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A short introduction to Generalized Hydrodynamics

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ABSTRACT

These are notes based on lectures given at the 2021 summer school on *Fundamental Problems in Statistical Physics XV*. Their purpose is to give a very brief introduction to *Generalized Hydrodynamics*, which provides a description of the large scale structure of the dynamics in quantum integrable models. The notes are not meant to be comprehensive or provide an overview of all relevant literature, but rather give an exposition of the key ideas for non-experts, using a simple fermionic tight-binding model as the main example.

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1. Introduction

The notes are based on lectures given at the summer school on *Fundamental Problems in Statistical Physics XV*. Their purpose is to give a very brief introduction to the exciting recent developments in *Generalized Hydrodynamics* [1,2]. They are not meant to be comprehensive or provide an overview of all relevant literature. They are several recent reviews and lecture notes [3–5] which provide much more detailed discussions and I strongly encourage the interested reader to consult in particular the following

1. B. Doyon, *Lecture notes on Generalized Hydrodynamics*, SciPost Phys. Lect. Notes p. 18 (2020).
2. V. Alba, B. Bertini, M. Fagotti, L. Piroli and P. Ruggeria, *Generalized-Hydrodynamic approach to Inhomogeneous Quenches: Correlations, Entanglement and Quantum Effects*, Journal of Statistical Mechanics: Theory and Experiment 2021(11), 114004 (2021).

1.1. Quantum quenches and experiments

A *quantum quench* is a particular protocol for driving many-particle quantum system out of equilibrium. It is defined as follows.

1. The starting point is a many-particle system in a large, finite volume L with Hamiltonian H.
2. The system is then prepared in an initial state $|\Psi(0)\rangle$, or an initial density matrix $\hat{\rho}(0)$, that has non-zero overlaps with exponentially many (in system size) eigenstates of H. The initial state should have good clustering properties[1] and is often taken to be a lowly entangled state.
3. At later times the quantum state describing the system is then given by the solution of the time-dependent Schrödinger equation
$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle. \tag{1}$$
4. The objective is to study expectation values of local operators $\mathcal{O}_A$ in the thermodynamic limit
$$\lim_{L\to\infty}\langle\Psi(t)|\mathcal{O}_A|\Psi(t)\rangle. \tag{2}$$

Here we define *local operators* as acting as the identity outside a finite, connected spatial region in the infinite volume limit. For a quantum spin chain operators of the form $\sigma_{j_1}^{\alpha_1}\dots\sigma_{j_\ell}^{\alpha_\ell}$ where $j_k\in[a,b]$ with a,b fixed are local. As we will see later locality is a very important concept in non-equilibrium dynamics. I note that this notion of locality is very strong and a weaker notion of *quasi-locality* [6] sufficies for our purposes[2] Often the Hamiltonian depends on a parameter h such a magnetic field or interaction strength, and a popular way of defining a quantum quench is then to take $|\Psi(0)\rangle$ as the ground state of $H(h_0)$, and consider time evolution under the Hamiltonian $H(h_1)$ with $h_1\neq h_0$. This corresponds to an instantaneous "quench" of h at time $t=0$ from h_0 to h_1.

The theoretical quantum quench protocol introduced above is inspired by cold atom experiments. In order to make the connection more concrete I now present a brief cartoon of experiments on ultra-cold bosonic Rb atoms carried out e.g. in Jörg Schmiedmayer's group in Vienna [7–11]. The Hamiltonian describing the atoms is to a good approximation
$$H(t)=\sum_j\left[-\frac{\hbar^2\nabla_j^2}{2m}+V(\boldsymbol{r}_j,t)\right]+\frac{g}{2}\sum_{j\neq k}\delta^{(3)}(\boldsymbol{r}_j-\boldsymbol{r}_k), \tag{3}$$

[1] By this we mean that connected correlation functions of local operators go to zero in the limit of large separations between them.

[2] We can loosely define quasi-local operators as having the property that they can be approximated by a sequence of local operators $\mathcal{O}_A$ defined on a finite, connected region A of linear size $|A|$ such that $\|\mathcal{O}-\mathcal{O}_A\|$ decays faster than any power of $|A|$.

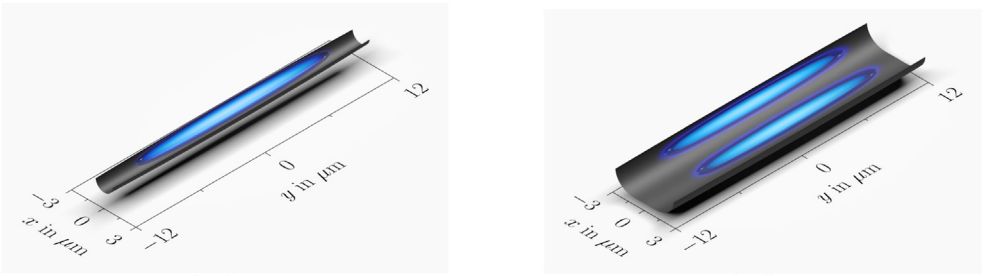

Fig. 1. By changing the transverse confining potential a one dimensional Bose gas (a) is "split into two" (b).

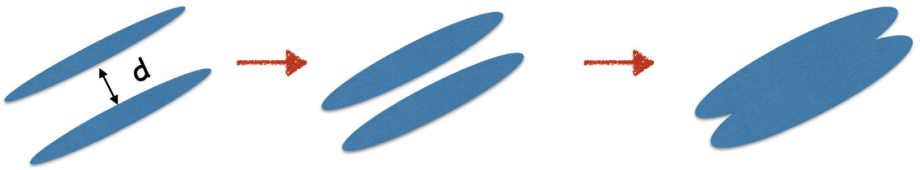

Fig. 2. After switching off the confining potential the atomic clouds expand in three dimensions and eventually overlap.

where $V(\boldsymbol{r}_j, t)$ is a confining potential that is varied in a time-dependent way. The potential is separable in the sense that $V(\boldsymbol{r}_j, t) = \frac{1}{2}m\omega_\parallel^2 x_j^2 + V_\perp(y_j, z_j, t)$ and $V_\perp$ can be tuned in such a way that the transverse degrees of freedom are essentially projected to the ground state(s) of the single-particle Hamiltonian

$$H_{\perp,0} = \sum_j -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}\right] + V_\perp(y_j, z_j, t). \tag{4}$$

By choosing the transverse confining potential to be very tight and having a single minimum the system at time $t = 0$ can be prepared in a low temperature thermal state of the one dimensional Hamiltonian

$$H(0) \approx \sum_j \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2}m\omega_\parallel^2 x_j^2\right] + c\sum_{j<k}\delta(x_j - x_k)\,, \tag{5}$$

where the interaction strength c is proportional to g and overlap integrals of the eigenfunctions of the confining potential, see e.g. Ref. [11]. In the experiments $V_\perp(y_j, z_j, t)$ is then changed in a time-dependent fashion so that one ends up with a tight double-well potential in the transverse direction, *cf.* [11] for a theoretical description. Neglecting the higher transverse modes (as they have very high energies) then leads to a Hamiltonian that in second quantization takes the form

$$H(t_0) \approx \int dx \sum_a \left[\Phi_a^\dagger(x)\Big(-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{m\omega_\parallel^2}{2}x^2\Big)\Phi_a(x) + c\Phi_a^\dagger(x)\Phi_a^\dagger(x)\Phi_a(x)\Phi_a(x)\right]. \tag{6}$$

Here $a = 1, 2$ label the two wells, *cf.* Fig. 1, and $\Phi_a(x)$ are canonical Bose fields obeying commutation relations $[\Phi_a(x), \Phi_b^\dagger(y)] = \delta_{a,b}\delta(x-y)$. The splitting process leaves the system in some initial state $|\Psi(t_0)\rangle$ (or more generally some initial density matrix $\hat{\rho}(t_0)$) that is not an eigenstate of $H(t_0)$: in this sense the situation is analogous to a quantum quench. The system is now left to evolve in time governed by the Hamiltonian (6). At a time t_1 the confining potential is switched off and the two clouds of atoms start to expand freely in three dimensions, *cf.* Fig. 2. Eventually they overlap and at a time t_2 the density of atoms is measured. The expansion can be easily modelled as the atoms effectively do not interact. One therefore can integrate the Heisenberg equations of motion for the measured observable (the particle density) backwards and relate it to an operator in the split one dimensional Bose gas at time t_1. One finds that the measured density is [12]

$$\hat{\rho}_{\rm tof}(x, \mathbf{r}, t_2) \approx \sum_{a,b=1}^{2}\int dx_1 dx_2\; g_{ab}(t_2 - t_1, \mathbf{r}, x - x_1, x - x_2)\Phi_a^\dagger(x_1, t_1)\Phi_b(x_2, t_1)\,, \tag{7}$$

where $\mathbf{r}$ denote the transverse directions and $g_{ab}(t, \mathbf{r}, x_1, x_2)$ are known functions. Repeating the experiment many times then provides access to e.g. the expectation value (7) in the split, one-dimensional Bose gas after a period of non-equilibrium evolution.

