mu^2 g^2)$ and consider the propagator given by

$$\Delta(\mathbf{k}) = -\frac{i}{\mathbf{k}^2 - \mu^2}, \tag{1.179}$$

with $\mathbf{k}^2 = \omega^2 - k^2$.

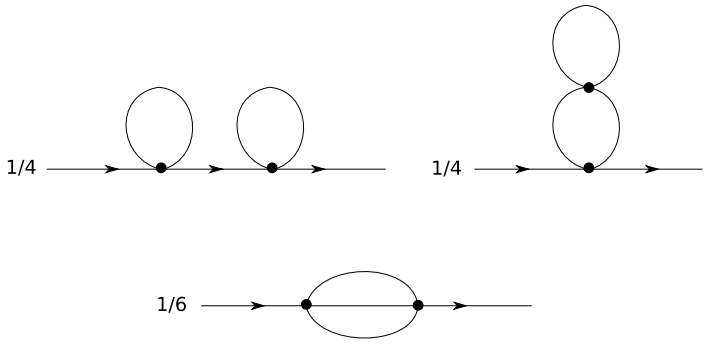


Figure 1.4: The 3 kinds of contributions at the second order to the two-point function.

1.4.3 Scattering matrix - introduction

The central object in the high-energy physics is a scattering matrix which describes amplitudes for possible scattering processes. As we shall see, the scattering matrix is also crucial in understanding IQFT's. Therefore we will spend some time introducing the scattering matrix and relating it to the correlation functions. Moreover, we shall see why the Sinh-Gordon theory is special.

Let us consider a scattering experiment. Say we have a bunch of particles approaching each other, scattering and after scattering being detected. Essentially, we can write $|in\rangle$ as a desription of the incoming state, $|out\rangle$ as a desription of the outgoing state. Furthermore, the evolution of quantum states is run by the Hamiltonian.

The time evolution of a quantum state is given by the Schroedinger equation

$$-i\partial_t|a\rangle = H|a\rangle. \tag{1.180}$$

We can formally solve this equation by defining the evolution operator

$$U(t) = \exp(-iHt), \tag{1.181}$$

which evolves the state by time t in time. Therefore the scattering experiment gives us access to amplitudes like this

$$\lim_{t\to\infty}\langle out|U(t)|in\rangle. \tag{1.182}$$

The question is, how to define the in and out states in an interacting theory. In principle it is a difficult task, because we would need to understand the particle content of the interacting theory, but this is just what we want to achieve studying the scattering matrix. Fortunately, assuming a short range interactions (natural in high-energy physics, reasonable in cond-mat) we can think about the in and out states as states in which particles are essentially free because they are far from each other. Such a state can be created with the free theory language. Essentially, what we need to do then is to take such a free state, evolve it in time to bring it to present. This evolution, because is done with a fully interacting theory can lead to any possible processes that we expect in an interacting theory, like creation and annihilation processes, scattering and so on. We can bring such an in state from the past and then compare it (compute overlap) with a similar out state brought from the future. Consider for example a two particle in and out states. They are defined as

$$|p_1,p_2\rangle_{\text{in}} = \lim_{t\to\infty} e^{-iHt}|p_1,p_2\rangle_0, \tag{1.183}$$

$${}_{out}\langle k_1,k_2| = \lim_{t\to\infty} {}_0\langle k_1,k_2|e^{-iHt} \tag{1.184}$$

The scattering matrix S is then

$$S_{(k_1,k_2|p_1,p_2)} = {}_{out}\langle k_1, k_2|p_1, p_2\rangle_{\text{in}}. \tag{1.185}$$

There is an intimate relation between the scattering matrix and correlation functions. Essentially, the correlation functions describes processes where particles are created, they propagate, scatter, and propagate and finally are annihilated. The scattering matrix describes the central part of the correlation functions: this involving the scattering. Therefore we might hope that there is a way of extracting the scattering from the correlation function. Indeed, this is the case. Essentially, we need to strip the correlation function from the processes of creation, annihilation and propagation of external particles. This is the content of the LSZ formula. Specifically, let us separate the no-scattering process by writing

$$S = 1 + T, \tag{1.186}$$

then,

$$T_{(\mathbf{k}_1,\ldots,\mathbf{k}_m|\mathbf{p}_1,\ldots,\mathbf{p}_n)} = \frac{\langle\phi(\mathbf{k}_1)\ldots\phi(\mathbf{k}_m)\phi(-\mathbf{p}_1)\ldots\phi(-\mathbf{p}_n)\rangle}{\prod_{j=1}^m \Delta(\mathbf{k}_j)\prod_{j=1}^n \Delta(\mathbf{p}_j)} \tag{1.187}$$

Because we are now dealing with actual particles all the momenta are on shell: $\mathbf{k}^2 = m^2$. The prescription is to first compute the nominator and then to take the external particles on-shell. Moreover, it is simple to formulate Feynmann rules for the scattering matrix. Proceed like for the correlation functions but do not plug propagators for external particles. We should also look only at the diagrams where the first processes are always interactions between the external particles.

Let us note also that, according to the LSZ reduction formula we can bring a particle from an outgoing state to the ingoing state by simply changing the sign of the momentum $\mathbf{k}$. This is called *crossing symmetry* and is an important feature of the S-matrix.

For details on deriving the LSZ formula we refer to any decent QFT book. Let us now analyze the scattering matrix using perturbation theory. Therefore we should start with the free theory first.

Free bosons

Let us start with a simple example of the scattering matrix for free massive bosons. Consider two particles scattering matrix. For that we need the 4-point function. The 4-point function contains a product of two propagators. In the LSZ reduction formula there are 4 propagators. When evaluted on-shell we get 0. Thus

$$T_{(\mathbf{k}_1,\mathbf{k}_2|\mathbf{p}_1,\mathbf{p}_2)} = 0 \tag{1.188}$$

This result makes sense: there is no scattering in a free theory. It is easy to generalize this results to any multi-particle process. In a free theory there are no creation and annihilation processes, therefore the particle number cannot change. There are no interactions, therefore the momenta of the particles cannot change either. Therefore the scattering matrix must be diagonal ($S = 1$). Analyzing the structure of the correlation functions in a free theory and LSZ reduction formula these observations can be confirmed.

Perturbation theory

Let us turn now the ϕ^4 interactions and see what changes. Of course, as we have seen the propagator itself changes, however this does not influence directly the S-matrix (at least not in the leading order). Let us look at the scattering process of 2 particles. To this end we need to consider the 4 point correlation function without assigning the propagators to external legs. Therefore the contribution comes solely from the vertex and

$$T_{(\mathbf{k}_1,\mathbf{k}_2|\mathbf{p}_1,\mathbf{p}_2)} = i\mu^2 g^2 (2\pi)^2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_1 - \mathbf{p}_2). \tag{1.189}$$

Because the momentum conserving δ always appears, sometimes it is convenient to write

$$T = \mathcal{M}(2\pi)^2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_1 - \mathbf{p}_2), \tag{1.190}$$

then

$$\mathcal{M}_{(\mathbf{k}_1,\mathbf{k}_2|\mathbf{p}_1,\mathbf{p}_2)} = i\mu^2 g^2. \tag{1.191}$$

This computation shows that including ϕ^4 term we let particles interact in a momentum and energy conserving way. Also the scattering matrix is a constant, it does not depend on the momenta.

Exercise 15

Show eq. (1.189) by considering the 4-point function using the perturbation theory.

Consider now a process in which one particle decays into 3 particles. Is such a process possible? Let us see

$$T_{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|\mathbf{p})} = \frac{\langle \phi(\mathbf{k}_1)\phi(\mathbf{k}_2)\phi(\mathbf{k}_3)\phi(-\mathbf{p})\rangle}{\Delta(\mathbf{k}_1)\Delta(\mathbf{k}_2)\Delta(\mathbf{k}_3)\Delta(\mathbf{p})} = i\mu^2 g^2 (2\pi)^2 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{p}). \tag{1.192}$$

We see that the process is possible as long as the δ-function can be resolved. Assume that the decaying particle is stationary $\mathbf{p} = (0, \mu)$. Then the three momenta of the outgoing particles must sum to zero: $k_1 + k_2 + k_3 = 0$. On the other hand the energy conservation implies that $\omega_1 + \omega_2 + \omega_3 = 0$ which is impossible to fulfill. Therefore such a decay process is not possible.

Note that the δ function in the $2 \to 2$ scattering process is easily resolved. Because of the conservation of energy and momentum we have that

$$\begin{cases} \mathbf{k}_1 = \mathbf{k}_3 \\ \mathbf{k}_2 = \mathbf{k}_4 \end{cases}, \qquad \text{or} \qquad \begin{cases} \mathbf{k}_1 = \mathbf{k}_4 \\ \mathbf{k}_2 = \mathbf{k}_3 \end{cases}. \tag{1.193}$$

The scattering acts as a permutation.

1.4.4 Why is Sinh-Gordon theory special?

Consider now a decay process $2 \to 4$. Due to the crossing symmetry this amplitude can be obtained from the amplitude $3 \to 3$ which is somehow simpler to compute and it's related to a 6-point correlation function. There are 2 classes of connected diagrams in the leading order in the perturbation theory and at the tree level. Writing the internal momentum as

$$\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k}_1, \tag{1.194}$$

$$\mathbf{r} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3. \tag{1.195}$$

The scattering amplitude, at the tree level, is

$$\mathcal{M}_{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)} = -\mu^4 g^4 \left(\Delta(\mathbf{r}) + \sum_{cycl(1,2,3)} \sum_{cycl(1',2',3')} \Delta(\mathbf{q}) \right). \tag{1.196}$$

Note that if we consider $2 \to 4$ process the middle particle is never on-shell. However for the $3 \to 3$ this particle might be on-shell in which case we have to regularize intermediate expressions by introducing $i\epsilon$ in propagator. As our aim is the $2 \to 4$ amplitude we don't need to do it. To analyze this expression we go to the light-cone coordinates

Remark 3: Light-cone coordinates

In $(1+1)$ dimensions light-cone coordinates are a convenient choice for the coordinate system. The left and right coordinate are defined as

$$p_l = \omega + p, \qquad p_r = \omega - p. \tag{1.197}$$

We will write $\mathbf{p} = (p_l, p_r)_{\text{lc}}$ The norm-squared reads then $\mathbf{p}^2 = \omega^2 - p^2 = p_l p_r$. The on-shell condition $\mathbf{p}^2 = \mu^2$ can be solved by writing $\mathbf{p} = \mu(a, a^{-1})_{\text{lc}}$, where a is a real number parametrising the momentum of the particle.

We introduce light-cone coordinates for all the external momenta writing

$$\mathbf{p}_1 = \mu(a_1, a_1^{-1}), \quad \mathbf{p}_2 = \mu(a_2, a_2^{-1}), \quad \mathbf{k}_1 = \mu(b_1, b_1^{-1}), \quad \text{etc.} \tag{1.198}$$

We find that the propagators are

$$\Delta(\mathbf{q}) = \frac{-i}{\mathbf{q}^2 - \mu^2} = \frac{-i}{\mu^2} \frac{1}{(a_1 + a_2 - b_1)(a_1^{-1} + a_2^{-1} - b_1^{-1}) - 1} \tag{1.199}$$

$$= \frac{-i}{\mu^2} \frac{-a_1 a_2 b_1}{(a_1 + a_2)(a_1 - b_1)(a_2 - b_1)}. \tag{1.200}$$

The other one is

$$\Delta(\mathbf{r}) = \frac{-i}{\mu^2} \frac{1}{(a_1 + a_2 + a_3)(a_1^{-1} + a_2^{-1} + a_3^{-1}) - 1} \tag{1.201}$$

$$= \frac{-i}{\mu^2} \frac{a_1 a_2 a_3}{(a_1 + a_2)(a_1 + a_3)(a_2 + a_3)}. \tag{1.202}$$

Therefore the scattering matrix is proportional to

$$H(a_1, a_2, a_3, b_1, b_2, b_3) = \frac{a_1 a_2 a_3}{(a_1 + a_2)(a_1 + a_3)(a_2 + a_3)} + \sum_{cycl(1,2,3)} \sum_{cycl(1',2',3')} \frac{-a_1 a_2 b_1}{(a_1 + a_2)(a_1 - b_1)(a_2 - b_1)}. \tag{1.203}$$

We should also impose total conservation of momentum and energy, or in the light cone coordinates

$$\sum_{j=1}^{3} (a_j - b_j) = 0, \qquad \sum_{j=1}^{3} (a_j^{-1} - b_j^{-1}) = 0. \tag{1.204}$$

Under this assumption

$$H(a_1, a_2, a_3, b_1, b_2, b_3) = -1. \tag{1.205}$$

Exercise 16

Show relation (1.205).

By the crossing symmetry we can bring now the third particle to the out state. In the light cone coordinates this amounts to change $a_3 \to -a_3$. The equality $H = -1$ remains intact. Therefore, we can conclude that the $2 \to 4$ amplitude in the leading order in the perturbation theory (and at the tree level) is a constant: doesn't depend on the momenta

$$\mathcal{M}_{(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)} = i\mu^2 g^4. \tag{1.206}$$

We can now cancel this amplitude by introducing ϕ^6 term in the action

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{\mu^2}{g^2}\left(1 + \frac{1}{2}g^2\phi^2(x,t) + \frac{1}{4!}g^4\phi^4(x,t) + \frac{1}{6!}g^6\phi^6(x,t)\right). \tag{1.207}$$

Computing the $2 \to 6$ amplitude in this theory we would find that it is again a constant and can be killed by introducing a ϕ^8 term. Following in this way we would finally get the Sinh-Gordon model

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{\mu^2}{g^2}\cosh(g\phi(x,t)). \tag{1.208}$$

Therefore, we see, that at least at the tree level, there are no annihilation or creation processes in the theory. What about the scattering? It can be computed, that the amplitude of the $3 \to 3$ process is non zero. To this end we would need to introduce $i\epsilon$ factors and recompute the amplitude. The result is that there is an extra δ function popping out which restricts the set of final moment to be (up to the permutation) the same as the set of initial momenta. Therefore, again, the scattering acts as a permutation.

All the computations considered so far were performed at the tree level. It is possible to include loop corrections, and at the one-loop level the conclusions are the same.

Finally, note that changing $g \to ig$ does not invalidate any of the perturbative computations. Therefore, we must conclude that the S-matrix of the Sine-Gordon theory also has these special features.

Chapter 2

Integrable Quantum Field Theories

2.1 S-matrix theory

Our aim is now to obtain the S-matrix without relying on the Lagrangean formulation. Our results won't rely also on the perturbative expansion. Instead, we will try to put physical constraints on the S-matrix and understand the resulting structure. We will do this in two steps. First we will explore a general features of the S matrix in $1+1$ dimensions. In the second step we will add an extra requirement on the scattering processes. Namely we will restrict to theories with elastic scattering processes.

The study of general features of the S matrix goes under the name of the S-matrix theory. This approach was originally introduced in 50's to circumvent problems with perturbative expansions and renormalization. It quickly became quite technical and eventually upon showing renormalizability of the QCD was abandoned. Its revival occurred in the 70's and 80's with the development of Integrable (and Conformal) Quantum Field Theories.

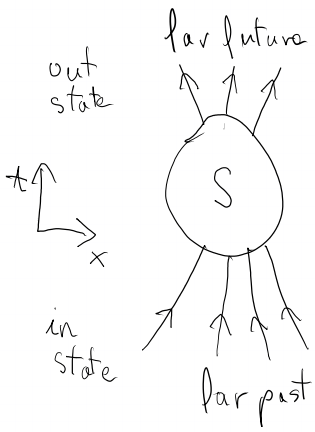


Figure 2.1: The S-matrix is a black-box that gives us an amplitude of a scattering process. There are complicated (and interesting) things happening inside, but we care (mostly) just about the output.

The main spirit of the S-matrix theory is in the assumption that the S-matrix should be maximally analytic function in the momenta of the particles. In other word, every non-analyticity in the S-matrix should have a good reason.

We will list now few general assumptions and in the following we will explore how they influence the structure of the S-matrix.

1. short range of interactions
2. superposition principle of QM
3. conservation of probability
4. relativistic invariance

5. crossing invariance
6. causality principle
7. analyticity principle

2.1.1 Short range of interactions

If the interactions are of short range, then the $|\text{in}\rangle$ and $|\text{out}\rangle$ states consist of free particles. We will assume that the particles are massive. Therefore to span the Hilbert spaces $\mathcal{H}_{\text{in}}$ of initial and $\mathcal{H}_{\text{out}}$ of final states we can use the free particle basis. In the free particle basis eigenstates can be labelled by their momenta and masses (and other quantum numbers if present, for example charge). We will denote such an eigenstate by $|\alpha\rangle$ where α abbreviates all these informations that we need to supply to identify the state. In the example of the free massive bosons of the Section 3, α specifies momenta of all the particles in the state (e.g. $\alpha = \mathbf{k}_1, \ldots, \mathbf{k}_n$ for a state with n particles).

The in and out states consist of actual particles, thus the momenta are on-shell $\mathbf{k}^2 = \mu^2$. We know that with the states of a free theory we can form an orthonormal basis. Therefore we have two relations [1]

$$\langle\alpha|\beta\rangle = \delta_{\alpha,\beta}, \qquad \sum_{\alpha} |\alpha\rangle\langle\alpha| = 1, \tag{2.1}$$

where $|\alpha\rangle$, $|\beta\rangle$ are both from $\mathcal{H}_{\text{in}}$ or $\mathcal{H}_{\text{out}}$. The spaces $\mathcal{H}_{\text{in/out}}$ are Hilbert spaces of the same theory, thus they are isomorphic.

2.1.2 Superposition principle

Let $|\text{in}\rangle$ be a chosen *in* state and $|\text{out}\rangle$ a chosen *out* state. We can now use the orthonormal basis of $\mathcal{H}_{\text{in}}$ to expand the $|\text{out}\rangle$ state. This expansion defines then a linear operator S bringing $|i\rangle$ into $|f\rangle$

$$|\text{out}\rangle = S|\text{in}\rangle, \qquad |\text{out}\rangle = \sum_{\alpha} \langle\alpha|S|\text{in}\rangle|\alpha\rangle. \tag{2.2}$$

Then $|\langle\alpha|S|\beta\rangle|^2$ gives probability that from $|\beta\rangle$ due to the scattering process we get $|\alpha\rangle$.

Note that this definition of the S-matrix does not rely on the Lagrangean (or Hamiltonian) describing the theory. We are guaranteed that the S-matrix exists solely from the principles of QM. The remaining problem is how to compute the S-matrix. The standard way is by falling back to the Lagrangean (or Hamiltonian) formalism. Here we will explore an alternative to that. We start by putting some constraints on this freshly defined S-matrix.

2.1.3 Conservation of probability

We take the initial state to be normalised ($|\text{in}\rangle, |\alpha\rangle \in \mathcal{H}_{\text{in}}$)

$$|\text{in}\rangle = \sum_{\alpha} a_\alpha |\alpha\rangle, \qquad \sum_{\alpha} |a_\alpha|^2 = 1. \tag{2.3}$$

The probability that $|\text{in}\rangle$ evolves into any final states is equal to 1, therefore

$$1 = \sum_{\beta} |\langle\beta|S|\text{in}\rangle|^2 = \sum_{\beta} \langle\text{in}|S^\dagger|\beta\rangle\langle\beta|S|\text{in}\rangle = \langle\text{in}|S^\dagger S|\text{in}\rangle = \sum_{\alpha,\beta} a^*_\alpha a_\beta \langle\alpha|S^\dagger S|\beta\rangle. \tag{2.4}$$

[1] These equations are schematic, for example, if the labelling is continuous the normalisation is with respect to the Dirac δ-function rather than Kronecker δ. The summation contains then also an integration

The coefficients a_α are arbitrary which implies that $S^\dagger S = 1$. We could redo this computation for a fixed (normalized) *out* state and summing over all the initial states. We would then find $SS^\dagger = 1$. Therefore, the S-matrix is a unitary operator.

2.1.4 Lorentz invariance

We assume now that the special relativity holds, that is there is a relativistic invariance: physical observables of the theory are independent of the observer's reference frame. An example of an invariant is a probability of the given outcome of the scattering process $|\langle\alpha|S|\beta\rangle|^2$. Let L be an arbitrary proper Lorentz transformation relating reference frames of two observers. Relativistic invariance implies that

$$|\langle\alpha'|S|\beta'\rangle|^2 = |\langle\alpha|S|\beta\rangle|^2, \tag{2.5}$$

with $|\alpha'\rangle = L|\alpha\rangle$ and $|\beta'\rangle = L|\beta\rangle$. Recall now, that there is an intrinsic arbitrariness of phases in QM. Therefore, we can impose a stronger condition

$$\langle\alpha'|S|\beta'\rangle = \langle\alpha|S|\beta\rangle, \tag{2.6}$$

which leads to the following relation

$$0 = \langle\alpha'|S|\beta'\rangle - \langle\alpha|S|\beta\rangle = \langle\alpha|L^{-1}SL - S|\beta\rangle. \tag{2.7}$$

This relation holds for any eigenstates α and β and hence $L^{-1}SL = S$, that is, the S-matrix is itself a relativistically invariant object. This implies that the S-matrix can depend only on relativistically invariant combinations of initial and final momenta (to be precise, 2-momenta (ω, k)).

Remark 4: Lorentz transformations in $(1+1)$d

Lorentz transformations are the one that leave invariant the metric. Recall, that in our notation, the metric is

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

Let L^μ_ν be a Lorentz transformation. Then

$$L^\mu_\rho L^\nu_\sigma \, \eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{2.9}$$

Consider an infinitesimal transformation, from which finite transformation can be built. We write $L^\mu_\nu \approx \delta^\mu_\nu + \omega^\mu_\nu$. From the above condition,

$$\eta_{\rho\sigma} = (\delta^\mu_\rho + \omega^\mu_\rho)(\delta^\nu_\sigma + \omega^\nu_\sigma)\,\eta_{\mu\nu} = \eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2). \tag{2.10}$$

This shows, that ω is an antisymmetric tensor in 2-dimensions. Any such tensor can be written as

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}, \tag{2.11}$$

where ϵ is a small real number parametrising the infinitesimal transformation. Using the metric tensor we find

$$\omega^\mu_\nu = \eta^{\mu\rho}\omega_{\rho\nu} = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \tag{2.12}$$

which shows that under the infinitesimal transformation $\mathbf{x}$ transforms as

$$x^\mu \to (\delta^\mu_\nu + \omega^\mu_\nu)\, x^\nu = x^\mu + \omega^\mu_\nu x^\nu = (t, x) + \epsilon(x, t) = (t + \epsilon x, x + \epsilon t). \tag{2.13}$$

This is a boost. We are in 1 spatial dimension so there are no rotations and the only Lorentz transformation is the boost.

It is a standard practice to separate from the S-matrix the "nothing happens" part describing the process in which the initial and final states are identical. This part of the S-matrix corresponds to evolution in a free theory. We write

$$S_{fi} = \delta_{fi} + i(2\pi)^2 \delta(\mathbf{k}_f - \mathbf{k}_i) T_{fi}. \tag{2.14}$$

where $S_{fi} = \langle \text{out}|S|\text{in}\rangle$ and T_{fi} are the scattering amplitudes. Writing the scattering amplitude we extracted the conservation law of momentum and energy. The relativistic invariance implies now that T_{fi} must be functions of relativistically invariant combinations of energy and momenta of incoming and outgoing particles.

Exercise 17: From the S-matrix formalism to the experiments

In this exercise we will relate the S-matrix to the basic experiments of hep: the decay process and the scattering of two particles.

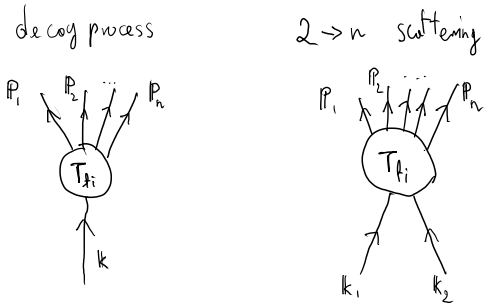


Figure 2.2: An illustration of the decay process (left figure) and scattering process of 2 particles (right figure).

- First we need to understand how to compute probabilities with the S-matrix. We are interested in *non-diagonal* scattering, the initial and final sets of particles are different. The S-matrix is then

$$S_{fi} = i(2\pi)^2 \delta(\mathbf{k}_f - \mathbf{k}_i) T_{fi}. \tag{2.15}$$

  Compute $|S_{fi}|^2$ by representing (one of the two) Dirac δ-functions as a Fourier transform of 1 over some large system of length L and time t. You should find that the transition probability per unit length and unit time is

$$P_{fi} = \frac{|S_{fi}|^2}{Lt} = (2\pi)^2 \delta(\mathbf{k}_f - \mathbf{k}_i) |T_{fi}|^2. \tag{2.16}$$

- We should now understand the density of states, namely given a momentum interval $(p, p + \mathrm{d}p)$ how many states are within such interval. The states are, by now nor-

malized to the δ-function (see e.g. (2.2)) involving the the 2-momentum. This is a relativistically invariant normalization [a]

$$\langle \mathbf{p}_1 | \mathbf{p}_2 \rangle = 2\pi \delta(\mathbf{p}_1 - \mathbf{p}_2). \tag{2.17}$$

Argue why this expression is relativistically invariant. This normalization is compatible with the following resolution of the identity

$$\int \frac{\mathrm{d}\mathbf{p}}{2\pi} |\mathbf{p}\rangle\langle\mathbf{p}| = 1. \tag{2.18}$$

- In scattering experiments the particles are on-shell and momentum and energy are not independent. Therefore, we can label the states just by the momentum $|p\rangle = |\mathbf{p}\rangle$ with $\mathbf{p}$ on-shell. Still we would like our expressions to be relativistically invariant. This can achieved by integrating in the resolution of the identity on the mass shell (the mass shell condition is relativistically invariant)

$$1 = \int \frac{\mathrm{d}\mathbf{p}}{2\pi} |\mathbf{p}\rangle\langle\mathbf{p}| \times \delta(\mathbf{p}^2 - m^2) = \int \frac{\mathrm{d}p}{4\pi E_p} |p\rangle\langle p|. \tag{2.19}$$

This gives the following normalization

$$\langle p_1 | p_2 \rangle = 4\pi E_{p_1} \delta(p_1 - p_2). \tag{2.20}$$

Therefore the density of states of on-shell particles with momentum in the interval $(p, p + \mathrm{d}p)$ is

$$\frac{dp}{4\pi E_p}. \tag{2.21}$$

We can now write down a probability of a decay of a particle of momentum $\mathbf{k}$ into an n-particle state with particles of momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ by multiplying the probability of such process by the density of final states. We find

$$d\Gamma = P_{\mathbf{k} \to \mathbf{p}_1 + \cdots + \mathbf{p}_n} \times \prod_{j=1}^{n} \frac{dp_i}{4\pi E_i}. \tag{2.22}$$

In a similar way, the probability that scattering of two particles of momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ results in n particles of momemta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ is

$$dP = P_{\mathbf{k}_1 + \mathbf{k}_2 \to \mathbf{p}_1 + \cdots + \mathbf{p}_n} \times \prod_{j=1}^{n} \frac{dp_i}{4\pi E_i}. \tag{2.23}$$

What are the units of $d\Gamma$ and dP?

[a] Factor 2π is a convention.

2.1.5 Crossing invariance

Recall the LSZ formula for the S-matrix in terms of the correlation functions

$$T_{(\mathbf{k}_1, \ldots, \mathbf{k}_m | \mathbf{p}_1, \ldots, \mathbf{p}_n)} = \frac{\langle \phi(\mathbf{k}_1) \ldots \phi(\mathbf{k}_m) \phi(-\mathbf{p}_1) \ldots \phi(-\mathbf{p}_n) \rangle}{\prod_{j=1}^{m} \Delta(\mathbf{k}_j) \prod_{j=1}^{n} \Delta(\mathbf{p}_j)}, \tag{2.24}$$

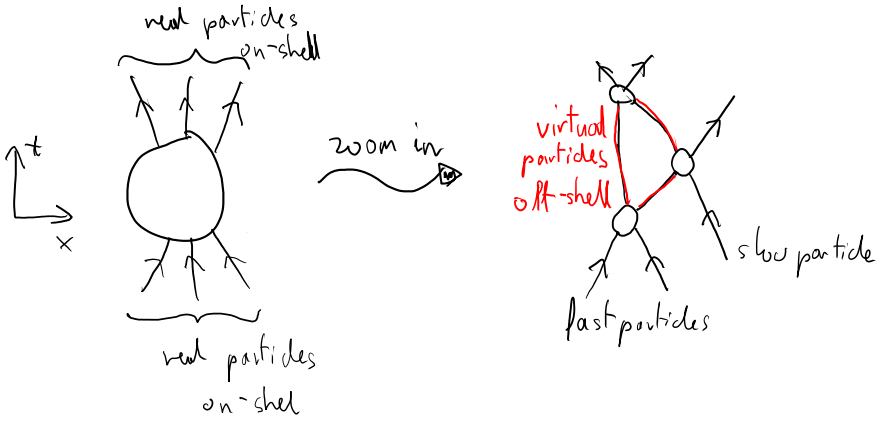


Figure 2.3: ...

with $S = 1 + T$. As we noted before, one can cross a particle from an *in* to an *out* state by negating it momentum. Additionally the particles changes to its antiparticle [2]. That is we have the following identity

$$T_{(\mathbf{k}_1,\ldots,\mathbf{k}_m|\mathbf{p}_1,\ldots,\mathbf{p}_n)} = T_{(\mathbf{k}_2,\ldots,\mathbf{k}_m|-\bar{\mathbf{k}}_1,\mathbf{p}_1,\ldots,\mathbf{p}_n)}. \tag{2.25}$$

where with bar we denote an antiparticle. This relation can be understood intuitively in the following way. Let us "subtract" a particle from the both sides of the scattering process. Therefore the particle vanishes from the *in* state and we get a "-" particle in the "out" state. This is an antiparticle. Why the negative momentum? This follows from the conservation of the 2-momentum. Initially we have: $\mathbf{k}_1 + \cdots + \mathbf{k}_m = \mathbf{p}_1 + \cdots + \mathbf{p}_m$. After the crossing, the particle appears on the right hand side thus it's momentum is negative: $\mathbf{k}_2 + \cdots + \mathbf{k}_m = -\mathbf{k}_1 + \mathbf{p}_1 + \cdots + \mathbf{p}_m$.
We assume the crossing invariance is a fundamental property of the S-matrix.

2.1.6 Causality

Causality means that things from the future do not influence the past. This is certainly important condition and it would be nice to pin-down how it restricts the structure of the S-matrix. However this turns out to be quite difficult. The first problem, is that there is no time in the S-matrix: we treat the S-matrix as a block-box that particles enter and then leave an we don't have an easy access to what happens inside.

Anyway, imagine that we try to look into the internals of the S-matrix. For example consider a $3 \to 3$ scattering process, see fig. 2.3, in which there are two particles that approach each other quickly and a third, slower, particle. It is then reasonable to expect that the scattering occurs in the sequence and that the causal structure leads to factorization of the scattering into 3 consecutive processes[3] Note however, that the scatterings, in the zoomed-in picture, involve also virtual particles. To account for such processes we would need to know the scattering matrix for the off-shell particles. But we don't know it. All we know is the S-matrix when in and out states consists of on-shell particles. Therefore to discuss causality we need to extend the definition of the S-matrix to the off-shell momenta.

[2] In Section 3 we were analyzing real scalar field theory in which particles are their own antiparticles and therefore we haven't seen this phenomena in full generality

[3] There might be more virtual particles and more scattering inside, what we drawn is the simplest diagram contributing to the $3 \to 3$ process.

It will turn out that going from on-shell momenta to off-shell corresponds to extending the S-matrix from a function of real arguments to complex ones. The causal structure of the S-matrix is then hidden in their analytic properties. The following exercise shows how causality is related to analyticity for a simple Green's function

Exercise 18: Causality and analyticity of the Green's functions

This problem is from "Statistical Field Theory" book by G. Mussardo.

Consider a linear system in which the output $b(t)$ depends on the input $a(t)$ as

$$b(t) = \int_{-\infty}^{t} \mathrm{d}t'\, G(t-t')a(t').$$

If the system is causal, the Green's function $G(t-t')$ vanishes when $t < t'$. Let

$$\hat{G}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, e^{i\omega\tau} G(\tau) = \int_{0}^{\infty} \mathrm{d}\tau e^{i\omega\tau} G(\tau),$$

be its Fourier transform. If $a(t)$ and $b(t)$ are both real, also $G(\tau)$ is a real function and

$$\hat{G}^*(\omega) = \hat{G}(-\omega^*).$$

- Show that, if $G(\tau)$ is a square integrable function, then $\hat{G}(\omega)$ is an analytic function in the upper half-plane $\mathrm{Im}\,\omega > 0$. This implies that $\hat{G}(\omega)$, for real ω, is a function obtained as a boundary value of an analytic function.
- Letting $\hat{G}(\omega) = \hat{G}_1(\omega) + i\hat{G}_2(\omega)$, use the Cauchy theorem to prove that these functions are related one to the other by the dispersion relations

$$\hat{G}_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\nu \frac{\hat{G}_2(\nu)}{\nu - \omega},$$
$$\hat{G}_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\nu \frac{\hat{G}_1(\nu)}{\nu - \omega},$$

  These are Kramers-Kronig relations.

If you made a serious attempt on solving this exercise and you can't solve it, have a look here: `https://en.wikipedia.org/wiki/Kramers%E2%80%93Kronig_relations`.

2.1.7 Analyticity

Finally let us discuss the analyticity condition. Once we analytically continue the S-matrix to be a function of complex variables we will enter the realm of complex analysis. Complex functions are to certain degree characterised by their non-analyticities like poles and branch-cuts. As we shall see, the existence of such singularities can be related to some physical phenomena. Therefore, our guiding principle will be that *any* singularity in the (analytically continued) S-matrix should have a physical explanation.

In the next section we will explore in depth the two-body scattering processes and the ideas presented so far will become more explicit. Before that, we recall some relevant aspects of complex analysis.

Remark 5: Complex analysis

We will deal with complex functions of a single complex variable z. Let us recall some terminology

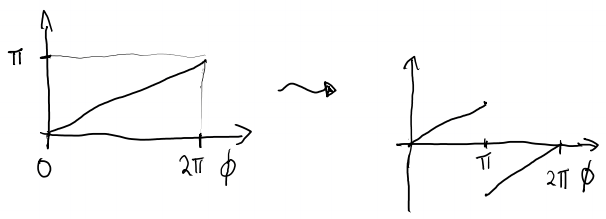
- *analytic function* - function that can locally be represented by a convergent Taylor series,
- *holomorphic function* - function that is locally differentiable,
- *meromorphic function* - function that is a ratio of two holomorphic functions, thus having poles at the zeroes of the denominator.