1.2. Homogeneous quantum quenches and local relaxation

We first focus on the simpler case of *homogeneous* systems where both the post-quench Hamiltonian and the initial state are translationally invariant [13].[3] The first question we ask is whether after a quantum quench a many-particle system somehow relaxes, i.e. whether if we wait long enough the quantum mechanical probability distributions describing the outcomes of measurements become time independent. This is equivalent to the question whether the double limit

$$\lim_{t\to\infty}\lim_{L\to\infty}\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle \tag{8}$$

exists for all Hermitian operators $\mathcal{O}$. We note that the order of limits is crucial here. It is easy to see that this limit cannot exist for all observables. Indeed, let $|n\rangle$ be the eigenstates of the Hamiltonian describing the time evolution of our system and E_n the corresponding energies. Then

$$|\Psi(t)\rangle = \sum_n e^{-iE_n t}\langle n|\Psi(0)\rangle\ |n\rangle. \tag{9}$$

Now we can choose "observables" that never relax, e.g.

$$\mathcal{O} = \mathcal{O}^\dagger = |1\rangle\langle 2| + |2\rangle\langle 1|. \tag{10}$$

Indeed, we have

$$\langle\Psi(t)|\mathcal{O}|\Psi(t)\rangle = A\cos\big((E_1 - E_2)t + \varphi\big), \tag{11}$$

which shows that the expectation value of this particular observable is a periodic function of time. However, the operator $\mathcal{O}$ is typically highly non-local in space. This suggests that we should restrict our attention to *local measurements* and concomitantly *local operators* $\mathcal{O}_A$. For these our double limit generally exists, i.e.

$$\lim_{t\to\infty}\lim_{L\to\infty}\langle\Psi(t)|\mathcal{O}_A|\Psi(t)\rangle = \langle\mathcal{O}_A\rangle_{\rm stat}. \tag{12}$$

The physical picture underlying this fact is as follows. As $\mathcal{O}_A$ is a local operator it acts like the identity outside some finite spatial region A. In the infinite volume limit the complement of A simply acts like a bath on A and eventually leads to relaxation. One can reformulate this observation in terms of density matrices as follows. The density matrix of the entire system $\hat\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ in our case is a pure state and hence can never become time independent. On the other hand, the *reduced density matrix*

$$\hat\rho_A(t) = {\rm Tr}_{\bar A}\big[\hat\rho(t)\big] \tag{13}$$

describing the region A on which our observable acts ($\bar A$ is the complement of A) is a mixed state and hence can become time-independent at late enough times in the thermodynamic limit. Physically this relaxation can be understood by noting that $\bar A$ acts like an infinite bath on the finite subsystem A. While local relaxation has been established in a number of particular examples and is compatible with numerous numerical studies, finding a general mathematical proof is an important open problem. A natural question to ask at this point whether it is possible to describe the late-time limits of the expectation values of local operators in terms of a statistical ensemble. In other words, is it possible to find a time-independent density matrix $\hat\rho_{\rm SS}$ such that for any local operator $\mathcal{O}_A$ acting non-trivially only on a finite subsystem A

$$\lim_{t\to\infty}\lim_{L\to\infty}\langle\Psi(t)|\mathcal{O}_A|\Psi(t)\rangle = \lim_{L\to\infty}{\rm Tr}\big[\hat\rho_{\rm SS}\mathcal{O}_A\big]. \tag{14}$$

We note that in analogy to equilibrium statistical mechanics (where micro-canonical, canonical and grand canonical ensembles provide equivalent descriptions of finite subsystems in the thermodynamic limit) $\hat\rho_{\rm SS}$ is not unique. In order to determine $\hat\rho_{\rm SS}$ we employ the following ergodicity principle: *under time evolution the reduced density matrix of a finite subsystem will retain the minimal possible amount of local information on the initial state.* As we are dealing with an isolated quantum system with a Hamiltonian that has a local density H_j one piece of local information that is always retained is the energy density $\langle\Psi(0)|H_j|\Psi(0)\rangle$.

1.2.1. Local conservation laws

A *local conservation law* is a Hermitian operator $I^{(n)}$ that commutes with the Hamiltonian of our system and has a density $I_j^{(n)}$ that is a local operator as defined above, i.e.

$$I^{(n)} = \sum_j I_j^{(n)}\ , \quad [H, I^{(n)}] = 0. \tag{15}$$

[3] For lattice models we may break translational invariance to translations by n sites, where n is fixed.

We will be particularly interested in the situation where we have many local conservation laws that are mutually compatible, i.e.

$$[I^{(n)}, I^{(m)}] = 0. \tag{16}$$

We stress that the conservation laws we have in mind here are extensive. The existence of a local conservation law has important consequences for the steady state density matrix $\hat{\rho}_{\rm SS}$ in translationally invariant cases. By (15) we have

$$\langle\Psi(t)|I^{(n)}|\Psi(t)\rangle = \text{time independent}. \tag{17}$$

Translational invariance then implies that

$$\lim_{L\to\infty}\frac{1}{L}\sum_j \langle\Psi(t)|I_j^{(n)}|\Psi(t)\rangle = \lim_{L\to\infty}\langle\Psi(t)|I_j^{(n)}|\Psi(t)\rangle. \tag{18}$$

Combining (18) with (17) we conclude that

$$\lim_{L\to\infty}\langle\Psi(0)|I_j^{(n)}|\Psi(0)\rangle = \lim_{t\to\infty}\lim_{L\to\infty}\langle\Psi(t)|I_j^{(n)}|\Psi(t)\rangle = {\rm Tr}\left[\hat{\rho}_{\rm SS}\, I_j^{(n)}\right], \tag{19}$$

where in the last step we have used that $I_j^{(n)}$ are local operators. This tells us that $\hat{\rho}_{\rm SS}$ *retains information about the expectation values of all local conservation laws in the initial state.*

1.2.2. Thermalization

As we are dealing with an isolated quantum system energy is always conserved

$$e_0 = \lim_{L\to\infty}\frac{\langle\Psi(t)|H|\Psi(t)\rangle}{L} = \lim_{L\to\infty}\langle\Psi(t)|H_j|\Psi(t)\rangle. \tag{20}$$

This is the minimal amount of information on the initial state $|\Psi(0)\rangle$ that gets retained under the dynamics. If there are no conserved quantities other than energy the system *thermalizes* at late times after a quantum quench. The steady state density matrix is then given by a finite temperature (equilibrium) ensemble constructed as follows. We define a Gibbs density matrix

$$\hat{\rho}_{\rm GE} = \frac{e^{-\beta_{\rm eff}H}}{Z_{\rm GE}}, \tag{21}$$

and fix the effective temperature $\beta_{\rm eff}^{-1}$ by requiring it to correspond to the energy density established by the choice of initial state

$$e_0 = \lim_{L\to\infty}\frac{{\rm Tr}\left[\hat{\rho}_{\rm GE}\, H\right]}{L}. \tag{22}$$

Under this choice we have

$$\hat{\rho}_{\rm SS} = \hat{\rho}_{\rm GE}\ . \tag{23}$$

We could have equally well chosen a micro-canonical description

$$\hat{\rho}'_{\rm SS} = \hat{\rho}_{\rm MC} = \frac{1}{\mathcal{N}}\sum_{|E_n - Le|<\epsilon}|n\rangle\langle n|, \tag{24}$$

where $|n\rangle$ are energy eigenstates with energy E_n and $\mathcal{N}$ is a normalization factor that ensures that ${\rm Tr}(\hat{\rho}_{\rm MC}) = 1$. While we of course have $\hat{\rho}_{\rm GE} \neq \hat{\rho}_{\rm MC}$, both density matrices lead to identical *reduced density matrices* for any finite subsystem A in the thermodynamic limit

$$\lim_{L\to\infty}{\rm Tr}_{\bar{A}}(\hat{\rho}_{\rm GE}) = \lim_{L\to\infty}{\rm Tr}_{\bar{A}}(\hat{\rho}_{\rm MC}). \tag{25}$$

Finally we note that averaging over a micro-canonical shell is not required as long we use a typical energy eigenstate to define our micro-canonical ensemble, which then takes the simple form

$$\hat{\rho}'_{\rm MC} = |n\rangle\langle n|\ . \tag{26}$$

Drawing an energy eigenstate at random out of our micro-canonical window provides us with a typical state with a probability that is exponentially close (in L) to one.