One of the main results of complex analysis is that holomorphic functions are analytic. The other important result is the base of the analytic continuation. Namely, when two holomorphic functions are equal in a neighbourhood of some point, they are equal everywhere where their domains intersect. This allows to extend (continue) the domain of a function in a unique way. You can find a nice example of analytic continuation on Wikipedia: `https://en.wikipedia.org/wiki/Analytic_continuation#Worked_example`.

Remark 6: Complex square root and branch cuts

We consider a complex square root function $f(z) = \sqrt{z}$. In polar coordinates $z = re^{i\phi+2\pi k}$ where we used that in polar coordinates point z have many representations labelled by integer k. Now, we might be tempted to write $f(z) = \sqrt{r}e^{i\phi/2+i\pi k}$. However this expression on its own shows that value of the square root depends on which (equivalent) representation of point z we take. In fact, we obtain two square root functions: $f_0(z) = \sqrt{r}e^{i\phi/2}$ and $f_1(z) = \sqrt{r}e^{i\phi/2+i\pi}$. These are called two branches of the complex square root.

Imagine now that we take a point z and vary its angle ϕ continuously. How does the value of the square root changes? If we make a full circle it will acquire an extra phase π which means that by going around we moved from one branch to the other. Making yet another circle we fall back to the branch hat we started with. Along the way the value of the function was changing continuously. Therefore in order to have a single square root with the same values for ϕ and $\phi + 2\pi$ we have to allow for the discontinuity. This discontinuity is called a branch cut.



Traditionally we choose this discontinuity to be along the negative part of the real axis ($\phi = \pi$) but it could be along any ray originating at the origin. We define the complex

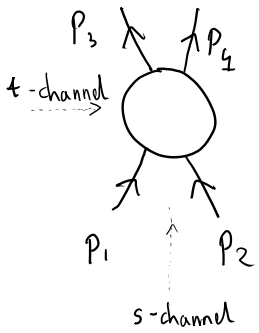


Figure 2.4: The two particle scattering process in the s- and t-channels.

square root to be

$$\sqrt{z} = \sqrt{r}e^{i\phi/2}, \qquad -\pi < \phi \leq \pi, \tag{2.26}$$

which indeed has now a discontinuity along the negative part of the real axis

$$\lim_{\phi\to\pi_-} \sqrt{z} = i\sqrt{r}, \qquad \lim_{\phi\to-\pi_+} \sqrt{z} = -i\sqrt{r} \tag{2.27}$$

The square root function just defined is holomorphic on the complex plane without the negative reals. This function is also *real analytic*: $f(z^*) = f(z)^*$. For a real analytic function the branch cut along the real line can be detected by looking at its imaginary part since $2\,\mathrm{Im} f(z) = f(z) - f(z)^* = f(z) - f(z^*)$. We then find

$$\lim_{\phi\to\pi_-} 2\,\mathrm{Im} f(z) = \lim_{\phi\to\pi_-} f(z) - \lim_{\phi\to\pi_-} f(z^*). = \lim_{\phi\to\pi_-} f(z) - \lim_{\phi\to\pi_+} f(z). \tag{2.28}$$

2.1.8 Two-body scattering matrix

We will analyse now in depth the 2-body scattering matrix. As we have discussed before, the S-matrix should be a function of relativistic invariants of momenta of incoming and outgoing particles. In the $2 \to 2$ scattering process it is customary to introduce Mandelstam variable

$$s = (\mathbf{p}_1 + \mathbf{p}_2)^2, \qquad t = (\mathbf{p}_1 - \mathbf{p}_3)^2, \qquad u = (\mathbf{p}_1 - \mathbf{p}_4)^2, \tag{2.29}$$

which are related through

$$s + t + u = 4m^2. \tag{2.30}$$

In $1+1$ dimensions $u = 0$ (why?) and therefore there is only one independent Mandelstam variable.

With the help of the crossing symmetry we can interpret the scattering process in 2 equivalent ways (see fig. 2.4)

- scattering in the s-channel (incoming particles p_1 and p_2)
- scattering in the t-channel (incoming particle p_1 and anti-particle p_3, out-going particle p_4 and antiparticle p_2),

The two processes have the same amplitude $S(s)$.

Exercise 19: Mandelstam variables

1. Show relation (2.30) employing conservations laws and on-shell relations (you can prove it in arbitrary number of spatial dimensions!).
2. Show that in both channels the corresponding Mandelstam variable is the total energy in the center of mass frame.
3. Find the range of values of s and t which correspond to physical process, i.e. this for which energies of particles are positive and momenta are real numbers, in the s-channel,

Analytic continuation

We consider $T(s)$ as describing the s-channel scattering. This defines the function for $s \geq 4m^2$. We will now analytically continue this function to other values of s. We start with extending range of s to complex numbers together with the prescription, that the physical amplitude should arise from falling back on the real axis from the upper half-plane. That is

$$T_{\text{phys}}(s) = \lim_{\epsilon \to 0_+} T(s + i\epsilon), \qquad \text{for} \quad s \geq 4m^2. \tag{2.31}$$

Note that $s + i\epsilon$ is the same regularization as $i\epsilon$ appearing in the propagator. Let us now analyze $T(s)$ as a function of complex s .

Production thresholds and branch cuts

We will analyze now the analytic structure of the S-matrix related to the production thresholds of particles. Consider the unitarity condition $SS^\dagger = 1$. In terms of the matrix T we find

$$2\text{Im}T_{fi} = i(2\pi)^2\delta(\mathbf{p}_f - \mathbf{p}_i)\sum_n T_{fn}T^*_{in}, \qquad S_{fi} = 1 + i(2\pi)^2\delta(\mathbf{p}_f - \mathbf{p}_i)T_{fi}. \tag{2.32}$$

This formula is correct for any initial and final state, but let us focus for now on the initial and final states consisting of two particles. The discontinuity in the imaginary part of the amplitude signals a branch cut. The right hand side gives a magnitude of this discontinuity. Let's analyse it more carefully. The variable s corresponds to the total energy in the s-channel and $s \geq 4m^2$ for the two particle state to exist in the asymptotic states. Note that when s exceeds $9m^2$ the asymptotic state can now contain 3 particles. It means that when $s > 9m^2$ then the 3 particle states starts to contribute to the right hand side. This appears as a new branch cut and the situation repeats whenever s exceeds the next particle production threshold.

Bound states and poles

Let us consider now an unphysical value of $s < 4m^2$. We assume that the equation (2.32) still holds and as T is an analytic function it can be continued in this region. This will now correspond to a virtual process. Imagine now a theory with a bound state of mass $m_b < 2m$ of the initial particles. Therefore, in the scattering matrix we

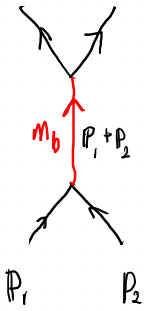


Figure 2.5 Formation of a bound state of

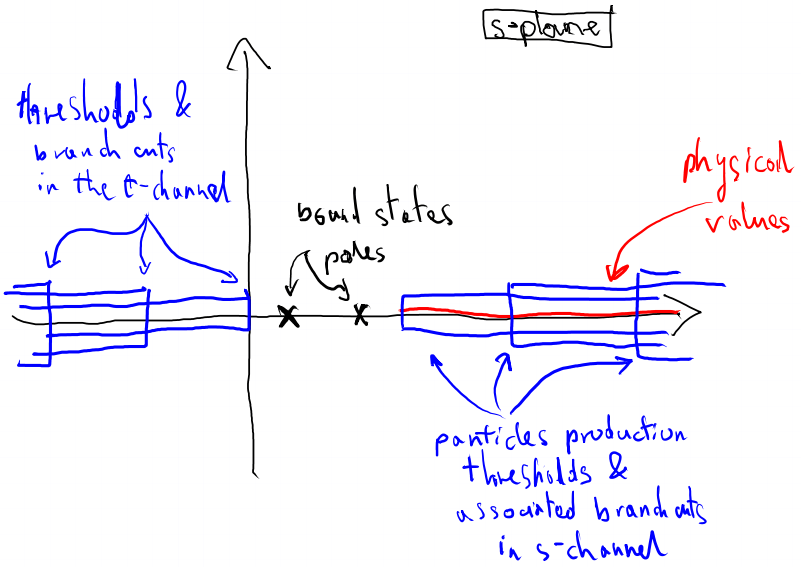


Figure 2.6: Analytic structure of the elastic two particle S-matrix in the s-plane. We have denoted the threshold values in the s-channel, threshold values in the t-channel and similarly for the bound states. The red line corresponds to the region of the physical scattering amplitude. Reproduced from G. Mussardo: "Statistical Field Theory"

have a process in which the initial particles form a bound state which then propagates, see fig. 2.5. The propagator of the bound state would be

$$\frac{-i}{(\mathbf{p}_1+\mathbf{p}_2)^2 - m_b^2 + i\epsilon}, \tag{2.33}$$

and would diverge when $s = (\mathbf{p}_1+\mathbf{p}_2)^2 = m_b^2$. This leads then to a pole in the S-matrix.

Crossing symmetry

The analysis performed so far in the s-channel can now be repeated in the t-channel. That is, we fix now s and consider S-matrix a function of t. Given the crossing invariance we would again found a series of branch cuts at $t = 4m^2, 9m^2, \ldots$ and poles corresponding to bound states at $t = m_b^2$. Using relation $s+t = 4m^2$ we can translate these results back to the s-plane. We find that the t-channel branch cuts correspond now to $s = 0, -5m^2, -12m^2, \ldots$ and the bound state poles appears for $s = 4m^2 - 3m_b^2$.

Figure 2.6 summarizes our findings. We will assume that there are no another non-analyticities of the 2-body scattering matrix. In the next section we will further develop the theory going in two directions: first we show that foe a class of theories we can built S-matrix of an arbitrary process from the knowledge of the 2-body S-matrix. Second, we will further explore the 2-body S-matrix and show how to further restrict its structure.

2.2 Purely elastic scattering matrices

We will now combine the S-matrix theory developed in the previous section with integrability. This will result in a powerful method of bootstrapping the scattering theory given some minimal information of its particle content. At the end of this section this will allow us to get the full S-matrix of the Sinh-Gordon theory. We start with two important remarks

Remark 7: Rapidity variable

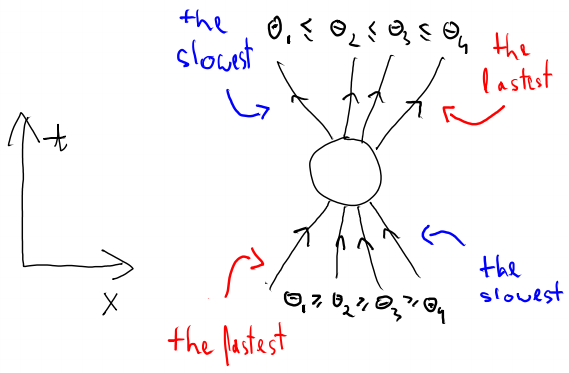
In $(1+1)$d It is convenient to parametrise the particle's momentum with rapidity variable θ defined through

$$\omega(\theta) = m\cosh\theta, \qquad p(\theta) = m\sinh(\theta). \tag{2.34}$$

With this parametrisation the on-shell condition: $\mathbf{p}^2 = m\cosh^2\theta - m\sinh^2\theta = m^2$ is automatically satisfied for real θ. Another convenient feature is that shifting θ by $i\pi$ we get a parametrization for an antiparticle. Yet another convenient accident is that Lorentz boost act simply as a shift of θ by a real parameter α. Life is simpler with rapidities!

Remark 8: Asymptotic states

So far we have been a bit vague with defining the asymptotic states (the *in* and *out* states). We simply stated that these are states in the far past or far future where particles are safely separated from each other. Given short range interactions in the theory we can treat them as free particles. There is small nuance to that story in $(1+1)$d.



Let consider an *in* state and imagine that we run the time evolution backwards. As we keep on doing that we do not want the particles to ever meet again. On the opposite, we want them to keep on getting more and more separated. But in one spatial dimension it is only possible if the fastest particle is the furthest to the left. The next to the fastest particle has to be to the right of it, and so on, till we get to the slowest particle which is furthest to the right. In other words: in asymptotic states particle are naturally ordered. The proper *in* state is therefore

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)\rangle_{\text{in}}, \qquad \theta_1 \geq \theta_2 \geq \dots \geq \theta_n, \tag{2.35}$$

whereas the proper *out* state is

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)\rangle_{\text{out}}, \qquad \theta_1 \le \theta_2 \le \dots \le \theta_n, \tag{2.36}$$

Here $A_{a_j}(\theta_j)$ denotes a particle of type a_j with rapidity θ_j. So far we have considered theories with a single particle type. From now on we give ourselves more freedom. For the completeness, the states are normalized as follows

$$\langle A_i(\theta_i)|A_j(\theta_j)\rangle = 2\pi\delta_{ij}\delta(\theta_i - \theta_j). \tag{2.37}$$

2.2.1 Conserved charges

In quantum theory conserved charges are operators whose expectation values are constant: do not evolve with time. If $\hat{Q}$ is such operator, then from the Heisenberg equation of motion we know that

$$[H, \hat{Q}] = 0, \tag{2.38}$$

where H is the Hamiltonian of a theory. An example of a conserved charge is the operator of the total momentum P, another conserved charge is the Hamiltonian itself. From the operatorial perspective, these two operators are rather special. They are both expressible as integrals over the system length of a *local density*. We shall call them *local conserved charges*. We will be interested in theories with an infinite (but countable) set of local conserved charges. QFT's with such feature are called *Integrable Quantum Field Theories*.

We can construct local conserved charge if we have a local charge density and local current that are related by the local conservation law: $\partial_t\rho = \partial_x j$. This implies that $Q = \int \mathrm{d}x\, \rho$ is conserved in time[4].

Let us consider now the light-cone components of the Hamiltonian and momentum

$$Q_1 = H + P, \qquad \bar{Q}_1 = H - P. \tag{2.39}$$

We know their action on the asymptotic states. For a single particle state

$$Q_1|A_a(\theta)\rangle = \frac{m_a}{2}e^{\theta}\,|A_a(\theta)\rangle, \quad \bar{Q}_1|A_a(\theta)\rangle = \frac{m_a}{2}e^{-\theta}\,|A_a(\theta)\rangle, \tag{2.40}$$

and extends to multiparticle states through additivity. Note that under the Lorentz boost, the rapidity is additive and hence it factorizes

$$e^{\theta} \to e^{\theta+\alpha} = e^{\alpha}e^{\theta} \tag{2.41}$$

Let us assume now that there exists an infinite family of local conserved charges Q_s and $\bar{Q}_s$ (with $s \in \mathbf{N}_+$) that generalize the light-cone components of the Hamiltonian and momentum such that

$$Q_s|A_a(\theta)\rangle = \chi_s(a)e^{s\theta}\,|A_a(\theta)\rangle, \quad \bar{Q}_s|A_a(\theta)\rangle = \bar{\chi}_s(a)e^{-s\theta}\,|A_a(\theta)\rangle, \qquad s \ge 1. \tag{2.42}$$

Functions $\chi_s(a)$ and $\bar{\chi}_s(a)$ are the single particle eigenvalues. Again, this expressions can be generalized to multi-particle states with additivity. Under the Lorentz boost the expectation values of these charges transform as s copies of Q_1 or $\bar{Q}_1$, that is

$$e^{s\theta} \to e^{s(\theta+\alpha)} = e^{s\alpha}e^{s\theta}. \tag{2.43}$$

[4]Up to boundary terms that can be neglected by either assuming that currents vanish at $\pm\infty$ or by imposing translational invariance with the help of periodic boundary conditions

The index s classifies the conserved charges from the perspective of Lorentz transformation.

We assumed that these are local conserved charges and hence there exist corresponding conserved densities and charges. These expressions are theory dependent and we don't specify them here [5]. Moreover, what we assume is that there is an infinite set of charges, but not that, for a given theory, there always exists conserved charge with every $s \in \mathbf{N}_+$. In fact, we will see later, that the set of integers corresponding to conserved charges is a good characteristic of a theory.

2.2.2 Scattering in the presence of conserved charges

We will now prove that the existence of an infinite set of local conserved charges has the following consequences on the scattering theory

1. the number of particles with mass m_a remains the same
2. the sets of final and initial momenta are the same
3. the scattering amplitude for a process with n-particles can be completely factorized in terms of the $n(n-1)/2$ elastic scattering processes.

Let us start with the first two statements. We consider action of the conserved charges on a multi-particle state

$$Q_s|A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\rangle = \sum_{j=1}^{n} \chi_s(a_j) e^{s\theta_j} |A_{a_1}(\theta_1)\dots A_{a_n}(\theta_n)\rangle. \tag{2.44}$$

Since $dQ_s/dt = 0$ the expectation values on the initial and final states must be the same, thus

$$\sum_{i\in \text{in}} \chi_s(a_i) e^{s\theta_i} = \sum_{j\in \text{out}} \chi_s(a_j) e^{s\theta_j}, \tag{2.45}$$

for an infinite set of s. The only solution, apart from permutation of particles of the same type, is when the final and initial sets of rapidities are equal and sets of particles are the same. Therefore there are no (real) annihilation and production processes.