1.2.3. Non-equilibrium steady states and generalized gibbs ensembles

If we have additional conservation laws with local densities $I_j^{(n)}$ the system cannot thermalize because

$$\lim_{L\to\infty}\langle\Psi(0)|I_j^{(n)}|\Psi(0)\rangle = {\rm Tr}\left[\hat{\rho}_{\rm SS}\, I_j^{(n)}\right], \tag{27}$$

which tells us that the system retains more information on the initial state than just its energy density. What should the ensemble describing the steady state then be? The answer to this question is provided by Rigol et al. [14] (see also Jaynes [15]): we should maximize the entropy under the constraints (27). This leads to a *generalized Gibbs ensemble*

$$\hat{\rho}_{\rm GGE} = \frac{1}{Z_{\rm GGE}} e^{-\sum_n \lambda_n I^{(n)}}, \tag{28}$$

where the Lagrange multipliers λ_n are fixed by

$$i^{(n)} = \lim_{L\to\infty} \langle \Psi(0)|I_j^{(n)}|\Psi(0)\rangle = {\rm Tr}\left[\hat{\rho}_{\rm GGE}\, I_j^{(n)}\right]. \tag{29}$$

We note that solving this system of equations is a difficult task in general. Alternatively we may employ a *generalized micro-canonical ensemble* [16,17]:

$$\hat{\rho}_{\rm GMC} = |\Phi_L\rangle\langle\Phi_L| \ , \tag{30}$$

where $|\Phi_L\rangle$ be a simultaneous eigenstate of all local conservation laws $I^{(n)}$ such that

$$\lim_{L\to\infty}\left[\frac{I^{(n)}}{L} - i^{(n)}\right]|\Phi_L\rangle = 0. \tag{31}$$

1.2.4. Approach to the steady state

In integrable models how fast the expectation value of a given operator approaches its steady state value depends on its locality properties *relative to the elementary excitations in the model* [18]. Expectation values of operators that are local relative to the elementary excitations approach their stationary values in a power-law fashion [13]. The spatial "size" ℓ of the operator under consideration[4] together with the maximal propagation velocity of elementary excitations $v_{\rm max}$ provides a time scale $t_\ell = \ell/2v_{\rm max}$ after which relaxation to the steady state begins [19]. The rule of thumb is that the more local an operator is, the faster its expectation value relaxes towards its steady-state value.

1.3. Summary

Integrable models with short-range interactions prepared in homogeneous initial states that have good clustering properties relax locally to non-thermal stationary states, which are completely specified by the expectation values of the (quasi)local conservation laws. In terms of the time-evolving density matrix $\hat{\rho}(t)$ of the system this statement reads

$$\lim_{t\to\infty}\lim_{L\to\infty} {\rm Tr}_{\bar{A}}\left[\hat{\rho}(t)\right] = \lim_{L\to\infty} {\rm Tr}_{\bar{A}}\left[\hat{\rho}_{\rm GGE}\right] = \lim_{L\to\infty} {\rm Tr}_{\bar{A}}\left[\hat{\rho}_{\rm GMC}\right]. \tag{32}$$

where $\bar{A}$ is the complement of an arbitrary, finite subsystem A. If the following we will be interested in the situation where our initial density matrix is inhomogeneous. The idea there is that we still have local relaxation, but in a *space and time dependent way*, i.e.

$$\lim_{t\to\infty}\lim_{L\to\infty} {\rm Tr}_{\bar{A}}\left[\hat{\rho}(t)\right] = \lim_{t\to\infty} \hat{\rho}_{{\rm GMC},A}(t). \tag{33}$$

2. Brief introduction to quantum integrable models I: free theories

The simplest integrable models are free theories and arguably the simplest free theory is the tight-binding chain

$$H = \sum_{j=1}^{L} -J(c_j^\dagger c_{j+1} + {\rm h.c.}) - \mu c_j^\dagger c_j = \frac{1}{L}\sum_k \epsilon(k)\hat{n}(k) \ , \tag{34}$$

where c_j and $c_j^\dagger$ are fermionic creation and annihilation operators on site j and

$$\epsilon(k) = -2J\cos(k) - \mu \ , \quad \hat{n}(k) = c^\dagger(k)c(k) \ , \quad c(k) = \sum_{j=1}^{L} c_j e^{-ikj}. \tag{35}$$

Imposing periodic boundary conditions quantizes the momenta

$$k_j = \frac{2\pi j}{L} \ , \quad j = 1, 2, \ldots, L. \tag{36}$$

The momentum space operators obey canonical anticommutation relations of the form

$$\{c(p), c^\dagger(q)\} = L\delta_{p,q} \ . \tag{37}$$

[4] We define this as the size of the connected spatial region on which the operator acts non-trivially.

The 2^L energy eigenstates are conveniently expressed in the momentum representation

$$|k_1,\ldots,k_N\rangle = \frac{1}{L^{N/2}}\prod_{j=1}^{N} c^\dagger(k_j)|0\rangle\ , \quad k_1<k_2<\cdots<k_N\ ,$$
$$H|k_1,\ldots,k_N\rangle = \left(\sum_{j=1}^{N}\epsilon(k_j)\right)|k_1,\ldots,k_N\rangle. \tag{38}$$

2.1. Local conservation laws and associated currents

It follows from the representation (34) that all mode occupation numbers are conserved

$$[H,\hat{n}(k_j)]=0=[\hat{n}(k_j),\hat{n}(k_\ell)]\ . \tag{39}$$

They are in one-to-one correspondence with mutually commuting, extensive conservation laws with local densities by[5]

$$Q^{(n,\alpha)} = -(-i)^\alpha J\sum_{j=1}^{L}\left[c_j^\dagger c_{j+n}+(-1)^\alpha c_{j+n}^\dagger c_j\right]\equiv\frac{1}{L}\sum_{j=1}^{L}\epsilon^{(n,\alpha)}(k_j)\,\hat{n}(k_j)\ , \quad \alpha=0,1. \tag{40}$$

We have $H=Q^{(1,0)}$, $[Q^{(n,\alpha)},Q^{(m,\beta)}]=0$ and

$$\epsilon^{(n,0)}(k)=-2J\cos(nk)\ , \quad \epsilon^{(n,1)}(k)=-2J\sin(nk)\ . \tag{41}$$

The mode occupation operators can be expressed as linear combinations of the conservation laws as

$$\hat{n}(k_j)=-\frac{1}{2J}\sum_{n=0}^{L-1}e^{-ik_jn}\left[Q^{(n,0)}+iQ^{(n,1)}\right]. \tag{42}$$

The Heisenberg equations of motion for the densities of the conservation laws take the form of continuity equations

$$\frac{d}{dt}Q_j^{(n,\alpha)}=J_{j-1}^{(n,\alpha)}-J_j^{(n,\alpha)}\ ,$$
$$J_j^{(n,\alpha)}=J^2i^{1-\alpha}\left[c_j^\dagger c_{j+n+1}-c_{j+1}^\dagger c_{j+n}+(-1)^\alpha(c_{j+n}^\dagger c_{j+1}-c_{j+n+1}^\dagger c_j)\right]. \tag{43}$$

2.2. Macro-states

In the thermodynamic limit physical properties are described in terms of *macro-states* defined as follows. Consider L, N very large with $D=N/L$ fixed and let $0\le\rho(k)\le 1$ be a given function. The family of energy eigenstates $\{|k_1,\ldots,k_N\rangle\}$ such that

$$\frac{2\pi}{L}\times\ \text{number of } k_j \text{ in } [k,k+\Delta k]=\rho(k)\Delta k \tag{44}$$

is called a *macro-state*. We have

$$\frac{1}{L}Q^{(n,\alpha)}|k_1,\ldots,k_N\rangle=\frac{1}{L}\sum_{j=1}^{N}\epsilon^{(n,\alpha)}(k_j)|k_1,\ldots,k_N\rangle\ ,$$
$$\frac{1}{L}\sum_{j=1}^{N}\epsilon^{(n,\alpha)}(k_j)=\int_{-\pi}^{\pi}\frac{dk}{2\pi}\,\rho(k)\epsilon^{(n,\alpha)}(k)+o(L^0). \tag{45}$$

We see that the extensive parts of the eigenvalues only depends on the density $\rho(k)$, i.e. is the same for all micro-states. Conversely, if we fix the expectation values of all local conservation laws in some quantum state in the thermodynamic limit

$$\lim_{L\to\infty}\frac{1}{L}\mathrm{Tr}\left[\hat{\rho}Q^{(n,\alpha)}\right]=q^{(n,\alpha)}\ , \tag{46}$$

then relation (42) fixes a unique macro-state

$$\rho(k)=\lim_{L\to\infty}-\frac{1}{2JL}\sum_{n=0}^{L-1}e^{-ikn}\left[q^{(n,0)}+iq^{(n,1)}\right]. \tag{47}$$

[5] In these notations $Q^{(0,0)}=-2J\hat{N}$, where $\hat{N}$ is the fermion number operator.

2.2.1. Counting micro-states

There are clearly many ways of fulfilling (44) for a given density $\rho(k)$. The total number of possible k_j values in the interval $[k, k+\Delta k]$ is $\Delta n_{\rm vac} = [L\Delta k/2\pi]$, where $[x]$ denotes the integer part of x. Of these "vacancies" $\Delta n_{\rm p} = [\rho(k)L\Delta k/2\pi]$ are occupied. The number of ways of distributing $\Delta n_{\rm p}$ particles among $\Delta n_{\rm vac}$ vacancies is

$$\binom{\Delta n_{\rm vac}}{\Delta n_{\rm p}} = \frac{(\Delta n_{\rm vac})!}{(\Delta n_{\rm p})!\,(\Delta n_{\rm vac}-\Delta n_{\rm p})!}. \tag{48}$$

This first of all shows that in general exponentially many (in L) micro-states correspond to each macro-state. Using that the entropy our macro-state is given by $S = \ln(\#\text{ of micro-states})$, the contribution arising from reordering momenta in the interval $[k, k+\Delta k]$ is

$$\Delta S = \ln\binom{\Delta n_{\rm vac}}{\Delta n_{\rm p}} \simeq \frac{-L\Delta k}{2\pi}\left[\rho(k)\ln\big(\rho(k)\big) + \big(1-\rho(k)\big)\ln\big(1-\rho(k)\big)\right], \tag{49}$$

where we have used Stirling's formula in the second step. Dividing the interval $[0, 2\pi]$ into bins of size Δk, using (49) for each bin, and finally approximating the sum over bins of size Δk by an integral, we obtain

$$S[\rho] = s[\rho]L = -L\int_0^{2\pi}\frac{dk}{2\pi}\left[\rho(k)\ln\big(\rho(k)\big) + \big(\rho_{\rm h}(k)\big)\ln\big(\rho_{\rm h}(k)\big)\right] + o(L), \tag{50}$$

where we have defined the density of holes as $\rho_{\rm h}(k) = 1-\rho(k)$.

2.2.2. Typical vs atypical states

It is clear from the above construction that any positive function $\rho(k)$ gives rise to a macro-state and these generically have finite entropy densities in the thermodynamic limit. Importantly these macro-states are in general not thermal. The thermal states are obtained by extremising the free energy per site

$$f[\rho] = \int_{-\pi}^{\pi}\frac{dk}{2\pi}\epsilon(k)\rho(k) - Ts[\rho]\ . \tag{51}$$

This gives

$$\frac{\delta f[\rho]}{\delta\rho(k)} = 0 \Longrightarrow \rho_{\rm th}(k) = \frac{1}{e^{\epsilon(k)/T}+1}\ . \tag{52}$$

Thermal states are by construction the maximal entropy states, i.e. the most likely states, at a given energy density. In free theories other macro-states exist at the same energy density, but their entropies are smaller than the one of the thermal state. This means in particular that if we select a micro-state with a given energy density at random, this state will be thermal with a probability that is exponentially close (in system size) to one. So typical states at a given energy density are thermal, but there are exponentially many "atypical" states as well, which differ from the thermal state by the values of the higher conservation laws and hence have different local properties. This situation generalizes to interacting integrable models [20], where such atypical states can have very interesting properties [21].

2.3. Expectation values of local operators in the thermodynamic limit

Let $|k_1,\ldots,k_N\rangle$ be a micro-state corresponding to a macro state with density $\rho(k)$. Then expectation values of fermion bilinears depend only on the macro-state up to finite-size corrections

$$\langle k_1,\ldots,k_N|c_j^\dagger c_\ell|k_1,\ldots,k_N\rangle = \frac{1}{L}\sum_{m=1}^{N}e^{-ik_m(j-\ell)} = \int_0^{2\pi}\frac{dk}{2\pi}\,\rho(k)\,e^{-ik(j-\ell)} + o(L^0). \tag{53}$$

By Wick's theorem this fact gets lifted to any multi-point correlation function involving a fixed, finite number of fermion operators. This in turn means that expectation values of any finite number of fermion operators calculated in different micro-states corresponding to the same macro-state differ only by finite-size corrections that vanish in the thermodynamic limit. Given a local operator $\mathcal{O}_j$ we introduce the following notations for later convenience

$$\langle\boldsymbol{\rho}|\mathcal{O}_j|\boldsymbol{\rho}\rangle = \lim_{\substack{N,L\to\infty\\ N/L=D\ \text{fixed}}}\langle k_1,\ldots,k_N|\mathcal{O}_j|k_1,\ldots,k_N\rangle\ . \tag{54}$$

2.4. "Excitations" over (finite entropy density) macro-states

Let us consider a micro-state $|k_1,\ldots,k_N\rangle$ corresponding to the macro-state with density $\rho(k)$. We can consider "excitations" over the micro-state by composing elementary particle and hole excitations:

- Particle excitations

$$c^\dagger(p)|k_1,\ldots,k_N\rangle\ , \quad p\notin\{k_j\}. \tag{55}$$

The energy and momentum of this excitation relative to those of $|k_1,\ldots,k_N\rangle$ are

$$\Delta E_p=\epsilon(p)\ , \qquad \Delta P_p=p\ , \tag{56}$$

where the dispersion $\epsilon(p)$ is given by (35).

- Hole excitations

$$c(q)|k_1,\ldots,k_N\rangle\ , \quad q\in\{k_j\}. \tag{57}$$

The energy and momentum of this excitation relative to those of $|k_1,\ldots,k_N\rangle$ are

$$\Delta E_h=-\epsilon(q)\ , \qquad \Delta P_h=-q. \tag{58}$$

In contrast to excitations over the ground state the energy of these "excitations" can be either positive or negative. In the thermodynamic limit we obtain families of excited states parametrized by their particle and hole momenta. Importantly they propagate with a group velocity that is simply the derivative of the "bare" dispersion relation

$$v(p)=\epsilon'(p)\ . \tag{59}$$

This is a special feature of free theories and is not the case for interacting integrable models.

3. Brief introduction to quantum integrable models II: interacting theories

We now want to generalize the results discussed above for free theories to interacting integrable models, i.e.

1. Identify local conservation laws $Q^{(n)}$.
2. Construct simultaneous eigenstates of H and $Q^{(n)}$.
3. Construct macro-states in the thermodynamic limit.
4. Work out stable excitations over these macro-states and determine their group velocities.