We now move to the third point, namely the factorization of the scattering process. The main idea is the following. Recall that the momentum operator is a generator of the translation: $e^{iPa}\psi(x) = \psi(x+a)$. The higher conserved charges generalize this property in a way that the resulting shift depends on the particle momentum. This has far reaching consequences that we will now illustrate.

Exercise 20: Moving wavepackets around

1. Let $|p\rangle$ denote a particle with momentum p (an eigenstate of the momentum operator). The associated wavefunction is $\psi(x) = \langle x|p\rangle = e^{ipx}$. Check that $e^{iaP}\psi(x) = \psi(x+a)$.
2. Consider now a spatial component $\tilde{Q}_s = (Q_s - \bar{Q}_s)/2$ of Q_s and $\bar{Q}_s$. What is its action on $|p\rangle$? Show that action of $e^{ia\tilde{Q}_s}$ on $\psi(x)$ yields $\psi(x + a f_s(p))$, where $f_s(p)$ is s dependent function of momentum p.
3. So far we were shifting a plane wave. Consider now a wavepacket centered in space

[5] In fact, that might be very difficult to specify!

around x_0 and with momentum centered around p_0. Its (not normalized) wavefunction is

$$\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}p e^{-\alpha(p-p_0)^2} e^{ip(x-x_0)}. \tag{2.46}$$

Show that $e^{ia\tilde{Q}_s}\psi(x)$ gives a wavepacket centered around a new position. What is this new position? Analyze how the answer depends on s.

Consider a scattering process in which 3 particles collide simultaneously. The conserved charges commute with the Hamiltonian and thus e^{iaQ_s} is a symmetry operator. Its action leads to physically equivalent situations. On the other hand action of e^{iaQ_s} shifts positions of particles in a momentum dependent way. It means that by its action the 3 particles don't collide at the same time anymore! Therefore instead of one scattering process we have now 3 scattering processes in a row with the same outcome. This shows that the $3 \to 3$ scattering process is factorizable in three $2 \to 2$ scattering processes, see fig. 2.7.

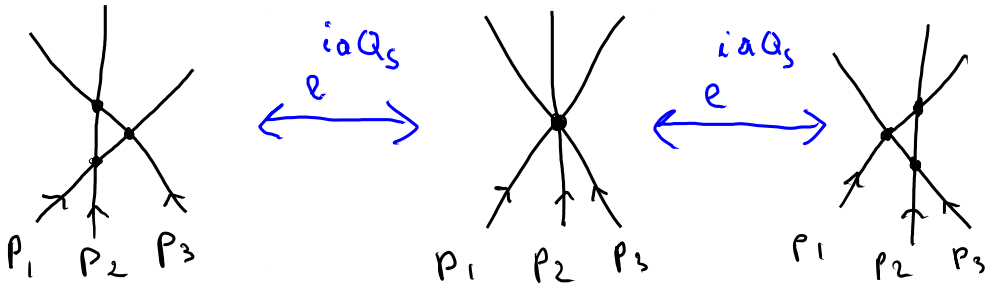


Figure 2.7: The 3 equivalent ways of describing the collision of 3 particles in an integrable field theory. The picture implies two facts: i) the 3-body scattering matrix is factorizable into a product of 2-body S-matrices. ii) the two ways of factorization are equivalent. The latter observation leads to a constrain on the 2-body S-matrix - the Yang-Baxter equation.

Because there are two ways of factorising the scattering of 3 particles, and both ways must be equivalent, we find an equation that the 2-body S-matrix must fulfil

$$S(p_1,p_2)S(p_1,p_3)S(p_2,p_3) = S(p_2,p_3)S(p_1,p_3)S(p_1,p_2). \tag{2.47}$$

This is the Yang-Baxter equation. Note that the S appearing there is in general a matrix. Its indices label different particles' types.

The factorization property naturally extends to scattering processes involving more particles. Moreover, the Yang-Baxter equation is enough to guarantee the equivalence of the factorizations also in these cases. Therefore we have a general statement: every n-body scattering matrix can be factorized into a product of 2-body scattering matrices with 2-body scattering matrices obeying the Yang-Baxter equation. This shows the central role of the 2-body scattering matrix in the IQFT's.

Exercise 21: Factorization of 4 particles scattering

Consider the scattering process of 4 particles. Show that different ways of factorizing the

process can be related assuming the Yang-Baxter relation holds.

In the following we will look again in the 2-body scattering matrix, further analyzing its analytic structure. Before doing so let us conclude the chapter on role of the conservation laws on the scattering theory by citing the Coleman-Mandula theorem.

Exercise 22: Coleman-Mandula theorem

Coleman-Mandula theorem is one of the milestones in our understanding of QFT's. The theorem says that a presence of a single (!) conserved charge (beyond Hamiltonian and momentum) related to a space-time symmetry[a] (hence generating translations) is enough for any theory in $(3+1)$d to be free. Based on our discussion argue against or in favour of the theorem.

[a]The supersymmetry (SUSY) is an exotic space-time symmetry that escapes the faith of Coleman and Mandula.

2.2.3 Two-body S-matrix again

Let us consider again the elastic collision of 2 particles,

$$A_i(\theta_1) + A_j(\theta_2) \longrightarrow A_k(\theta_2) + A_l(\theta_1), \tag{2.48}$$

where i, j, k, l label possible particles' types and θ_i are rapidities. We allow the particles to be of different types and hence have for example different masses. The 2-momenta are then

$$\mathbf{p}_1 = m_i(\cosh\theta_1, \sinh\theta_1), \qquad \mathbf{p}_2 = m_j(\cosh\theta_2, \sinh\theta). \tag{2.49}$$

The Mandelstam variables can be now expressed through a difference of rapidities

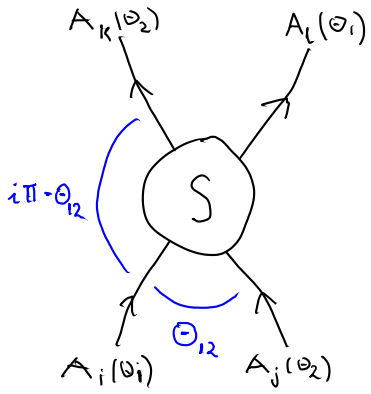


Figure 2.8: ...

$$\begin{aligned} s &= (\mathbf{p}_1 + \mathbf{p}_2)^2 = (m_i\cosh\theta_1 + m_j\cosh\theta_2)^2 - (m_i\sinh\theta_1 + m_j\sinh\theta_2)^2 \\ &= m_i^2 + m_j^2 + 2m_im_j(\cosh\theta_1\cosh\theta_2 - \sinh\theta_1\sinh\theta_2) \\ &= m_i^2 + m_j^2 + 2m_im_j\cosh(\theta_1 - \theta_2), \end{aligned} \tag{2.50}$$

and in a similar way

$$t = (\mathbf{p}_1 - \mathbf{p}_2) = m_i^2 + m_j^2 - 2m_i m_j \cosh(\theta_1 - \theta_2). \tag{2.51}$$

Note that both expressions are relativistically invariant because depend on the difference of rapidities. Lorentz boost shifts rapidities by a constant and therefore has no effect on their difference. Moreover, the variables $s(\theta)$ and $t(\theta)$ are related by a simple transformation of rapidities, namely

$$t(\theta) = s(i\pi - \theta). \tag{2.52}$$

Finally, we note that physical values of s ($s \geq (m_1 + m_2)^2$) correspond to real value of $\theta_1 - \theta_2$.

Let us define now the S-matrix in this language. We define it as the following matrix

$$|A_i(\theta_1)A_j(\theta_j)\rangle = S_{ij}^{kl}(\theta_1 - \theta_2)|A_k(\theta_2)A_l(\theta_1)\rangle, \tag{2.53}$$

with $\theta_1 > \theta_2$. Note that according to the ordering of the rapidities in the asymptotic states, the S-matrix just defined is $|\text{in}\rangle = S|\text{out}\rangle$, hence it's an inverse of our previous definition. We will stick to this new convention from now on.

We will now translate the analytic properties of the S-matrix as a function of s, discussed in the previous section, to analytic properties of S-matrix as a function of θ. We will see that the resulting structure is much simpler. For that we need a map from s to θ:

$$\theta = \log\left[\frac{s - m_i^2 - m_j^2 + \sqrt{(s-(m_i+m_j)^2)(s-m_i-m_j)^2}}{2m_i m_j}\right]. \tag{2.54}$$

The effect of this map on the analytic structure of the S-matrix is visualized in fig. 2.9. Few

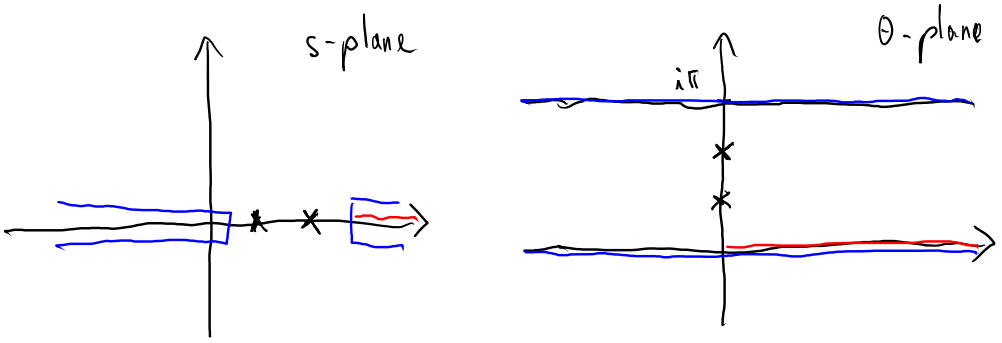


Figure 2.9: The comparison of the analytic structure of the 2-body scattering matrix in IQFT in the s-plane and in the θ-plane.

comments are in place.

First, in the previous section we have seen that there is a series of branch cuts originating at the production thresholds. In IQFT there are no production processes in the scattering and therefore there is only a single branch cut originating at the 2-particle threshold.

Second, we see that the whole s-plane is mapped to a strip in θ with $0 < \text{Im}\,\theta < \pi$. Inside this strip (physical strip) the S-matrix has only isolated poles due to the bound states. All those poles lie on the imaginary axis. Within the physical strip, $S(\theta)$ is a meromorphic function.

Third, let us express the unitarity and crossing invariance conditions in terms of $S(\theta)$. We have

$$\text{unitarity}: \quad S_{ij}^{nm}(\theta)S_{nm}^{kl}(-\theta) = \delta_i^k \delta_j^l. \tag{2.55}$$

$$\text{crossing}: \quad S_{ij}^{kl}(\theta) = S_{i\bar{l}}^{k\bar{j}}(i\pi - \theta). \tag{2.56}$$

Remark 9: CPT symmetries

There are 3 discrete transformations that play an important role in physics:

1. charge conjugation - changes particles into antiparticles,
2. parity transformation - flips the direction of x,
3. time reversal - flips the direction of time.

If any of these transformations is a symmetry of the theory, this puts additional restrictions on the S-matrix. We have

$$\begin{aligned} \mathrm{C}: &\quad S_{ij}^{kl}(\theta) = S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta), \\ \mathrm{P}: &\quad S_{ij}^{kl}(\theta) = S_{ji}^{kl}(\theta), \\ \mathrm{T}: &\quad S_{ij}^{kl}(\theta) = S_{lk}^{ji}(\theta). \end{aligned}$$

2.2.4 The S-matrix of the Sinh-Gordon theory

We have now enough tools to conjecture the S-matrix of the Sinh-Gordon theory. To apply the formalism that we just introduced, we need to have some idea of the particle content of the theory. We can conjecture, that in the Sinh-Gordon theory there is only a single neutral particle and this particle does not form the bound states (like it could in the ϕ^3 theory). So what does this tell us about the S-matrix.

Since there is only one particle type and the particle is its own anti-particle, the scattering matrix is actually just a single function. The unitarity and crossing symmetries imply then

$$S(\theta)S(-\theta) = 1, \qquad S(\theta) = S(i\pi - \theta). \tag{2.57}$$

We are looking for a meromorphic function in the physical strip obeying these two equations. Since there are no bound states, the S-matrix shouldn't have poles in the physical strip. However, let us give us a a bit of more freedom and allow also for the poles.

We start by noting two properties of $S(\theta)$ that follow from the unitarity and crossing conditions. First, by combining unitarity and crossing symmetry we can show that $S(\theta) = S(\theta - 2\pi i)$, hence $S(\theta)$ is $2\pi i$ periodic. Second, if we have two function solving the constraint, their product will also be a solution. Therefore, a general solution is of the form

$$S(\theta) = \prod_{\gamma} f_\gamma(\theta), \tag{2.58}$$

where $f_\gamma(\theta)$ are some elementary solutions to the unitarity and crossing relations. The product goes here over some countable set. $f_\gamma(\theta)$ should be an elementary solution meaning that its analytic structure should be as simple as possible.

Recall the Liouville theorem that roughly speaking says that the only functions without poles are constant functions. If we don't want the scattering matrix to be a constant we need to allow for the poles. Given the $2\pi i$ periodicity, there must be a pole lying in the strip $0 < \mathrm{Im}\,\theta < 2\pi$. We have now two options, either the pole is within the physical strip $0 < \mathrm{Im}\,\theta < \pi$ or outside of it. The crossing relation shows that given a pole at θ_0 there is a pole at $i\pi - \theta_0$, the unitarity relation shows that given a pole at θ_0 there is a zero at $-\theta_0 = 2\pi i - \theta_0$ (where we used the periodicity). Fig. 2.10 summarizes this findings.

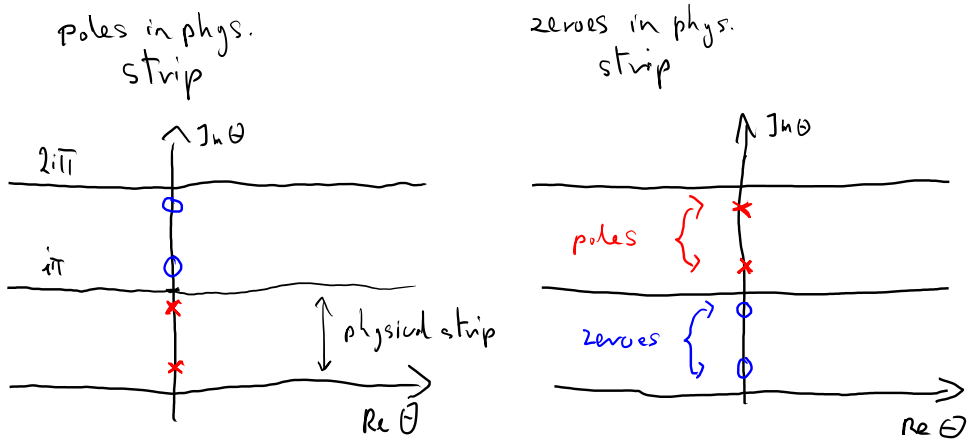


Figure 2.10: The analytic structure of the elementary solution $f_x(\theta)$.

Let us denote the position of one of the zeroes by $i\gamma$. Then an elementary solution to the unitarity and crossing relation having all the desired properties is given by

$$f_\gamma(\theta) = \frac{\sinh\theta - i\sin\gamma}{\sinh\theta + i\sin\gamma}. \tag{2.59}$$

Given the periodicity of trigonometric functions we see that the range of γ can be limited to $0 \leq \gamma \leq 2\pi$. For application to the Sinh-Gordon theory we don't want poles in the physical region, hence $\gamma < \pi$.

Exercise 23: Elementary solution

In this exercise we will derive eq. (2.59). We will do this in two steps. First we will find the function that has right poles and zeros in the strip $0 < \mathrm{Im}\,\theta < 2\pi$. Then we will enforce $2\pi i$ periodicity on this function

1. Construct the simplest function $f_\gamma(\theta)$ that has a zero at $\theta = i\gamma$ and fix the other zeroes and poles from the unitarity and crossing symmetry.

2. Given function $g(\theta)$ we can turn it into a periodic function in a brutal way: simply multiply it by all its values with shifted arguments:

$$g(\theta) \longrightarrow \prod_k g(\theta + 2\pi i k). \tag{2.60}$$

Use such periodization and formula

$$\sinh x = x \prod_{k=1}^{\infty} \left[1 + \left(\frac{x}{k\pi}\right)^2\right] \tag{2.61}$$

to show that $g(\theta) = \theta - i\gamma$ can be turned into

$$\sinh\left(\frac{\theta - i\gamma}{2}\right) \times 2\prod_{k=1}^{\infty}(2\pi k)^2. \tag{2.62}$$

The infinite product in this expression is ill-defined but it will disappear once we take ratio of such functions in the S-matrix.

3. By combining the two points, show that the elementary solution takes the following form

$$f_x(\theta) = \frac{\sinh\left(\frac{\theta - i\gamma}{2}\right)\sinh\left(\frac{\theta - i\pi + i\gamma}{2}\right)}{\sinh\left(\frac{\theta + i\gamma}{2}\right)\sinh\left(\frac{\theta - i\pi - i\gamma}{2}\right)}. \tag{2.63}$$

Finally, use properties of hyperbolic functions to arrive at (2.59).

Let us analyze the result in some detail. Observe that for $\gamma = 0, \pi$ the S-matrix is equal to 1 what corresponds to a free theory. Therefore we can conjecture that parameter γ controls the interactions in the theory. Let us then consider $S(0)$. For $\gamma = 0, \pi$ we find that $S(0) = 1$ whereas for other cases $S(0) = -1$. This is amazingly deep result, let us see why.

Amplitude of the scattering process with $\theta = 0$ corresponds to scattering of two identical particles. Such a scattering can be thought of as interchanging two particles. Therefore we arrive at the conclusion that $S(0)$ determines the statistics of particles, for $S(0) = 1$ particles follow bosonic statistics, for $S(0) = -1$ the fermionic ones. Coming back to our theory, we see that due to interactions the particle's nature changes drastically from bosons of non-interacting theory, to fermions once interaction are turned on. This is a non-perturbative effect.

Still, we can try to study our theory perturbatively by expanding the scattering matrix in small γ. We find then

$$S(\theta) = 1 - \frac{2i\gamma}{\sinh\theta} - \frac{2\gamma^2}{\sinh^2\theta} + \mathcal{O}(\gamma^3). \tag{2.64}$$

This expansion could be compared with the standard QFT computations based on Feynman diagrams. Recall the Sinh-Gordon Lagrangean

$$\mathcal{L}_{SG} = \frac{1}{2}(\partial_t\phi(x,t))^2 - \frac{1}{2}(\partial_x\phi(x,t))^2 - \frac{\mu^2}{g^2}\cosh g\phi(x,t). \tag{2.65}$$

By computing the S-matrix and comparing with (2.64) we would find the relation between γ and g

$$\gamma = \frac{\pi g^2}{8\pi + g^2}. \tag{2.66}$$

We can observe one more feature, namely weak/strong duality. The scattering matrix does not change if $\gamma \to \pi - \gamma$, however coupling constant does change from g to $1/g$. Therefore Sinh-Gordon models with g and $1/g$ describe the same scattering theory. This symmetry is impossible to see from our Lagrangean formulation.

2.2.5 Bound states

We will now display some general features of the S-matrix if the theory supports bound states. That is the interactions in the theory allow for forming bound states. These might be realized in many ways, for example (and schematically)

$$A_1 \times A_1 \to A_1, \qquad A_1 \times A_1 \to A_2, \quad A_1 \times A_2 \to A_3, ... \tag{2.67}$$

We will treat bound state particles on equal footing with asymptotic particles - *nuclear democracy.*

Kinematic constrains

We have seen that bound states in the spectrum correspond to a pair (due to the crossing symmetry) of poles. Let us assume that in the scattering matrix of two particles of types i and j we observe a simple pole for $\theta_{ij} = iu_{ij}^n$, that is, in the vicinity, the S-matrix looks like

$$S(\theta) = i\frac{R}{\theta - iu_{ij}^n}, \tag{2.68}$$

where $R = f_{ij}^n f_{kl}^n$ is related to the on-shell vertex function of the incoming and outgoing particles and the bound state. In the s-channel this pole is for $s = m_n^2$ and

$$m_n^2 = s(iu_{ij}^n) = m_i^2 + m_j^2 + 2m_i m_j \cos u_{ij}^n. \tag{2.69}$$

We could now consider another process in which particles A_i and A_n are in the incoming state. We expect then to see a pole at $s = m_j^2$ and corresponding $\theta = iu_{in}^j$. These are again related

$$m_j^2 = s(iu_{in}^j) = m_i^2 + m_n^2 + 2m_i m_n \cos u_{in}^j. \tag{2.70}$$

By considering the third process in which particles A_j and A_n are in the incoming state we arrive to an analogous equation. From trigonometric arguments we can then conclude that

$$u_{ij}^n + u_{in}^j + u_{jn}^i = 2\pi. \tag{2.71}$$

Remark 10: Bullogh-Dodd model

Let us apply result (2.71) to the case when all the 3 particles are of the same kind. Then the position of the pole is uniquely fixed to $2\pi i/3$. Therefore we can immediately write down an S-matrix compatible with this condition and unitarity and crossing symmetries. In the notation of the previous section, the S-matrix is $f_{4/3\pi}(\theta)$.

This S-matrix corresponds to an integrable extension of the ϕ^3 theory. We have seen that starting with perturbative expansion of ϕ^4 we can construct the integrable Sinh-Gordon theory. In the same way we can start with ϕ^3 theory and by killing production processes with higher interaction terms we will arrive to the Bullogh-Dodd model

$$\mathcal{L}_{\text{BD}} = \frac{1}{2}(\partial\phi)^2 - \frac{\mu^2}{6g^2}\left(e^{2g\phi} + 2e^{-g\phi} - 3\right). \tag{2.72}$$

The Bullogh-Dodd model, just like the Sinh-Gordon model also has a weak/strong duality, and its S-matrix at the self-dual point $g = \sqrt{4\pi}$ is exactly $f_{4/3\pi}(\theta)$. To describe the Bullogh-Dodd model for arbitrary g the S-matrix has to be further extended but it still contains $f_{4/3\pi}(\theta)$ piece.

Bootstrap principle

The integrability of the theory allows also to bootstrap scattering matrices of some processes from scattering matrices of different processes. Let us see this in an example. We consider

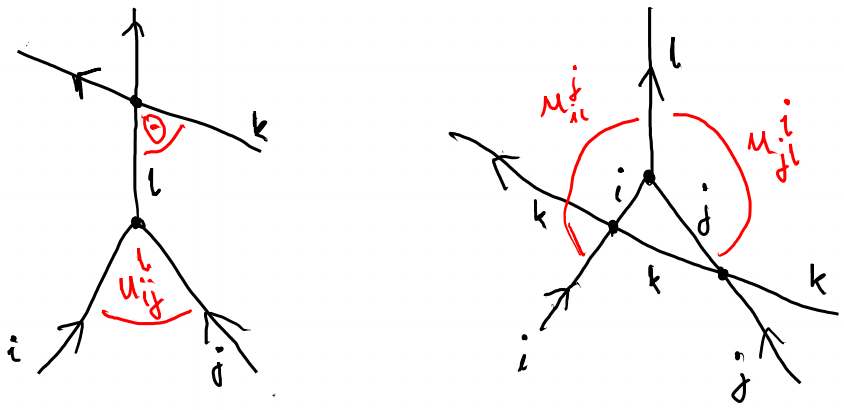


Figure 2.11: The bootstrap principle - scattering with a bound state is equivalent to the scattering with constituents.

theory with a *diagonal scattering* - only transmission, no reflection and particles types do not chage. This assumption is not crucial for the bootstrap principle but simplifies the reasoning. We consider now two scattering processes, see fig. 2.11. Using the integrability of the theory, we can shift the lines, just like we did in showing the factorization property of the 3-particle scattering matrix. Such operation is a symmetry of the theory, hence the amplitudes must be equal

$$S_{lk}(\theta) = S_{jk}(\theta - i\pi + iu^i_{jl})S_{ki}(\theta + i\pi - iu^j_{il}). \tag{2.73}$$

This shows that a scattering matrix with a bound state can be determined from the scattering matrix with its constituents.

2.3 The S-matrix of the Sine-Gordon model

In this lecture we will describe the S-matrix of the Sine-Gordon model with the method that we developed in the previous sections. To apply these methods the theory needs to be integrable and we need to have a basic understanding of the particle content of the theory. Perturbative computations at the end of Section 1.4 suggest the integrability, but what is the particle content?

To get some idea what is the particle content of the Sine-Gordon theory we will use two tools. First, we will study the corresponding classical theory, with an eye on the consequences in quantum theory. Independently of this approach, we will consider an apparently completely different theory, the massive Thirring model. We will see, based on perturbation theory, that both Sine-Gordon theory and massive Thirring model are just two ways of looking at the same theory. Equipped with these knowledge we will attack the question of the S-matrix.

2.3.1 Semiclassical analysis

The action of the Sine-Gordon model in the Minkowski space is

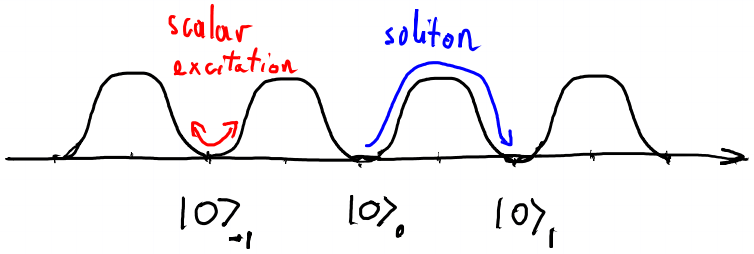
$$S = \int \mathrm{d}x\,\mathrm{d}t\,\left(\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right), \tag{2.74}$$

with the potential

$$V(\phi) = \frac{\mu^2}{g^2}(1-\cos g\phi). \tag{2.75}$$

In $V(\phi)$ we have included, for a convenience, a constant term μ^2/g^2.

The potential $V(\phi)$ has an infinite number of degenerate minima at $\phi = 2\pi n/g$. In a quantum theory each of the minima corresponds to a vacuum state denoted by $|0\rangle_n$. Around each of the minimum the potential behaves like $\mu^2\phi^2$ so μ can be thought of as a mass of the scalar particle created out of the vacuum. Therefore we expect that in the quantum theory there will be scalar massive particles.



Beside these scalar particles there are also excitations that connect two vacua. We label these excitations by two integers (n_1, n_2) such that field configurations reach the corresponding vacua (minima of the potential) for large x

$$\lim_{x\to+\infty}\phi(x) = 2\pi n_1/g, \qquad \lim_{x\to-\infty}\phi(x) = 2\pi n_2/g. \tag{2.76}$$

We can associate to such field configurations a (topological) charge

$$Q_t = n_1 - n_2 = \frac{g}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\frac{\partial\phi(x)}{\partial x}. \tag{2.77}$$

We call this charge a topological because it does not depend on the details of the field configuration, it cares only about the "endpoints". In other words, we can modify (deform) the field configuration, and as long as we don't touch the end points, we won't change the charge Q_t. In the classical field theory, the allowed configurations solve equations of motion. We will now look for such topological excitation, that is a solution of the equations of motion connecting two different minima of the potential.

The energy of a generic configuration $\phi(x,t)$ is

$$E[\phi] = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \right]. \tag{2.78}$$

and the equations of motion are

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial V}{\partial \phi}. \tag{2.79}$$

We can note at this point that given $\phi(x,t)$ solving equations of motion, also $\phi(x,t) + 2\pi n/g$ with $n \in \mathbf{Z}$ is a solution.

Let us look for a static solution, $\partial\phi/\partial t = 0$. For such static solution to have a finite energy we need

$$V(\phi) \to 0, \qquad \frac{\partial \phi}{\partial x} \to 0, \qquad \text{for } x \to \pm\infty. \tag{2.80}$$

The equations of motion for a static solution are

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial V}{\partial \phi}. \tag{2.81}$$

and the solution is

$$\phi(x) = \frac{4}{g} \text{atan} \left(\exp \left(\pm \mu (x - x_0) \right) \right). \tag{2.82}$$

Derivation 3: Static solutions to the classical Sine-Gordon theory

We solve the static equation of motion (2.81) by mapping it to a problem of classical mechanics. The equations of motion look like equations of motion for a fictitious particle described by a coordinate $\phi(x)$ in the potential $-V(\phi)$ and with time labelled by x. Translating

$$\phi \to x, \qquad x \to t, \qquad V(\phi) \to -V(x), \tag{2.83}$$

we find

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V(x)}{\partial x}. \tag{2.84}$$

The force is conservative, so the total energy of this fictitious particle is a constant of motion

$$W = \frac{1}{2} \left(\frac{\partial x}{\partial t} \right)^2 - V(x). \tag{2.85}$$

The conditions, that the static solution $\phi(x)$ has a finite energy translates into a condition that at $t \to \pm\infty$ the particle has to be in one of the maxima of the potential $V(x)$ and with zero velocity $\partial x/\partial t$. The maximum of the potential $V(x)$ is zero and therefore $W = 0$. This gives

$$\frac{\partial x}{\partial t} = \pm \sqrt{2V(x)}. \tag{2.86}$$

We can now separate the variables to find

$$t - t_0 = \pm \int^x \frac{\mathrm{d}x}{\sqrt{2V(x)}} = \pm \frac{1}{\mu} \log\left(\tan\frac{gx}{4}\right), \tag{2.87}$$

Inverting this relation we find

$$x(t) = \frac{4}{g}\mathrm{atan}\left(\exp\left(\pm\mu(t-t_0)\right)\right). \tag{2.88}$$

Translating back from the problem of mechanics of a fictitious particle to a static solution of the field equations we get (2.82).

Note that

$$\mathrm{atan}\left(e^{-x}\right) = \mathrm{acot}\left(e^{x}\right) = \frac{\pi}{2} - \mathrm{atan}\left(e^{x}\right). \tag{2.89}$$

Therefore, up to a shift of $2\pi/g$, the static solutions (2.82) are

$$\phi_{\pm}(x) = \pm\frac{4}{g}\mathrm{atan}\left(\exp\left(\mu(x-x_0)\right)\right). \tag{2.90}$$

These two solutions interpolate between the neighbouring minima. Function $\phi_+(x)$ interpolates between $|0\rangle_0$ and $|0\rangle_1$ while $\phi_-(x)$ interpolates between $|0\rangle$ and $|0\rangle_{-1}$. Given the freedom to modify the field configuration by adding $2\pi k/g$, we obtain static solutions interpolating between any two neighbouring minima. Solutions $\phi_+(x)$ have topological charge $+1$, while for $\phi_-(x)$ the topological charge is -1. Let us analyze these solutions closer.

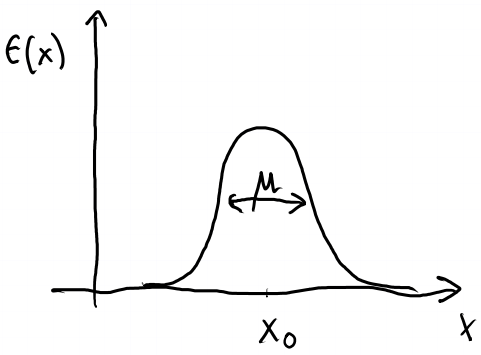
The energy of such configuration is

$$E[\phi] = \int_{-\infty}^{\infty} \mathrm{d}x\epsilon(x) = \frac{8\mu}{g^2}, \tag{2.91}$$

where

$$\epsilon(x) = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi) = \frac{4\mu^2}{g^2}\frac{1}{\cosh^2\mu(x-x_0)}. \tag{2.92}$$

The distribution of energy $\epsilon(x)$ is localized around x_0 so this looks like a particle with a mass $M_s = 8\mu/g^2$. Note that the mass contains the inverse of the coupling g in the theory. We have found classical non-perturbative solution to a field theory.



So far our solution is quite boring, because it is static. However we can use Lorentz transformation to boost it

$$\phi(x-x_0) \to \phi\left(\frac{(x-x_0)-vt}{\sqrt{1-v^2}}\right). \tag{2.93}$$

The energy of such configuration is

$$E[\phi] = \frac{M_s}{\sqrt{1-v^2}}. \tag{2.94}$$

We interpret these field configurations as massive particles - solitons. In this classical description these are waves that propagate without dispersion or dissipation, keeping the shape intact.

There are also solutions in other topological sectors. For instance there is a solution with $Q_t = 0$ which can be thought of as a bound state of soliton and anti-soliton (**check these equations**)

$$\phi_{s\bar{s}}(x,t) = \frac{4}{g}\text{atan}\left(\frac{\sinh(\mu v t/\sqrt{1-v^2})}{\cosh(\mu x/\sqrt{1-v^2})}\right). \tag{2.95}$$

At $t \to \pm\infty$

$$\phi_{s\bar{s}}(x,t) \to \phi_s\left(\frac{x+v(t\pm\Delta_{s\bar{s}}/2)}{\sqrt{1-v^2}}\right) + \phi_{\bar{s}}\left(\frac{x-v(t\pm\Delta_{s\bar{s}}/2)}{\sqrt{1-v^2}}\right). \tag{2.96}$$

The time evolution brings soliton and anti-soliton together, they collide and fly away. The only effect of the interactions is in the time shift $\Delta_{s\bar{s}} = (1-v^2)v\log v$. An example of yet another solution, this time of charge $Q_t = 2$, is the combination of two solitons

$$\phi_{ss}(x,t) = \frac{4}{g}\text{atan}\left(\frac{\sinh(\mu x/\sqrt{1-v^2}}{\cosh(\mu v t/\sqrt{1-v^2}}\right). \tag{2.97}$$

2.3.2 Massive Thirring model and Sine-Gordon model

Perturbation theory computations show that the Sine-Gordon model is equivalent to the massive Thirring. The massive Thirring model is a theory of a Dirac field

$$\mathcal{L}_{\text{MTM}} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m_0\bar{\Psi}\Psi - \frac{g_T}{2} : \left(\bar{\Psi}\gamma^\mu\Psi\right)^2 :, \tag{2.98}$$

where Ψ is a Dirac spinor with components (cause we are in $(1+1)$d), γ^μ are two γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.99}$$

m_0 is the bare mass and g_T is the coupling constant. The Dirac conjugation is $\bar{\Psi} = \Psi^\dagger\gamma^0$

Based on the perturbative computations Sidney Coleman conjectured the massive Thirring model is related to the Sine-Gordon model and that there is a following relation between the couplings in the respective theories

$$\frac{g_T}{\pi} = \frac{4\pi}{g^2} - 1. \tag{2.100}$$

Not only two seemingly different theories are related, moreover this relation is another example of the weak-strong duality that we observed in the Sinh-Gordon model.

We will not go into the details of the perturbative analysis that support this statements, instead let us observe that depending on value of g in the Sine-Gordon theory the sign of the interaction terms in the massive Thirring model changes such that

$$g^2 < 4\pi \qquad \text{attractive}, \tag{2.101}$$

$$g^2 > 4\pi \qquad \text{repulsive}, \tag{2.102}$$

and for $g^2 = 4\pi$ the massive Thirring model is non-interactive, still the Sine-Gordon model looks as complicated as for any other value of the coupling.