The example we will work with is the δ-function Bose gas [22], also known as the Lieb–Liniger model [23]

$$H=\int dx\ \Phi^\dagger(x)\left[-\frac{\hbar^2}{2m}\partial_x^2\right]\Phi(x)+c\int dx\big(\Phi^\dagger(x)\big)^2\big(\Phi(x)\big)^2\ . \tag{60}$$

Here $\Phi(x)$ is a complex Bose field with canonical commutation relations $[\Phi(x),\Phi^\dagger(y)]=\delta(x-y)$. In first quantization the Hamiltonian for N bosons reads

$$H=\sum_{j=1}^N-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2}+2c\sum_{j<k}\delta(x_j-x_k)\ . \tag{61}$$

It is customary to work in notations where $\hbar=1=2m$ and we do this from here on. The model (60) is approximately realized in cold-atom experiments [24–26], the main additional ingredient being the trapping potential, *cf.* our discussion in Section 1.1. The Lieb–Liniger model is integrable and local conservation laws can be constructed using the quantum inverse scattering method [22], and the first few read [27]

$$\begin{aligned}
&Q^{(0)}=\int dx\ \Phi^\dagger(x)\Phi(x)\ , \qquad Q^{(1)}=-i\int dx\ \Phi^\dagger(x)\partial_x\Phi(x)\ ,\\
&Q^{(2)}=H\ , \qquad Q^{(3)}=i\int dx\left[\Phi^\dagger(x)\partial_x^3\Phi(x)-\frac{3c}{2}\big(\Phi^\dagger(x)\big)^2\partial_x\big(\Phi(x)\big)^2\right].
\end{aligned} \tag{62}$$

The Heisenberg equations of motion for the densities of these conserved quantities take the form of continuity equations

$$\frac{\partial}{\partial t}Q^{(n)}(x)=i[H,Q^{(n)}(x)]=-\frac{\partial}{\partial x}J^{(n)}(x)\ . \tag{63}$$

The first two currents are

$$\begin{aligned}
&J^{(0)}=-i\int dx\left[\big(\partial_x\Phi^\dagger(x)\big)\Phi(x)-\Phi^\dagger(x)\partial_x\Phi(x)\right],\\
&J^{(1)}=-\frac{1}{2}\int dx\left[\partial_x^2\Phi^\dagger\Phi+\Phi^\dagger\partial_x^2\Phi-2\partial_x\Phi^\dagger\partial_x\Phi-2c\big(\Phi^\dagger\big)^2\big(\Phi\big)^2\right].
\end{aligned} \tag{64}$$

3.1. Simultaneous eigenstates of $Q^{(n)}$

Simultaneous eigenstates of the Hamiltonian and the conservation laws are constructed as follows. Working in the position representation

$$|\chi\rangle = \frac{1}{\sqrt{N!}} \int dx_1 \dots dx_N \ \chi(x_1, \dots, x_N)\ \Phi^\dagger(x_1) \dots \Phi^\dagger(x_N)|0\rangle \tag{65}$$

the time-independent Schrödinger equation for N-particle states reads

$$\left[\sum_{j=1}^{N} -\frac{\partial^2}{\partial x_j^2} + 2c \sum_{k<j} \delta(x_j - x_k)\right] \chi(x_1, \dots, x_N) = E\chi(x_1, \dots, x_N)\ . \tag{66}$$

When the separation between any two particles is large the solution is simply a superposition of plane waves. The structure of the interaction term is such that the plane-wave solutions can be consistently matched whenever two particles meet such that the exact eigenfunctions take the form of the celebrated *Bethe ansatz*

$$\chi_\lambda(x_1, \dots x_N) = \frac{1}{\mathcal{N}} \sum_{P\in S_N} \mathrm{sgn}(P) e^{i\sum_{j=1}^N \lambda_{P_j} x_j} \prod_{j>k} \left[\lambda_{P_j} - \lambda_{P_k} - ic\right]\ , \tag{67}$$

where the normalization is $\mathcal{N}^2 = N! \prod_{j>k}[(\lambda_j - \lambda_k)^2 + c^2]$. If the model were not integrable we would have a superposition of plane waves with a set single-particle momenta $\{k_1, \dots, k_N\}$ in the sector $x_1 < x_2 < \dots < x_N$, but with a different set $\{p_1, \dots, p_N\}$ in the sector $x_2 < x_1 < x_3 < \dots < x_N$, such that the total energy and the total momentum are the same

$$E = \sum_{j=1}^{N} k_j^2 = \sum_{j=1}^{N} p_j^2\ , \quad P = \sum_{j=1}^{N} k_j = \sum_{j=1}^{N} p_j\ . \tag{68}$$

The presence of the higher conservation laws $Q^{(n)}$ ensures that the set of single-particle momenta does not change between different sectors. The states $|\chi_\lambda\rangle$ corresponding to the wave functions (67) are in fact simultaneous eigenstates of all conserved charges

$$Q^{(n)}|\chi_\lambda\rangle = \left(\sum_{j=1}^{N} q^{(n)}(\lambda_j)\right) |\chi_\lambda\rangle\ , \quad q_n(\lambda) = \lambda^n. \tag{69}$$

As we need a description of the system in a large, finite volume we now impose periodic boundary conditions

$$\chi_\lambda(x_1, \dots, x_j + L, x_N) = \chi_\lambda(x_1, \dots x_N)\ . \tag{70}$$

As usual these lead to the quantization of the single-particle momenta, but unlike in the free case the resulting quantization conditions are very non-trivial

$$e^{i\lambda_j L} = -\prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}\ , \quad j = 1, \dots, N. \tag{71}$$

This set of equations is referred to as *Bethe equations*. Importantly, as a result of the interactions the "single-particle momenta" k_j are *state dependent* and correlated with one another via the quantization conditions (71). Taking the logarithm of these equations we obtain

$$\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi I_j\ , \qquad \theta(x) = 2\ \arctan\left(\frac{x}{c}\right)\ , \tag{72}$$

where I_j are integers (half-odd integers) for N odd (even). The key point is that each set $\{I_j\}$ of distinct (half-odd) integers is in one-to-one correspondence with a solution $\{\lambda_j\}$ of the Bethe equations, which in turn provides the wave-function $\chi_\lambda(x_1, \dots, n_N)$ of a simultaneous eigenstate of the Hamiltonian and the conserved charges. This set of states is complete.

3.2. Macro-states

We now want to construct macro-states along the same lines as for free theories. The main complication is that the quantization conditions (71), (72) are non-trivial and state-dependent. The way to deal with this complication is to work with the (half-odd) integers I_j, which are independent from one another. In analogy[6] with (44) we may define a density for $z_j = I_j/L$ through

$$L\chi(z)\Delta z = \ \text{number of}\ \frac{I_j}{L}\ \text{in}\ [z, z+\Delta z]. \tag{73}$$

[6] We adopt the customary normalization of the density $\rho(\lambda)$ in the interacting case, which differs from (44) by a factor of 2π.

A positive function $\chi(z)$ specifies a macro-state, and corresponding micro-states can the obtained by choosing sets $\{I_j\}$ distributed according to $\chi(z)$. In practice we require a formulation in terms of the "particle" distribution function of the roots λ_j of (71) defined by

$$L\rho(\lambda)\Delta\lambda = \text{ number of } \lambda_j \text{ in } [\lambda, \lambda+\Delta\lambda]. \tag{74}$$

This can be related to $\chi(z)$ by turning the sum over roots in (72) into an integral over the particle density $\rho(\lambda)$ in the thermodynamic limit

$$z_j = \frac{I_j}{L} = \frac{\lambda_j}{2\pi} + \frac{1}{2\pi L}\sum_{k=1}^{N}\theta(\lambda_j-\lambda_k) \simeq \frac{\lambda_j}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\ \theta(\lambda_j-\mu)\ \rho(\mu)\ . \tag{75}$$

In the thermodynamic limit we have

$$z(\lambda) = \frac{\lambda}{2\pi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu\ \theta(\lambda-\mu)\ \rho(\mu)\ . \tag{76}$$

The function $z(\lambda)$ is called *counting function*. It is easy to see that $z(\lambda)$ is a strictly monotonically increasing function. We are now in a position to relate $\chi(z)$ to $\rho(\lambda)$ by equating the numbers of roots and integers in corresponding intervals

$$\rho(\lambda)d\lambda = \underbrace{\chi(z(\lambda))}_{\vartheta(\lambda)}\frac{dz}{d\lambda}d\lambda\ . \tag{77}$$

The function $\vartheta(\lambda)$ is known as *occupation function*. This gives

$$\frac{1}{2\pi} + \int\frac{d\mu}{2\pi}K(\lambda-\mu)\ \rho(\mu) = \frac{\rho(\lambda)}{\vartheta(\lambda)}\ , \qquad K(\lambda) = \frac{2c}{c^2+\lambda^2}. \tag{78}$$

It is customary to define a *hole density* $\rho_h(\lambda)$ by

$$\frac{\rho(\lambda)}{\vartheta(\lambda)} = \rho(\lambda) + \rho_h(\lambda). \tag{79}$$

This establishes that rather than specifying a macro-state by the corresponding function $\chi(z)$, we can specify it through its root density $\rho(\lambda)$.

3.2.1. Expectation values of conserved charges in the thermodynamic limit

Let $|\lambda_1,\ldots,\lambda_N\rangle$ be a micro-state corresponding to a macro-state with root density $\rho(\lambda)$.[7] By translational invariance the densities $Q^{(n)}(x)$ of the conserved charges fulfil

$$\langle\lambda_1,\ldots,\lambda_N|Q^{(n)}(x)|\lambda_1,\ldots,\lambda_N\rangle = \frac{1}{L}\sum_{j=1}^{N}q^{(n)}(\lambda_j) = \int_{-\infty}^{\infty}d\lambda\ q_n(\lambda)\ \rho(\lambda) + o(L^0). \tag{80}$$

In the thermodynamic limit these expectation values do not depend on the choice of micro-state. For later convenience we introduce the notation

$$\langle\boldsymbol{\rho}|\mathcal{O}(x)|\boldsymbol{\rho}\rangle = \lim_{\substack{N,L\to\infty\\ N/L=D\ \text{fixed}}}\langle\lambda_1,\ldots,\lambda_N|\mathcal{O}(x)|\lambda_1,\ldots,\lambda_N\rangle\ . \tag{81}$$

For local operators these expectation values are independent of the sequence of micro-states chosen in the limiting procedure and depend only on the root density $\rho(\lambda)$ that specifies the macro-state [22,28,29].

3.3. Stable excitations over macro-states

We now turn to the construction of excitations over macro-states. Our discussion follows the textbook [22]. We start by specifying a macro-state through its root density $\rho(\lambda)$. We then determine the counting function from (76), invert it, and determine $\chi(z)$ from (77)

$$\chi(z) = \rho(\lambda(z))\frac{d\lambda}{dz}. \tag{82}$$

Finally we use $\chi(z)$ to generate a distribution of I_j/L that corresponds to our given macro-state.[8] We can then construct families of "excited states" as we now explain for the particular example of a "particle–hole" excitation. The latter is

[7] More precisely we consider $N, L \gg 1$, $N/L = D$ and impose that $\{\lambda_1,\ldots,\lambda_N\}$ solve the Bethe equations (72).

[8] The corresponding set of roots $\{\lambda_j\}$ is obtained by numerically solving the logarithmic form (72) of the Bethe equations, which is straightforward to do (and is much more stable than numerically solving (71)).

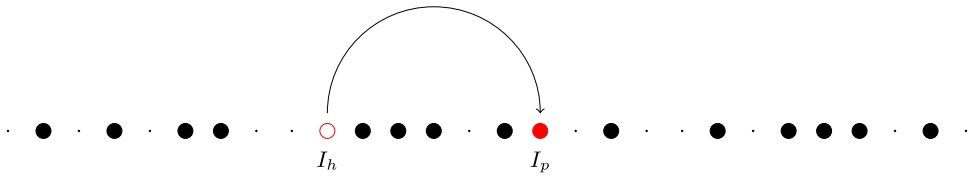

Fig. 3. Set of (half-odd) integers $\tilde{I}_j$ corresponding to a particle–hole excitation over a micro-state characterized by $\{I_j\}$: one (half-odd) integer is changed from I_h to I_p.

obtained by changing one of the (half-odd) integers I_j as shown in Fig. 3. Allowing I_h and I_p to vary gives rise to a two-parameter family of eigenstates. The Bethe equations for the eigenstates characterized by $\{I_j\}$ and $\{\tilde{I}_j\}$ are respectively

$$\begin{aligned}\lambda_j L + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) &= 2\pi I_j \ , \\ \widetilde{\lambda}_j L + \sum_{k=1}^{N} \theta(\widetilde{\lambda}_j - \widetilde{\lambda}_k) &= 2\pi \tilde{I}_j \ . \end{aligned} \tag{83}$$

The two equations involving I_p and I_h simply fix the positions of the corresponding momenta λ_p and λ_h. Turning sums into integrals and using that both micro-states correspond to the same macro-state, i.e. are described by the same root distribution $\rho(\lambda)$, we have

$$\lambda_a + \int_{-\infty}^{\infty} d\mu \ \theta(\lambda_a - \mu)\rho(\mu) = \frac{2\pi I_a}{L} + \mathcal{O}(L^{-1}) \ , \quad a = p, h. \tag{84}$$

Focusing on the subset of $N-2$ equations for which $I_j = \tilde{I}_j$ and taking differences we have

$$\lambda_j - \widetilde{\lambda}_j + \frac{1}{L}\sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) - \theta(\widetilde{\lambda}_j - \widetilde{\lambda}_k) = 0. \tag{85}$$

Next we use that $\lambda_j - \widetilde{\lambda}_j = \mathcal{O}(L^{-1})$ to Taylor-expand, which gives

$$L(\lambda_j - \widetilde{\lambda}_j)\left[1 + \frac{1}{L}\sum_k K(\lambda_j - \lambda_k)\right] - \sum_{k=1}^{N} K(\lambda_j - \lambda_k)(\lambda_k - \widetilde{\lambda}_k) = \theta(\lambda_j - \lambda_p) - \theta(\lambda_j - \lambda_h). \tag{86}$$

Defining a *shift function* by

$$F(\lambda_j) = \frac{\lambda_j - \widetilde{\lambda}_j}{\lambda_{j+1} - \lambda_j} \ , \tag{87}$$

we can turn (86) into an integral equation in the thermodynamic limit

$$F(\lambda|\lambda_p, \lambda_h) - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \ \vartheta(\lambda)K(\lambda - \mu) \ F(\mu|\lambda_p, \lambda_h) = \frac{\vartheta(\lambda)}{2\pi}\left[\theta(\lambda - \lambda_p) - \theta(\lambda - \lambda_h)\right]. \tag{88}$$

We can write this as

$$F(\lambda|\lambda_p, \lambda_h) = f(\lambda, \lambda_p) - f(\lambda, \lambda_h) \ , \tag{89}$$

where

$$f(\lambda|\lambda') - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \ \vartheta(\lambda)K(\lambda - \mu) \ f(\mu|\lambda') = \frac{\vartheta(\lambda)}{2\pi}\theta(\lambda - \lambda'). \tag{90}$$

Using the shift function we can work out the difference in eigenvalues of any of the conserved charges (69)

$$\begin{aligned}\Delta Q^{(n)} &\equiv \sum_{j=1}^{N}\left(q^{(n)}(\widetilde{\lambda}_j) - q^{(n)}(\lambda_j)\right) \approx q^{(n)}(\lambda_p) - q^{(n)}(\lambda_h) + \sum_{j=1}^{N-2} q^{(n)'}(\lambda_j)(\widetilde{\lambda}_j - \lambda_j) \\ &= \lambda_p^n - \lambda_h^n - \int_{-\infty}^{\infty} d\mu \ F(\mu) \ n\mu^{n-1} + \mathcal{O}(L^{-1}). \end{aligned} \tag{91}$$

In particular $n = 1, 2$ correspond to the excitation momentum and energy respectively. We stress that the latter can be positive or negative. We see that $\Delta Q^{(n)}$ can be expressed in the form

$$\boxed{\begin{aligned}&\Delta Q^{(n)} = q_n(\lambda_p) - q_n(\lambda_h)\ ,\\ &q_n(\lambda) = \lambda^n - \int_{-\infty}^{\infty}\frac{d\mu}{2\pi} n\mu^{n-1} f(\mu|\lambda)\ .\end{aligned}} \tag{92}$$

Crucially, we observe that the contributions of the particle and the hole excitation are *additive*. It also should be clear that the above construction straightforwardly generalizes to excitations involving multiple particles and holes. Specifying $n = 1, 2$ in (92) tells us that the energy and momentum of a particle with rapidity λ_p are given by $q_2(\lambda_p)$ and $q_1(\lambda_p)$ respectively. The corresponding group velocity is then

$$\boxed{v_\rho(\lambda_p) = \frac{q_2'(\lambda_p)}{q_1'(\lambda_p)}.} \tag{93}$$

Here we have introduced the index v_ρ to indicate that, in contrast to the case of non-interacting models, the group velocity now depends on the macro-state under consideration. Inspection of (93) and (92) shows that the group velocity in fact depends on the macro-state only through the occupation function $\vartheta(\lambda)$ (77).