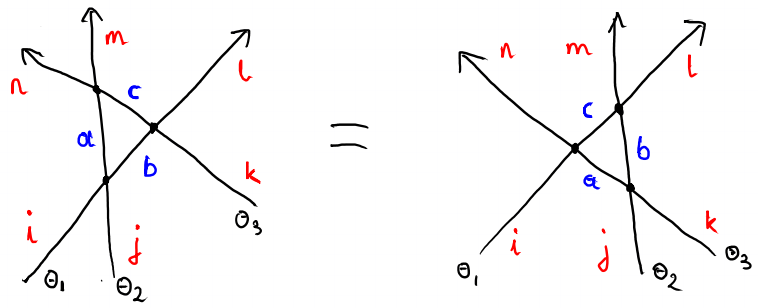


Figure 2.12: Yang-Baxter equation

2.3.3 Scattering matrix of the Sine-Gordon model

In the classical analysis we have seen that there are two types of particles: solitons and anti-solitons, having an opposite topological charge. Let us denote by $A(\theta)$ an annihilation operator for a soliton and $\bar{A}(\theta)$ an annihilation operator for the anti-soliton. What we mean by this is the following.

Recall that in the scattering theory we have well defined *in* and *out* states. What we want is to give life to the symbols appearing there outside of the states, thus defining operators

$$|A(\theta_1)A(\theta_2)\rangle_{\text{in}} = A^\dagger(\theta_1)A^\dagger(\theta_2)|0\rangle_{\text{in}}. \tag{2.103}$$

This way we can think of the S-matrix as implementing the commutation relations between the operators instead of being an operator relating *in* and *out* states. In an integrable theory the S-matrix factorizes and therefore it is enough to give the commutation relations for pairs of the operators. The Yang-Baxter equation implies then the associativity of the just defined algebra. This algebra is known as Faddeev-Zamolodchikov algebra, we write

$$A_i(\theta_1)A_j(\theta_2) = S_{ij}^{kl}(\theta_2-\theta_1)A_k(\theta_2)A_l(\theta_1). \tag{2.104}$$

With this idea we now get back to the Sine-Gordon theory. The topological charge is conserved thus we can write the following relations

$$A(\theta_1)\bar{A}(\theta_2) = S_T(\theta_2-\theta_1)\bar{A}(\theta_2)A(\theta_1) + S_R(\theta_2-\theta_1)A(\theta_2)\bar{A}(\theta_1), \tag{2.105}$$
$$A(\theta_1)A(\theta_2) = S(\theta_2-\theta_1)A(\theta_2)A(\theta_1), \tag{2.106}$$
$$\bar{A}(\theta_1)\bar{A}(\theta_2) = S(\theta_2-\theta_1)\bar{A}(\theta_2)\bar{A}(\theta_1). \tag{2.107}$$

where $S_T(\theta)$ is the amplitude for a *transmission* process and $S_R(\theta)$ is the amplitude for the *reflection*. We can collect the amplitudes into 4×4 matrix

$$S^{\text{SG}}(\theta) = \begin{pmatrix} S(\theta) & & & \\ & S_T(\theta) & S_R(\theta) & \\ & S_R(\theta) & S_T(\theta) & \\ & & & S(\theta) \end{pmatrix} = S(\theta)\begin{pmatrix} 1 & & & \\ & T(\theta) & R(\theta) & \\ & R(\theta) & T(\theta) & \\ & & & 1 \end{pmatrix}. \tag{2.108}$$

The S-matrix is constrained by the unitarity condition, crossing relation and Yang-Baxter equation. We start with the later. The Yang-Baxter equation, in the full glory with the indices, see Fig. 2.12, reads

$$S_{ij}^{ab}(\theta_{12})S_{bk}^{cl}(\theta_{13})S_{ac}^{nm}(\theta_{23}) = S_{jk}^{ab}(\theta_{23})S_{ia}^{nc}(\theta_{13})S_{cb}^{ml}(\theta_{12}), \tag{2.109}$$

where $\theta_{ij} = \theta_i - \theta_j$, $\{i, j, k, l, n, m\}$ specify types of external particles and $\{a, b, c\}$ types of virtual, internal, particles. In general, in a theory with r types of particles, Yang-Baxter equation gives r^6 equations for r^4 unknowns. Therefore, solutions exists only for special functional forms of $S_{ij}^{kl}(\theta)$. Note also, that the Yang-Baxter equation can only fix the ratios of the scattering amplitudes.

The solution to the Yang-Baxter equation for the S-matrix of (2.108) is the following

$$T(\theta) = \frac{\sinh(\pi\theta/\xi)}{\sinh(\pi(i\pi - \theta)/\xi)}, \qquad R(\theta) = \frac{i\sin(\pi^2/\xi)}{\sinh(\pi(i\pi - \theta)/\xi)}, \tag{2.110}$$

where $\xi \in \mathbf{R}$ is a parameter. Function $S(\theta)$ can be found from the crossing and unitarity relations. The crossing symmetry implies

$$S(\theta) = S_T(i\pi - \theta), \qquad S_R(\theta) = S_R(i\pi - \theta), \tag{2.111}$$

whereas the unitarity leads to the following constraints

$$S(\theta)S(-\theta) = 1, \qquad S_T(\theta)S_T(-\theta) + S_R(\theta)S_R(-\theta) = 1, \tag{2.112}$$

$$S_T(\theta)S_R(-\theta) + S_R(\theta)S_T(-\theta) = 0. \tag{2.113}$$

Using the solution to the Yang-Baxter equation together with unitarity and crossing conditions we can find a minimal set of equations for $S(\theta)$

$$S(\theta)S(-\theta) = 1, \tag{2.114}$$

$$S(i\pi - \theta) = \frac{\sinh\pi\theta/\xi}{\sinh\pi(i\pi - \theta)/\xi}\, S(\theta). \tag{2.115}$$

These are similar equations to the ones we encountered in the previous section when considering the Sinh-Gordon model. However, due the presence of the ratio of hyperbolic functions the solution is much more complicated. We will not derive, merely present, the answer

$$\begin{aligned} S(\theta) = \prod_{k=0}^{\infty} & \frac{\Gamma(1 + (2k+1)\pi/\xi - i\theta/\xi)}{\Gamma(1 + (2k+1)\pi/\xi + i\theta/\xi)} \frac{\Gamma(1 + 2k\pi/\xi + i\theta/\xi)}{\Gamma(1 + 2k\pi/\xi - i\theta/\xi)} \\ & \times \frac{\Gamma((2k+1)\pi/\xi - i\theta/\xi)}{\Gamma((2k+1)\pi/\xi + i\theta/\xi)} \frac{\Gamma((2k+2)\pi/\xi + i\theta/\xi)}{\Gamma((2k+2)\pi/\xi - i\theta/\xi)}, \end{aligned} \tag{2.116}$$

where $\Gamma(z)$ is Gamma function, that generalises the factorial,

$$\Gamma(z) = \int_0^\infty \mathrm{d}t t^{z-1} e^{-t}, \qquad \Gamma(z+1) = z\Gamma(z), \qquad \Gamma(1) = 1. \tag{2.117}$$

Having the S-matrix we can look at its analytic structure to say something about the possible bound states. We first note that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$ and has no zeroes. Therefore the poles must come from numerator of (2.116). We look for the poles in the physical sheet: $0 < \theta < i\pi$. We see that poles appear when

$$\theta = i\xi n, \qquad n = 1, 2, 3 \ldots. \tag{2.118}$$

There are also poles at $\theta = i(\pi - \xi n)$ coming from the denominator of $R(\theta)$.

These poles are in the physical sheet only when $\xi < \pi$. When $\xi > \pi$ there are no poles in the physical sheet, hence there are no bound states. We could say that the interactions are then

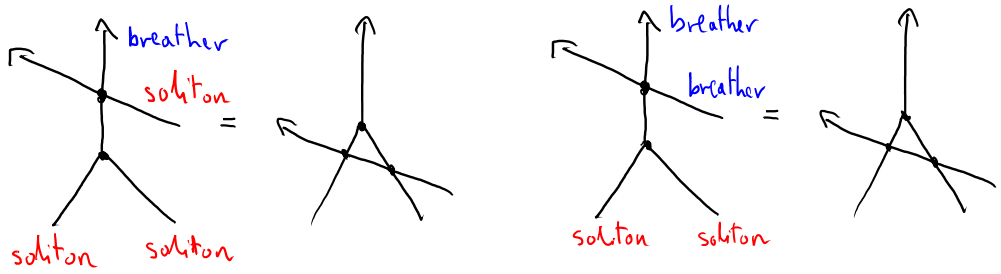


Figure 2.13: We use the bootstrap principle to find breather scattering matrix with solitons and the scattering matrix of two breathers.

repulsive. For $\xi < \pi$ poles, and hence bound states appear – the interactions are attractive. Number of bound states depends on ξ and is given by $[\pi/\xi]$ where $[x]$ is integer part of x.

Recall that in the s-channel the pole is at $s = m^2$ where m is the mass of bound state. Consider poles at $\theta = i(\pi - \xi n)$, in the process of colliding soliton and anti-soliton. Let M denotes their masses. Then

$$m_m^2 = s(i(\pi - \xi n)) = 2M^2(1 - \cos n\xi) = 4M^2 \sin^2 \frac{n\xi}{2}. \tag{2.119}$$

These are masses of (different types of) bound states of soliton and anti-soliton, we call these bound state *breathers* and label B_n.

We can now use the bootstrap principle to compute the scattering matrix $S^{(n)}(\theta)$ of breathers with solitons

$$S^{(n)}(\theta) = S(\theta + in\xi)\, S(\theta - in\xi). \tag{2.120}$$

and also scattering matrix $S^{(n,m)}(\theta)$ of breathers with breathers

$$S^{(n,m)}(\theta) = S^{(m)}(\theta + in\xi)\, S^{(m)}(\theta - in\xi). \tag{2.121}$$

The computations are again quite lengthy with the final results

$$S^{(n)}(\theta) = \frac{\sinh\theta + i\cos n\xi/2}{\sinh\theta - i\cos n\xi/2} \prod_{k=1}^{n-1} \frac{\sin^2\left(\frac{(n-2k)\xi}{4} - \frac{\pi}{4} + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{(n-2k)\xi}{4} - \frac{\pi}{4} - i\frac{\theta}{2}\right)}, \tag{2.122}$$

and

$$S^{(n,m)}(\theta) = \frac{\sinh\theta + i\sin\frac{(n+m)\xi}{2}}{\sinh\theta - i\sin\frac{(n+m)\xi}{2}} \frac{\sinh\theta + i\sin\frac{(n-m)\xi}{2}}{\sinh\theta - i\sin\frac{(n-m)\xi}{2}} \tag{2.123}$$

$$\times \prod_{k=1}^{n-1} \frac{\sin^2\left(\frac{(m-n-2k)\xi}{4} - \frac{\pi}{4} + i\frac{\theta}{2}\right)}{\sin^2\left(\frac{(m-n-2k)\xi}{4} - \frac{\pi}{4} - i\frac{\theta}{2}\right)} \frac{\cos^2\left(\frac{(m+n-2k)\xi}{4} - \frac{\pi}{4} + i\frac{\theta}{2}\right)}{\cos^2\left(\frac{(m+n-2k)\xi}{4} - \frac{\pi}{4} - i\frac{\theta}{2}\right)}. \tag{2.124}$$

To make sense of this lengthy formulas let us evaluate scattering matrix of the lightest breather B_1 with itself, we get

$$S^{(1,1)}(\theta) = \frac{\sinh\theta + i\sin\xi}{\sinh\theta - i\sin\xi}. \tag{2.125}$$

Expanding now this scattering matrix in ξ and comparing with the perturbative computations based on the Sine-Gordon Lagrangean we would 1) find that breathers correspond to the excitations of the elementary field ϕ, 2) the map between ξ and the interaction parameter g

$$\xi = \frac{g^2}{8}\frac{1}{1-\frac{g^2}{8\pi}}. \tag{2.126}$$

This leads to an interesting observation, for ξ (or g) large enough the breathers disappear from the spectrum. It means that the resulting particles that are associated with the field ϕ disappear from the spectrum!

Exercise 24

Show that for $\xi = \pi$ the scattering matrix $S^{\mathrm{SG}}(\theta)$ becomes that of a free theory. Value of $\xi = \pi$ corresponds to $g = 2\sqrt{\pi}$. Go back to the Lagrangean of the Sine-Gordon theory, for example eq. (1.68), and contemplate this result.

2.4 Statistical physics and Ising model

In this section we will consider a canonical model of statistical physics, namely the Ising model. The final aim is to show how the methods of integrability solve the 2d Ising model in certain regimes. We will approach this problem gradually, starting with discussion of the physics of the Ising model. We will then show that the problem of computing its partition function is related to the problem of $(1+1)d$ quantum mechanics. This is an example of a more general relation between statistical and quantum physics. We will comment on this as well. We will then prove a strong/weak duality of the Ising model. Finally, we will show that the Ising model can be mapped to the theory of fermions. The technique of mapping spins to fermions we have also already seen in early sections of these notes.

2.4.1 $2d$ Ising model

We are now in the realm of classical statistical physics. We consider a $2d$ square lattice of size $N \times M$. At each vertex of the lattice there is a binary variable - classical spin - taking values $+1$ or -1. In total there are NM spins and 2^{NM} configurations. We denote a given configuration $\boldsymbol{\sigma}$ and to each configuration we associate an energy $E(\boldsymbol{\sigma})$. The central object in statistical physics is the partition function, given by a sum over configurations weighted with the Boltzmann factor

$$Z = \sum_{\boldsymbol{\sigma}} e^{-\beta E(\boldsymbol{\sigma})}, \tag{2.127}$$

with $\beta = 1/T$, the inverse temperature. To fully specify the problem we have to make a choice for $E(\boldsymbol{\sigma})$. We imagine that spins interact with each other and there might be an external field that tend to align the spins one way or the other

$$E(\boldsymbol{\sigma}) = \sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i. \tag{2.128}$$

J_{ij} controls how strongly spins at site i and j are coupled. It seems reasonable to assume that the spins that are closer to each other are coupled stronger than the distant ones. We can take a "limit" of this and assume that only the neighbouring spins are coupled. On the $2d$ square lattice every spins has 4 neighbours. We can also simplify the model by assuming certain homogeneity - every site is the same, so it is reasonable to assume that the strength of interaction does not depend on the site. However we leave an option for the different interaction strength in vertical and horizontal directions. Finally, we take the magnetic field h_i to be the same at every site. After these simplifications,

$$E(\boldsymbol{\sigma}) = \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i, \qquad J_{ij} = \begin{cases} \bullet_i \!-\!\!-^{J}\!\!-\! \bullet_j \\ \bullet_j \\ \;\;\vdots\;J' \\ \bullet_i \end{cases} \tag{2.129}$$

with $\langle i,j\rangle$ denoting the sum over the nearest neighbours. This setup is summarized in Fig. 2.14.

We will focus on the ferromagnetic case ($J, J' < 0$) in which spins tend to be ordered, say all are $+1$.

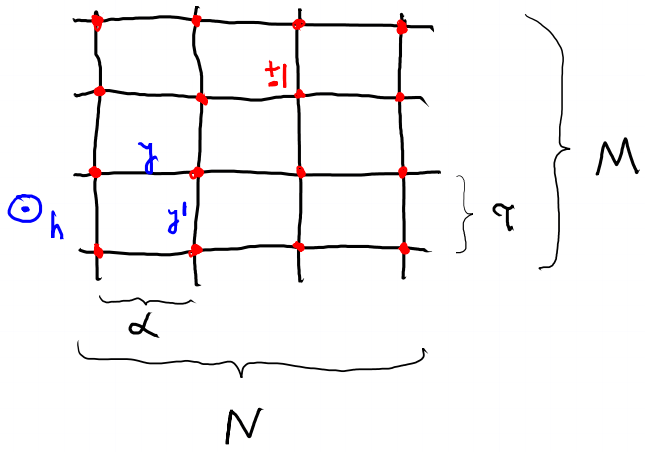


Figure 2.14: The Ising model on the $2d$ lattice

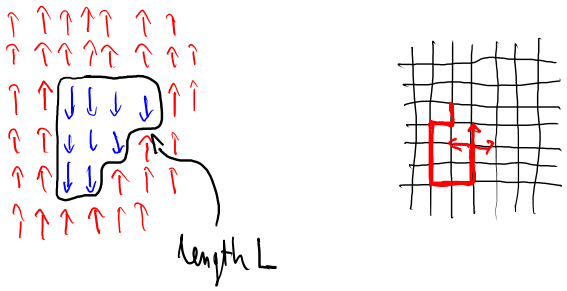
2.4.2 Quantitative picture

Before attempting a full solution it is usually beneficial to try to understand quantitatively what's going on. In this case we would like to understand qualitative features of configurations dominating the physics (that is the partition function). The tool is the free energy,

$$F[\boldsymbol{\sigma}] = E[\boldsymbol{\sigma}] - TS[\boldsymbol{\sigma}]. \tag{2.130}$$

which should be minimised. Here $S[\boldsymbol{\sigma}]$ is the entropy, that is (the logarithm of) the number of configurations which have the same energy as the chosen configuration $\boldsymbol{\sigma}$. At low temperatures, the entropic factor in the $F[\boldsymbol{\sigma}]$ is not important and to minimize the free energy we need to simply minimize the energy. In the opposite regime of large temperatures, we need to maximize the entropy. At intermediate temperatures, the dominating configurations result from the competition between these two terms. And that's where potentially interesting things happen.

In the case of the ferromagnetic Ising model, the configurations minimizing the energy are with all spins ordered in the direction of the magnetic field. This is the leading configuration which is stable upon cranking up the temperature till the temperature reaches a critical value T_C where the order melts, the entropy takes over and spins are distributed more or less randomly. This is a phase transition from ordered to disordered phase. How can we estimate its critical temperature T_C? Here is the *Peierl's argument*.



We are in the zero magnetic field case. Imagine that a thermal fluctuations creates a pocket of flipped spins. The increase in the energy is proportional to the perimeter, say L, of the

pocket and is given by $\Delta E = 2JL$. What's the entropy? To estimate it we need to count loops of length L. This looks like quite a complicated problem but we can brutal. To create a loop of length L you need to choose L times which direction you move on the lattice. On the $2d$ square lattice every time you move you have 3 choices (the 4-th direction is where you came from). Roughly speaking (ignoring the fact the some paths might cross themselves) there are 3^L options. Of course not all of them lead to a loop. On the other hand, on the average (and for large loop) we always have more than one choice. So let us say that we have on average 2 possible choices. Thus there is 2^L loops and the change to the entropy is $\Delta S = L \ln 2$. The change to the free energy is then

$$\Delta F = L \times (2J - T \ln 2), \tag{2.131}$$

and for temperatures $T > T_C = 2J/\ln 2$ the creation of the pockets of flipped spins is favourable. The critical temperature, in the units of J reads $T_C = 2.89\, J$.

Remark 11: Free energy and phase transitions

It should be obvious that if the lattice is very large there are many configurations with approximately the same energy and also the energy, as an extensive quantity, scales with the area (in $2d$). Number of configurations with the same energy is counted by (the exponential of) entropy. Instead of summing over configurations we can evaluate the partition function by summing over representative configurations by grouping together configurations of the same energy

$$Z = \sum_{\bar{\boldsymbol{\sigma}}} e^{F[\bar{\boldsymbol{\sigma}}]/T}, \tag{2.132}$$

with $F[\boldsymbol{\sigma}]$ the free energy. The free energy is an extensive quantity, it scales with the area and for large system we can evaluate the partition function by the saddle point method

$$Z = \sum_{\bar{\boldsymbol{\sigma}}} e^{Af[\bar{\boldsymbol{\sigma}}]/T} \sim e^{Af[\boldsymbol{\sigma}_{\rm sp}]/t}, \tag{2.133}$$

where $\boldsymbol{\sigma}_{\rm sp}$ is the configuration minimizing the free energy $F[\boldsymbol{\sigma}] = Af[\boldsymbol{\sigma}]$ where $f[\boldsymbol{\sigma}]$ is the free energy density and $A = \alpha N \times \tau M$ is the area (volume) of the system. The phase transition happens when, upon changing the temperature, the saddle point configurations changes $\boldsymbol{\sigma}_{\rm sp}$, for example from ordered to disordered.

The $2d$ classical Ising model is exactly solvable and the critical temperature can be computed. The exact result is

$$\sinh^2 \frac{2J}{T_C} = 1 \tag{2.134}$$

which gives $T_C \approx 2.27\, J$. The Peierl's estimation is quite good.

2.4.3 Transfer matrix and the Hamiltonian limit

We will reformulate the problem of computing the partition function into a problem of diagonalizing matrices. A key concept is the *transfer matrix*. We assume now the periodic boundary conditions, That is after the M-th row the 1st row follows, and similarly for columns. Let us look at the Ising model and the underlying lattice not focusing on single spins but on rows. We

denote $\mu_a = \{\sigma_1, \ldots, \sigma_N\}$ set of spins of the a-th row (for $a = 1, \ldots, M$). There are then two types of contributions to $E(\boldsymbol{\sigma})$: intra-row and inter-row. For the former we have

$$E(\mu) = J \sum_{k=1}^{N} \sigma_k \sigma_{k+1} - h \sum_{k=1}^{N} \sigma_k, \tag{2.135}$$

for the latter we have

$$E(\mu, \mu') = J' \sum_{k=1}^{N} \sigma_k \sigma'_k. \tag{2.136}$$

The partition function can be now expressed as a sum over different configurations of each row

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_M} \exp\left[-E(\mu_1, \mu_2, \ldots, \mu_M)/T\right], \tag{2.137}$$

where

$$\begin{aligned} E(\mu_1, \mu_2, \ldots, \mu_M) &= \sum_{a=1}^{M} \left(E(\mu_a, \mu_{a+1}) + E(\mu_a)\right) \\ &= \sum_{a=1}^{M} \left(E(\mu_a, \mu_{a+1}) + \frac{1}{2}E(\mu_a) + \frac{1}{2}E(\mu_{a+1})\right). \end{aligned} \tag{2.138}$$

We can now define the transfer matrix T_N. It's a matrix in the configurational space of rows.

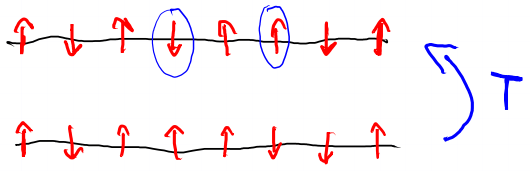


Figure 2.15: Transfer matrix relates two spin configurations, here differing by two spin-flips.

We will define it by specifying all its matrix elements.

$$\langle \mu | T_N | \mu' \rangle = \exp\left(-(E(\mu, \mu') + E(\mu))/T\right). \tag{2.139}$$

Then the partition function is

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_M} \langle \mu_1 | T_N | \mu_2 \rangle \langle \mu_2 | T_N | \mu_3 \rangle \cdots \langle \mu_M | T_N | \mu_1 \rangle = \sum_{\mu_1} \langle \mu_1 | T_N | \mu_1 \rangle = \mathrm{tr}(T_N^M). \tag{2.140}$$

The transfer matrix T_N is diagonalizable and its spectrum consists of positive numbers which can be ordered: $\lambda_1 > \lambda_2, \ldots$. In the limit of the large system only the largest eigenvalue λ_1 matters and we have

$$Z = \left(\lambda_1^M + \lambda_2^M + \ldots\right) \approx \lambda_1^M \times \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^M + \ldots\right). \tag{2.141}$$

Hamiltonian limit: We start with an observation that we can formally write

$$T = e^{-\tau H}, \tag{2.142}$$

and think of the transfer matrix as an evolution operator in the imaginary time. Then going from a row to row looks like evolving the configuration of spins for a time τ with Hamiltonian H. Knowing T, in principle we can find H,

$$H = -\frac{1}{\tau}\log T. \tag{2.143}$$

To make the connection with ordinary physics, we should take the limit $\tau \to 0$ in which "time" becomes continuous. We have to identify now the correct perspective. The leading process is "nothing happens" - the configuration of spins does not change. The subleading processes will be 1 spin flip, 2 spin flips, etc. Let us denote by $T(k)$ the transfer matrix between the configurations with k-spin flips. We have

$$T(0) = e^{-J'N/T} \times e^{-E(\mu)/T}, \tag{2.144}$$

$$T(1) = e^{-J'N/T} \times e^{2J'/T} e^{-\frac{1}{2}(E(\mu)+E(\mu'))/T}, \tag{2.145}$$

$$T(k) = e^{-J'N/T} \times e^{2kJ'/T} e^{-\frac{1}{2}(E(\mu)+E(\mu'))/T}. \tag{2.146}$$

We see that the prefactor changes with the number of spin-flips. We scale out $\exp(-J'N/T)$ to have nicer expressions — this is equivalent to adjusting the zero energy level. Identifying $\tau \sim \exp(2J'/T)$ (recall that $J' < 0$) we can start recovering the Hamiltonian. For example,

$$T(1) = \tau e^{-\frac{1}{2}(E(\mu)+E(\mu'))/T} = -\tau H_1 \tag{2.147}$$

where H_1 is the Hamiltonian evaluated between two states differing by 1-spin flip. We could find H_k in a similar way, but it would be multiplied by τ^k and we can neglect it the continuous time limit. What about $T(0)$. We should have

$$T(0) = e^{-E(\mu)/T} = 1 - \tau H_0, \tag{2.148}$$

where H_0 comes now from identifying a small parameter and expanding the exponent in it. Let us first assume that $h = 0$. We can then take J to be a small parameter which gives

$$H_0 = \frac{1}{\tau}\frac{J}{T}\sum_{k=1}^{N}\sigma_k\sigma_{k+1}, \tag{2.149}$$

upon identification $\tau \sim -J/T$. We introduce now number λ such that

$$-J/T = \lambda \exp(2J'/T), \tag{2.150}$$

and identify the vertical lattice spacing $\tau = \exp(2J'/T)$. This implies that $-J/T = \tau\lambda$. Now upon, sending $\tau \to 0$ we see that the horizontal coupling J goes to zero, while vertical coupling J' becomes very large (and negative).

We can now get back to identifying the Hamiltonian. From $T(1)$ we find

$$H_1 = -e^{-\frac{1}{2}(E(\mu)+E(\mu'))/T} = -1 + \mathcal{O}(J/T), \tag{2.151}$$

where configurations μ and μ' differ by one spin flip. The Hamiltonian is therefore

$$H = H_0 + H_1 = -\sum_{k=1}^{N} \left(\lambda \sigma_k^z \sigma_{k+1}^z + \sigma_k^x\right), \tag{2.152}$$

where σ_k^α are Pauli matrices: σ^z reads out the spin and thus gives the classical value ± 1, σ^x instead flips the spin. Observe that this is a quantum $1d$ Ising chain with z-components of spin coupled and magnetic field in x direction. The coupling constant $\lambda = -J/T \exp(-2J'/T)$ is given by the interaction parameters and the temperature of the $2d$ model. At the critical temperature $\lambda_c(T_C) = 1$. This is a critical value of the interaction strength at which quantum phase transition takes place. For $\lambda > 1$ the ground state is in ordered phase, whereas for $\lambda < 1$ is disordered.

Magnetic field: Turning on the magnetic field leads to an extra term in H_0

$$H_0 = \frac{1}{\tau}\left(\frac{J}{T}\sum_{k=1}^{N} \sigma_k \sigma_{k+1} - \frac{h}{T}\sum_{k=1}^{N} \sigma_k\right), \tag{2.153}$$

and in order to have a finite $\tau \to 0$ limit we need to let $h \to 0$. Redefining $h \to \tau h$ we find

$$H = -\sum_{k=1}^{N} \left(\lambda \sigma_k^z \sigma_{k+1}^z + \sigma_k^x + h\sigma_k^z\right). \tag{2.154}$$

This is a quantum $1d$ chain in the magnetic field in both x and z directions.

2.4.4 Order and disorder duality

We consider now the $1d$ quantum Ising chain with magnetic field only in the x-direction. We also consider an infinite chain. Let us define two operators

$$\mu_{k+1/2}^z = \prod_{j \le k} \sigma_j^x, \tag{2.155}$$

$$\mu_{k+1/2}^x = \sigma_k^z \sigma_{k+1}^z. \tag{2.156}$$

This operators are defined on the dual lattice. It is easy to show that

$$\mu_{k-1/2}^z \mu_{k+1/2}^z = \sigma_k^x. \tag{2.157}$$

With the help of this expression we can easily express the Hamiltonian in terms of these new operators

$$H = -\sum_k \left(\mu_{k-1/2}^z \mu_{k+1/2}^z + \lambda \mu_{k+1/2}^x\right) = -\lambda \sum_k \left(\lambda^{-1} \mu_{k-1/2}^z \mu_{k+1/2}^z + \mu_{k+1/2}^x\right). \tag{2.158}$$

This implies that

$$H(\sigma, \lambda) = \lambda H(\mu, \lambda^{-1}). \tag{2.159}$$

One can show that operators μ_k obey the same algebraic relations like σ_k and therefore each eigenvalue of the Hamiltonian satisfies

$$E(\lambda) = \lambda E(\lambda^{-1}). \tag{2.160}$$

We can use this relation to find the critical value of λ_c of the quantum phase transition. Recall that the quantum phase transition occurs when two lowest eigenenergies degenerate. That is, that mass gap $m(\lambda_c) = 0$. Given the above relation we must have that $m(\lambda_c^{-1}) = 0$ as well. Based on our understanding of $2d$ classical Ising model, we expect only a single critical point at thus

$$\lambda_c = \lambda_c^{-1} \longrightarrow \lambda_c = 1. \tag{2.161}$$

The duality relation is known as Kramers-Wannier duality and is yet another example, this time very explicit, of a weak-strong duality.

Exercise 25: Disorder operators

In this exercise we will confirm that the disorder operators $\mu^{\alpha}_{k+1/2}$ follow the same algebra as order operators σ^{α}_{k}. Start with defining $\mu^{y}_{k+1/2}$ from the commutation relations of the other two operators. Show the remaining commutation and anticommutation relations that you would expect from spin operators.

2.4.5 Fermionic field theory

Finally, we would like to formulate the quantum field theory of the Ising model. We will first map, still at the lattice, the Ising model to the model of Majorana fermions, defined by the following algebra

$$\{a_j, a_k\} = 2\delta_{jk}, \qquad a_j^{\dagger} = a_j. \tag{2.162}$$

We consider the following map

$$a_{2j} = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_k^y, \qquad a_{2j-1} = \left(\prod_{k=1}^{j-1} \sigma_k^z\right) \sigma_k^x, \tag{2.163}$$

which leads to the following expression for the Hamiltonian

$$H = -iJ \sum_{j=1}^{M} a_{2j}(\lambda a_{2j+1} - a_{2j-1}) + (\text{boundary} - \text{terms}), \tag{2.164}$$

where we will neglect the boundary terms.

Now, towards the continuum. Let us define two real fields

$$\phi_1(j\alpha) = \frac{1}{\sqrt{2\alpha}} a_{2j}, \qquad \phi_2(j\alpha) = \frac{1}{\sqrt{2\alpha}} a_{2j-1}. \tag{2.165}$$

The normalization factor $\sqrt{2\alpha}$ makes them anti-commute with the δ-function in the limit $\alpha \to 0$,

$$\{\phi_a(x)\phi_b(y)\} = \delta_{a,b}\delta(x-y). \tag{2.166}$$

In terms of them the Hamiltonian is

$$H = -2iJ\lambda\alpha \sum_{j=1}^{M} \phi_1(j\alpha) \left(\phi_2((j+1)\alpha) - \phi_2(j\alpha) + \frac{\lambda - 1}{\lambda} \phi_2(j\alpha)\right), \tag{2.167}$$

where we conveniently regrouped the terms. Taking now the continuum limit, we find

$$H = -2iJ\lambda \int \mathrm{d}x\, \phi_1(x) \left(\partial_x \phi_2(x) + \frac{(\lambda - 1)}{\lambda} \phi_2(x)\right), \tag{2.168}$$

We can introduce now $-2J = v = 1$ and $-2J(\lambda - 1) = m$ to write

$$H = i \int \mathrm{d}x \, \left(\phi_1(x) \partial_x \phi_2(x) + m \phi_1(x) \phi_2(x) \right), \tag{2.169}$$

Note that for $h = 1$ the mass is 0. At the point of the quantum phase transition the theory is massless.

As a final step, let us rewrite this theory in terms of chiral fermions

$$\Psi_{L,R}(x) = \frac{\phi_1(x) \pm i\phi_2(x)}{\sqrt{2}}, \qquad \Psi_L^\dagger(x) = \Psi_R(x). \tag{2.170}$$

We find

$$H = \frac{1}{2} \int \mathrm{d}x \, \left(\Psi_L \partial_x \Psi_L - \Psi_R \partial_x \Psi_R + 2m \Psi_R \Psi_L \right). \tag{2.171}$$

We see that the left and right movers become decoupled when $m = 0$.

2.5 Ising field theory and Integrability

In this section we will explore the integrability of the Ising field theory. We will see that there are two deformations of the critical (massless) Ising model. In the classical $2d$ case they correspond to tuning away the temperature from its critical value T_C and to turning on the external magnetic field. In the quantum $(1+1)$d case they correspond to changing interaction parameter λ from its critical value $\lambda_c = 1$ and to turning on the magnetic field in the z-direction.

However, we will start by developing one more piece of IQFT.

2.5.1 Conserved charges and bound states

We start this section be revealing a relation between the set of integers s defining the local conserved charges and the structure of the bound states. Recall the action of the conserved charges (their light-cone components) on the single particle states

$$Q_s|A_a(\theta)\rangle = \chi_s^{(a)} e^{s\theta}|A_a(\theta)\rangle. \tag{2.172}$$

Imagine now that in our IQFT a pair of particles of type a and b can form a bound state, which is a particle of type c. We can formally write this as the following identity

$$\lim_{\epsilon\to 0}|A_a(\theta + iu_{ab}^c + \epsilon)A_b(\theta - iu_{ba}^c)\rangle = |A_c(\theta)\rangle. \tag{2.173}$$

Applying Q^s to the both sides we obtain an infinite system of linear equations for the eigenvalues χ_s^a

$$\chi_s^{(a)} e^{isu_{ab}^c} + \chi_s^{(b)} e^{-isu_{ba}^c} = \chi_s^{(c)}. \tag{2.174}$$

These equations have non-trivial ($\chi_s^{(a)} \neq 0$) solutions for particular values of the resonance angles u_{ab}^c. Non-trivial solutions implies that the bound state structure supports existence of the conserved charges with spin s. This leads to the following relation

$$\text{bound states structure} \quad \leftrightarrow \quad \text{spectrum of spins of cons. charges} \tag{2.175}$$

We will now analyze this relation in few simple examples.