4. Hydrodynamic description of non-equilibrium dynamics

In order to see how the non-equilibrium dynamics of many-particle quantum systems relates to classical hydrodynamics we follow Ref. [3].

Classical hydrodynamics in 1+1 dimensions describes the dynamics of fluids on intermediate time and length scales. It combines the continuity equation expressing mass conservation with the Euler equation

$$\begin{aligned}&\partial_t \rho(x,t) + \partial_x\left[v(x,t)\ \rho(x,t)\right] = 0\ ,\\ &\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = -\frac{1}{\rho(x,t)}\partial_x P[\rho(x,t)]\ .\end{aligned} \tag{94}$$

Here $\rho(x,t)$ is the density of the fluid, $v(x,t)$ the velocity field and the pressure P is assumed to be a function of the density only. Classical hydrodynamics can be reformulated in terms of continuity equations for conserved quantities by introducing the momentum $p(x,t) = v(x,t)\rho(x,t)$ and its associated current $j(x,t) = P + v^2(x,t)\rho(x,t)$

$$\begin{aligned}&\partial_t \rho(x,t) + \partial_x p(x,t) = 0\ ,\\ &\partial_t p(x,t) + \partial_x j(x,t) = 0\ .\end{aligned} \tag{95}$$

In these lectures we want to discuss how to obtain similar descriptions for many-particle quantum systems. The general idea is as follows. Let $\hat\rho(0)$ be the initial density operator of a many-particle system with (local) time-independent Hamiltonian H. Then the time evolution operator is

$$U(t) = e^{-iHt}\ , \tag{96}$$

and the Schrödinger equation implies that the density matrix at time t is

$$\hat\rho(t) = U(t)\hat\rho(0)U^\dagger(t)\ . \tag{97}$$

The quantities of interest are e.g. expectation values of local operators

$$\mathrm{Tr}\left[\hat\rho(t)\hat O(x)\right] = \mathrm{Tr}\left[\hat\rho(0)\hat O(x,t)\right]\ , \tag{98}$$

where we have defined the Heisenberg picture operator

$$\hat O(x,t) = U^\dagger(t)\hat O(x)U(t)\ . \tag{99}$$

As the time evolution (97) is unitary energy is always conserved. Let us assume that in addition to energy we have other local conservation laws $[Q^{(n)}, H] = 0 = [Q^{(n)}, Q^{(m)}]$

$$Q^{(n)} = \int dx\ Q^{(n)}(x)\ ,\quad Q^{(n)}(x) \text{ a local operator.} \tag{100}$$

The Heisenberg equations of motion for the density of the conservation laws take the form of continuity equations

$$\partial_t Q^{(n)}(x,t) = i[H, Q^{(n)}(x,t)] = -\partial_x J^{(n)}(x,t)\ , \tag{101}$$

where $J^{(n)}(x)$ are currents associated with the conserved quantities $Q^{(n)}$. Let us now consider the expectation values in some quantum mechanical state

$$\begin{aligned} q_n(x,t) &= \mathrm{Tr}\left[\hat{\rho}(t)Q^{(n)}(x)\right] , \\ j_n(x,t) &= \mathrm{Tr}\left[\hat{\rho}(t)J^{(n)}(x)\right] . \end{aligned} \tag{102}$$

The basic idea is that we now choose to look at our system at length and times scales that are large compared to some mesoscopic "Euler" scale. We will then assume that our system has *relaxed locally* to a state that by our "minimal information principle" is fully characterized by the expectation values of the conserved quantities, i.e. the $q_n(x,t)$. This implies in particular that the currents are functionals of the densities $q_m(x,t)$

$$j_n(x,t) = j_n[\{q_m(x,t)\}] , \tag{103}$$

and hence

$$\partial_x j_n(x,t) = \frac{\delta j_n[\{q_m\}]}{\delta q_m}\frac{\partial q_m(x,t)}{\partial x} \equiv \sum_m A_{n,m}[\{q_m(x,t)\}]\ \partial_x q_m(x,t) . \tag{104}$$

Assuming that the matrix A is diagonalizable

$$RAR^{-1} = \mathrm{diag}(v_1, v_2, \dots) , \quad R = R[\{q_m(x,t)\}] , \quad v_\alpha = v_\alpha[\{q_m(x,t)\}] , \tag{105}$$

and changing variables (locally) from $q_n(x,t)$ to $\vartheta_n(x,t)$ defined by $R_{n,m} = \frac{\delta q_n}{\delta \vartheta_m}$ we arrive at evolution equations for the "hydrodynamic normal modes"

$$\boxed{\partial_t \vartheta_n(x,t) + v_n(x,t)\partial_x \vartheta_n(x,t) = 0 .} \tag{106}$$

The difficulty in this description is hidden in the fact that the normal model velocities $v_n(x,t)$ depend on the conservation laws in an unknown way. As we will see, it is however possible to work them out exactly in the case of integrable models. This then allows one to determine $\vartheta_n(x,t)$, which by our assumption of local relaxation/minimal information completely fixes the density operator and thus expectation values of local operators!

4.1. "Derivation" of GHD in free theories using a local density approximation

In a homogeneous macro-state with density $\rho(k)$ we have (using translational invariance) in the notations of (54)

$$\begin{aligned} \langle \boldsymbol{\rho}|\mathrm{Q}_j^{(n,\alpha)}|\boldsymbol{\rho}\rangle &= \int \frac{dk}{2\pi}\epsilon^{(n,\alpha)}(k)\rho(k) , \\ \langle \boldsymbol{\rho}|\mathrm{J}_j^{(n,\alpha)}|\boldsymbol{\rho}\rangle &= \int \frac{dk}{2\pi}\epsilon'(k)\,\epsilon^{(n,\alpha)}(k)\rho(k) . \end{aligned} \tag{107}$$

Here we have used (40) and (43). Now imagine that we prepare our system in an equilibrium density matrix $\hat{\rho}$ that is not homogeneous but varies very slowly in space. Locally this density matrix will "look like" a macro-state with a position-dependent density $\rho_{x,0}(k)$, $x = ja_0$ with a_0 the lattice spacing, and in particular

$$\begin{aligned} \mathrm{Tr}\left[\hat{\rho}\mathrm{Q}_j^{(n,\alpha)}\right] &\approx \int \frac{dk}{2\pi}\epsilon^{(n,\alpha)}(k)\rho_{x,0}(k) , \\ \mathrm{Tr}\left[\hat{\rho}\mathrm{J}_j^{(n,\alpha)}\right] &\approx \int \frac{dk}{2\pi}\epsilon'(k)\,\epsilon^{(n,\alpha)}(k)\rho_{x,0}(k) . \end{aligned} \tag{108}$$

Let us now turn to the situation at sufficiently late times after our quantum quench. We expect that our system will have relaxed locally, and as long as the system varies very slowly in space we again can employ a description in terms of an appropriately chosen macro-state, which however now will depend on x and t

$$\begin{aligned} \mathrm{Tr}\left[\hat{\rho}\mathrm{Q}_j^{(n,\alpha)}(t)\right] &= \mathrm{Tr}\left[\hat{\rho}(t)\mathrm{Q}_j^{(n,\alpha)}\right] \approx \int \frac{dk}{2\pi}\epsilon^{(n,\alpha)}(k)\rho_{x,t}(k) , \\ \mathrm{Tr}\left[\hat{\rho}\mathrm{J}_j^{(n,\alpha)}(t)\right] &= \mathrm{Tr}\left[\hat{\rho}(t)\mathrm{J}_j^{(n,\alpha)}\right] \approx \int \frac{dk}{2\pi}\epsilon'(k)\,\epsilon^{(n,\alpha)}(k)\rho_{x,t}(k) . \end{aligned} \tag{109}$$

Using the equations of motion (43) for the charge densities in (109) we obtain an evolution equation for $\rho_{x,t}(k)$

$$\int \frac{dk}{2\pi}\epsilon^{(n,\alpha)}(k)\left[\partial_t\rho_{x,t}(k) + \epsilon'(k)\partial_x\rho_{x,t}(k)\right] = 0 . \tag{110}$$

As the $\epsilon^{(n,\alpha)}(k)$ form a basis of periodic functions on the interval $[-\pi,\pi]$ the term in brackets must vanish, i.e.

$$\boxed{\partial_t\rho_{x,t}(k) + \epsilon'(k)\partial_x\rho_{x,t}(k) = 0 .} \tag{111}$$

These are indeed the GHD equations for free fermions. The argument above is based on the principle of local relaxation. We stress that there is no finite time scale over which integrable models relax locally, instead (109) become exact only in the so-called *Euler scaling limit* $x, t \to \infty$, x/t fixed. At finite times there are power-law corrections to (109) and at sufficiently short times (109) generically do not hold.