Consider for example the case when all three particles are of the same type $a(= b = c)$. This gives

$$\cos(s\, u_{aa}^a) = \frac{1}{2} \tag{2.176}$$

On the other hand, from the mass triangle we know that $u_{aa}^a = \pi/3$, thus $\cos(s\pi/3) = 1/2$, for $s \in \mathbf{Z}$ is solved for

$$s = 1, 5 \,(\mathrm{mod}\, 6) = 1, 5, 7, 11, 13, 17, 19, \ldots \tag{2.177}$$

Thus we see that if the theory supports particles that are boundstates of themselves only conserved charges of certain spins are allowed. Equivalently, knowing that conserved charges exists for such values of spins we expect that there is a particle in the theory that is a bound state of itself. We can represent the bound state structure in such theory writing $A_a \times A_a \to A_a$. An example of such theory is Bullogh-Dodd model.

Let us analyze now theories with two types of particles a and b with the following two options for the bootstrap structure

$$X: \begin{cases} A_a \times A_a \to A_b \\ A_b \times A_b \to A_a \end{cases} \qquad Y: \begin{cases} A_a \times A_a \to A_a + A_b, \\ A_b \times A_b \to A_a. \end{cases} \tag{2.178}$$

Let us start with the first case X. We have two types of consistency equations

$$2\chi_s^{(a)}\cos(su_{aa}^b)=\chi_s^{(b)},\quad 2\chi_s^{(b)}\cos(su_{bb}^a)=\chi_s^{(a)}, \tag{2.179}$$

which can be combined to find

$$4\cos(su_{aa}^b)\cos(su_{bb}^a)=1. \tag{2.180}$$

Recall, that this equation should hold for every s belonging to some infinite subset of $\mathbf{Z}$. Solving for u_{bb}^a we find

$$su_{bb}^a=\pm\operatorname{acos}\left(\frac{1}{4\cos su_{aa}^b}\right). \tag{2.181}$$

We can use $s=1$ case to solve for u_{bb}^a. This gives (still an infinite) set of equations for u_{aa}^b

$$s\operatorname{acos}\left(\frac{1}{4\cos u_{aa}^b}\right)=\operatorname{acos}\left(\frac{1}{4\cos su_{aa}^b}\right),\qquad u_{bb}^a=\pm\operatorname{acos}\left(\frac{1}{4\cos u_{aa}^b}\right). \tag{2.182}$$

We can look for solutions step by step by inspecting small values of s. With the help of Mathematica we find the following values

$$(s=2,u_{aa}^b=\pi/3,a_{bb}^a=\pi/3),\quad (s=3,u_{aa}^b=2\pi/5,u_{bb}^a=\pi/5),$$
$$(s=4,u_{aa}^b=5\pi/12,u_{bb}^a=\pi/12).$$

Continuing the computations for higher s we find the following two families

$$u_{aa}^b=2\pi/5,u_{bb}^a=\pi/5,\qquad s=1,3,7,9,11,13,17,19,\cdots=1,3,7,9\,(\mathrm{mod}\,10) \tag{2.183}$$
$$u_{aa}^b=5\pi/12,u_{bb}^a=\pi/12,\qquad s=1,4,5,7,8,11,13,16,17,19,21,23,\ldots$$
$$=1,4,5,7,8,11\,(\mathrm{mod}\,12). \tag{2.184}$$

The first set of conserved charges is realized in the Toda field theories (that we don't discuss here).

Consider now the bound state structure Y. We can view it as the combination of the bound state structure X with the $A_a\times A_a\to A_a$ case discussed above. The consistent set of conserved charges comes from combining the two sets. For example, combining $s=1,3,7,9\,(\mathrm{mod}\,10)$ with $s=1,5\,(\mathrm{mod}\,6)$ we find

$$u_{aa}^a=\pi/3,u_{aa}^b=2\pi/5,u_{bb}^a=\pi/5,\quad s=1,7,11,13,17,19,23,29\,(\mathrm{mod}\,30). \tag{2.185}$$

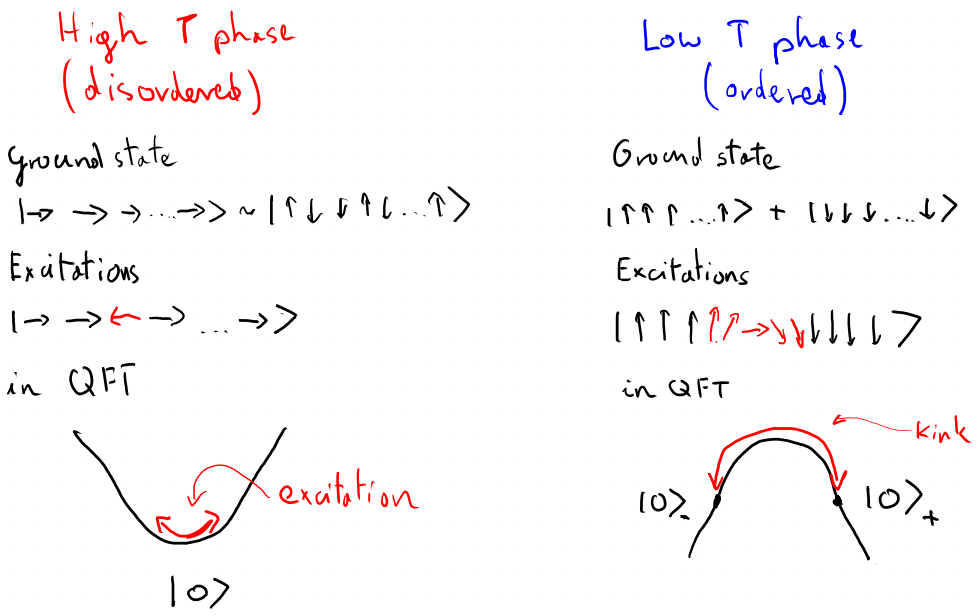
As we shall see this set of conserved charges (and therefore the corresponding bound state structure) is relevant for the magnetic deformation of the Ising model.

2.5.2 Thermal deformation

As we have seen in the previous section, the Ising model in the absence of the magnetic field can be described by the field theory of free massive fermions

$$H=\frac{1}{2}\int \mathrm{d}x\left(\Psi_L\partial_x\Psi_L-\Psi_R\partial_x\Psi_R+2m\Psi_R\Psi_L\right), \tag{2.186}$$

with the mass parameter $m=-2J(\lambda-\lambda_c)\sim T-T_C$. This is a free theory of fermions, thus $S(\theta)=-1$. However the interpretation of the particle content is different for $m>0$ (high temperature phase) and for $m<0$ (low-temperature phase).



High temperature phase: The mass parameter is positive, so the state of the lowest energy is the usual QFT vacuum. On top of this vacuum we can create massive particles (excitations). These particles are free fermions. To better understand their nature, let us revoke the Ising Hamiltonian

$$H = -\sum_{k=1}^{N} \left(\lambda \sigma_k^z \sigma_{k+1}^z + \sigma_k^x\right), \tag{2.187}$$

with $\lambda < 1$ in the high temperature phase. The structure of the ground state is then dominated by the σ_k^x operators which tend to order the spins in the x-direction. Thus the ground state looks like

$$|\mathrm{GS}\rangle = |\rightarrow\rightarrow\cdots\rightarrow\rangle. \tag{2.188}$$

The excitations are simply the spin-flips along the x-axis: $|\rightarrow\rangle \Rightarrow |\leftarrow\rangle$, and corresponds to the action of σ_k^z operator. In the QFT limit the ground state becomes the vacuum, and the spin-flips become massive particles.

Low temperature phase: The mass parameter is now negative, which means that the ground state is not the naive vacuum. We can thing about it as a theory with a quadratic inverted potential. Because of the Pauli principle we cannot roll down the potential. Instead the ground state is doubly degenerate: the two states correspond to either all spins up or down configurations (the Ising Hamiltonian is now with $\lambda > 1$). In the QFT the low-energy excitations are then kinks connecting the two vacua.

2.5.3 Magnetic deformation

We consider now the second deformation of the critical theory. This time the temperature stays at $T = T_C$ but we turn n the magnetic field h. Equivalently, from the point of view of the quantum chain the interaction parameter $\lambda = \lambda_c = 1$ and we turn magnetic field in the z direction. Remember that in this picture there is always magnetic field in the x direction of strength 1.

In future lectures we will show that the resulting theory is characterized by the set

$$s = 1, 7, 11, 13, 17, 19, 23, 29 \,(\mathrm{mod}\, 30), \tag{2.189}$$

of integrals of motion. This hints about the particle content of the theory, as we have seen in the beginning of this section. We shall follow these hints to construct the scattering matrix and recover the full particle content. This is similar to what we did in the case of the Sine-Gordon model. There we were guided by semi-classical analysis, here we are guided by a set of conserved charges.

As we have seen, this set of conserved charges is compatible with the existence of two particles A_1 and A_2 (we change the labels from a, b to $1, 2$). Knowing that $u_{11}^2 = 2\pi/5$ we find

$$\frac{m_2}{m_1} = 2\cos\frac{\pi}{5} \approx 1.62, \tag{2.190}$$

where we used that

$$m_2^2 = 2m_1^2 + 2m_1^2 \cos u_{11}^2 = 4m_1^2 \cos^2 \frac{u_{11}^2}{2}. \tag{2.191}$$

Let us consider now the scattering matrix of the particle 1, that is $S_{11}(\theta)$. In view of these results it should have poles at $\theta = iu$ with $u = 2\pi/5$ and $u = \pi/3$. Thus the minimal S-matrix is

$$S_{11}(\theta) = f_{2/5}(\theta) f_{1/3}(\theta), \tag{2.192}$$

with $f_\gamma(\theta)$ the minimal solution that we explored earlier (we use now slightly different notation to save on writing π's and shifts of γ)

$$f_\gamma(\theta) = \frac{\sinh\theta + i\sin\pi\gamma}{\sinh\theta - i\sin\pi\gamma} = \frac{\tanh\frac{\theta+i\pi\gamma}{2}}{\tanh\frac{\theta-i\pi\gamma}{2}} \tag{2.193}$$

In what follows it is useful to keep in mind the following relations

$$f_\gamma(\theta + i\pi\delta) = \frac{f_{\gamma+\delta}(\theta)}{f_{\gamma-\delta}(\theta)}, \qquad f_{\gamma+1}(\theta) = f_\gamma^{-1}(\theta), \qquad f_{-\gamma}(\theta) = f_\gamma^{-1}(\theta). \tag{2.194}$$

To shorten the notation we can write $f_\gamma(\theta) = (\gamma)$, so that the scattering matrix is

$$S_{11}(\theta) = (2/5)(1/3). \tag{2.195}$$

This is our ansatz for the S-matrix, we should check if it is compatible with the bootstrap. For example, given the process $A_1 \times A_1 \to A_1$ the following equation should hold

$$S_{11}(\theta) = S_{11}(\theta + 2\pi/3i) S_{11}(\theta - 2\pi/3i). \tag{2.196}$$

To check this, let us multiply $S_{11}(\theta)$ by some yet unknown function $f_x(\theta) = (x)$ and substitute to this equation. We find

$$(2/5)(1/3)(x) = \frac{(2/5+2/3)}{(2/5-2/3)}\frac{(1/3+2/3)}{(1/3-2/3)}\frac{(x+2/3)}{(x-2/3)}. \tag{2.197}$$

Simplifying the fractions and using the properties of $f_\gamma(\theta)$ we find

$$(2/5)(1/3)(x) = \frac{(16/15)}{(-4/15)}\frac{(1)}{(-1/3)}\frac{(x+2/3)}{(x-2/3)} = \frac{(11/15)}{(1/15)}(1/3)\frac{(x+2/3)}{(x-2/3)}, \tag{2.198}$$

where we used that $(1) = 1$. Simplyfing

$$(2/5)(x) = \frac{(11/15)}{(1/15)} \frac{(x + 2/3)}{(x - 2/3)}. \tag{2.199}$$

This should have s solution for $x = 1/15$. [**Warning: I made a mistake somewhere in the computations, which I can't localize. The answer is correct but the solution is not.**] Therefore the scattering amplitude $S_{11}(\theta)$ has a pole at $\theta = i\pi/15$. This signals a presence of a new particle. It's mass is

$$m_3 = 2m_1 \cos \frac{\pi}{30} \approx 1.99 m_1. \tag{2.200}$$

The final result for the scattering amplitude of two particles A_1 is

$$S_{11}(\theta) = f_{1/3}(\theta) f_{2/5}(\theta) f_{1/15}(\theta). \tag{2.201}$$

One can continue this way, solving now for $S_{12}(\theta)$ using the bootstrap equation

$$S_{12}(\theta) = S_{11}(\theta + iu_{11}^2) S_{11}(\theta - iu_{11}^2). \tag{2.202}$$

$S_{12}(\theta)$ has then 3 poles corresponding to already known particles A_1, A_2 and A_3 and a new pole, at position $u_{12}^4 = 7\pi/15$ which corresponds to a new particle A_4 with mass

$$m_4 = 2m_1 \cos \frac{7\pi}{30} \approx 2.41 m_1. \tag{2.203}$$

One can continue this way finding consistently new poles. Amazingly, this procedure closes and final theory consists of 8 massive particles. The masses of the remaining ones are

$$m_5 \approx 2.96 m_1, \quad m_6 \approx 3.22 m_1, \quad m_7 \approx 3.89 m_1, \quad m_8 \approx 4.78 m_1. \tag{2.204}$$

Recall, that we are describing a 1d spin chain with nearest neighbour interactions in the magnetic field. If we are able to find a material in which the effective physics is describe by such a chain we should be able to test this theory. When we perturb such a system, the masses of the particles should appear as resonant energies. In fact, such an experiment was performed and the measurement confirmed the masses. It was a great success of the IQFT's.

Let us finish this section with few bibliographical notes. The IQFT describing Ising model in the magnetic field was developed by A. B. Zamolodchikov in the paper: "Integrable Field Theory from Conformal Field Theory" in 1989, the relevant experiment was described in "Quantum criticality in an Ising chain: Experimental evidence for emergent E8 symmetry" published in 2010. The titles of these papers hint on two aspects: first there is a relation between this Ising Field Theory in Conformal Field Theories, second there is a relation with an exceptional Lie algebra E_8. A nice summary of these aspects and the whole story can be found in "Did a 1-Dimensional Magnet Detect a 248-Dimensional Lie Algebra?". All these 3 papers will appear together with these notes.

We will conclude this section with a bit of an overview of what we have done.

2.5.4 Multiple deformation

As the title of the Zamolodchikov's paper claims, the IQFT's can come from Conformal Field Theories. We will introduce CFT in more details in the future. The Ising model at $T = T_C$ and in the absence of the magnetic field is right at the phase transition, we say that it is critical or massless theory. In QFT mass gives a characteristic length scale with which

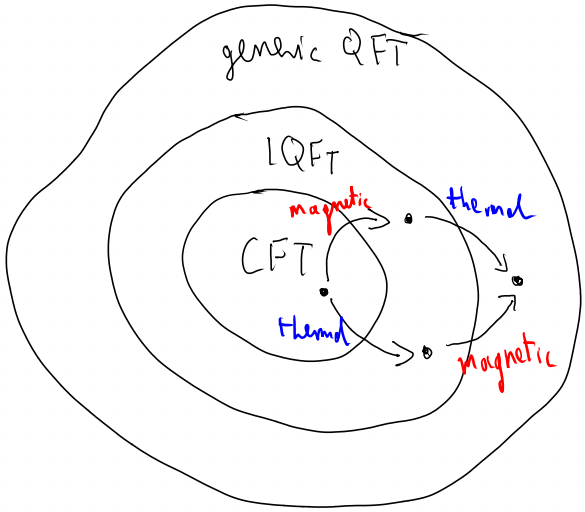


Figure 2.16: Landscape of the field theories in $1+1$d.

the correlation functions decay exponentially. In the massless limit, correlation functions can exhibit power law decay. Critical theories are described by Conformal Field Theories, there is a CFT corresponding to Ising - it is simply the theory of free, massless fermions.

From this perspective we can think of the massive theory as a deformed (perturbed) CFT. We have seen that in the case of Ising model there are two deformations: thermal and magnetic, that lead to IQFT's. We will see that in general, there is a kind of universal construction that shows which deformations of the CFT leads to IQFT. This was the main idea that Zamolodchikov explored in his work. The essential point is that CFT is more than an integrable theory. So we have a lot of conserved charges. Deforming it we might kill them all, or end up with a "smaller" but still infinite set of them. Following this idea one can show that magnetic deformation leads to the set of conserved charges that we used before

$$s = 1, 7, 11, 13, 17, 19, 23, 29 \,(\mathrm{mod}\, 30), \tag{2.205}$$

We will show this in the future.

For now let us ask what kind of theory we find when we do both deformation at the same time. This actually makes all the conserved charges but momentum and energy to disappear and we end up with a "standard" QFT. In general it is believed (however no proofs are available) that there are no multiple deformations of CFT that would lead to IQFT.

Chapter 3

Further developments - in progress

3.1 Thermodynamic Bethe Ansatz

3.2 Generalized hydrodynamics

3.3 Form factors

Chapter 4

Conformal field theories