4.2. Dynamics for weakly inhomogeneous initial states in free theories

The above "derivation" was rather hand waiving, and we now want to do a more proper job. The solution of the Heisenberg equation of motion for the fermion annihilation operator in momentum space is

$$c(p,t) = e^{-i\epsilon(p)t}c(p)\ . \tag{112}$$

Using this we can calculate the two-point functions for an arbitrary initial density matrix $\hat{\rho}(0)$

$$\begin{aligned} G(j,\ell;t) &= \mathrm{Tr}\left[\hat{\rho}(t)c_j^\dagger c_\ell\right] = \frac{1}{L^2}\sum_{p,q}\mathrm{Tr}\left[\hat{\rho}(0)c^\dagger(p)c(q)\right]e^{it[\epsilon(p)-\epsilon(q)]-ijp+i\ell q}\ , \\ F(j,\ell;t) &= \mathrm{Tr}\left[\hat{\rho}(t)c_j c_\ell\right] = \frac{1}{L^2}\sum_{p,q}\mathrm{Tr}\left[\hat{\rho}(0)c(p)c(q)\right]e^{-it[\epsilon(p)+\epsilon(q)]+ijp+i\ell q}\ . \end{aligned} \tag{113}$$

Assuming that $\hat{\rho}(0)$ is Gaussian all higher point correlators can then simply be obtained using Wick's theorem. This is the full description of the dynamics. In order to analyse weakly inhomogeneous initial states we introduce a Wigner function (for $L \to \infty$) by

$$\begin{aligned} W(z,p;t) &= \mathrm{Tr}\left[\hat{\rho}(0)\hat{W}(z,p;t)\right]\ , \\ \hat{W}(z,p;t) &= \int\frac{dq}{2\pi}c^\dagger\big(p-\frac{q}{2}\big)c\big(p+\frac{q}{2}\big)e^{iqz+it[\epsilon(p-\frac{q}{2})-\epsilon(p+\frac{q}{2})]}\ , \quad 2z\in\mathbb{Z}. \end{aligned} \tag{114}$$

The Wigner function is defined for half-integer values of z and has the following properties:

[W1.] The single-particle Green's function can be recovered from the Wigner function using

$$\mathrm{Tr}\left[\hat{\rho}(t)c^\dagger(p_1)c(p_2)\right] = \frac{1}{2}\sum_{2z\in\mathbb{Z}}e^{iz(p_1-p_2)}W(z,\frac{p_1+p_2}{2};t)\ . \tag{115}$$

[W2.] For homogeneous states $\hat{\rho}(0)$ the Wigner function reduces to the density of states in momentum space

$$\begin{aligned} W(z,p;t) &= \int\frac{dq}{2\pi}e^{iqz+it[\epsilon(p-\frac{q}{2})-\epsilon(p+\frac{q}{2})]}\underbrace{\mathrm{Tr}\left[\hat{\rho}(0)c^\dagger\big(p-\frac{q}{2}\big)c\big(p+\frac{q}{2}\big)\right]}_{2\pi\delta(q)\rho(p)} \\ &= \rho(p) = \sum_j\mathrm{Tr}\left[\hat{\rho}(0)c_{j+1}^\dagger c_1\right]e^{ipj}. \end{aligned} \tag{116}$$

[W3.] For states $\hat{\rho}(t)$ that are slowly varying in space and have short-ranged correlations the Wigner function can be interpreted as a position-dependent density of states in momentum space. To see this we note that the Wigner function can be expressed as

$$W(z,p;t) = \sum_{j,\ell}\mathrm{Tr}\left[\hat{\rho}(t)\ c_j^\dagger c_\ell\right]e^{ip(j-\ell)}\ \frac{\sin\big(\pi(z-\frac{j+k}{2})\big)}{\pi\big(z-\frac{j+k}{2}\big).} \tag{117}$$

Let A be a region centred around z such that its linear size $|A|$ is large compared to the correlation length $\xi(z)$ of the state $\hat{\rho}(t)$, but small compared to the scale $\zeta(z)$ on which $\hat{\rho}(t)$ becomes inhomogeneous in space. Let us further define an integer $m(z)$ such that $\xi(z) \ll m(z) \ll \zeta(z)$. All length scales depend on the position z. Then we have

$$\begin{aligned} W(z,p;t) &= \sum_n\sum_m\mathrm{Tr}\left[\hat{\rho}(t)\ c_{n+m}^\dagger c_n\right]e^{ipm}\frac{\sin\big(\pi(z-n-\frac{m}{2})\big)}{\pi\big(z-n-\frac{m}{2}\big)} \\ &\approx \sum_{m=-m(z)}^{m(z)}\mathrm{Tr}\left[\hat{\rho}(t)\ c_{[z]+m}^\dagger c_{[z]}\right]e^{ipm}\sum_{n\in A}\frac{\sin\big(\pi(z-n-\frac{m}{2})\big)}{\pi\big(z-n-\frac{m}{2}\big)} \\ &\approx \sum_{m=-m(z)}^{m(z)}\mathrm{Tr}\left[\hat{\rho}(t)\ c_{[z]+m}^\dagger c_{[z]}\right]e^{ipm}\ . \end{aligned} \tag{118}$$

As within $|A|$ our state is by assumption approximately homogeneous and has a finite correlation length this corresponds to a position-dependent density of states in momentum space, as advertised.

[W4.] The Heisenberg equations of motion for the Wigner function are linear

$$\partial_t \hat{W}(z,p;t) + \epsilon'(p)\left[\hat{W}(z+\frac{1}{2},p;t) - \hat{W}(z-\frac{1}{2},p;t)\right] = 0. \tag{119}$$

Given some initial values $W(z,p;0)$ the solution to (119) is obtain by Fourier methods

$$W(z,p;t) = \sum_{z'} W(z',p;0)\int_{-2\pi}^{2\pi}\frac{dQ}{4\pi}e^{iQ(z-z')-2it\epsilon'(p)\sin(Q/2)}\ . \tag{120}$$

4.2.1. Euler scaling limit

Let us now consider the Euler scaling limit

$$z = \Lambda\xi\ ,\quad t = \Lambda\tau\ ,\ \Lambda\to\infty\ ,\xi,\tau\ \text{fixed}\ , \tag{121}$$

and assume that

$$\lim_{\Lambda\to\infty} W(\Lambda\xi,p;\Lambda\tau) = \rho_{\xi,\tau}(p) \tag{122}$$

exists. Then we have from (119)

$$\boxed{\partial_\tau\rho_{\xi,\tau}(p) + \epsilon'(p)\partial_\xi\rho_{\xi,\tau}(p) = 0\ .} \tag{123}$$

This has the same form as the evolution equation (111), and by virtue of [W3] $\rho_{\xi,\tau}(p)$ can indeed be thought of as a "ray-dependent" mode occupation function. We note that the solutions to (123) have simple expressions in terms of the "initial conditions" at $\tau = 0$

$$\rho_{\xi,\tau}(p) = \rho_{\xi-\epsilon'(p)\tau}(p)\ . \tag{124}$$

In order for the Euler scaling limit to be non-trivial (in the sense that one obtains ray-dependent results) the initial momentum space Green's function $\mathrm{Tr}[\hat{\rho}(0)c^\dagger(p)c(q)]$ must exhibit some singularities as (113) can otherwise be determined by a simple stationary-phase approximation with saddle points only along the real axis.

4.2.2. Expectation values of local operators in free theories

Let us consider expectation values of local operators along rays in spacetime. The basic building block is the single-particle Green's function

$$G(j,\ell,t) = \mathrm{Tr}\left[\hat{\rho}(t)c_j^\dagger c_\ell\right] = \int\frac{dp_1dp_2}{(2\pi)^2}e^{-ip_1j+ip_2\ell}\underbrace{\mathrm{Tr}\left[\hat{\rho}(0)c^\dagger(p_1,t)c(p_2,t)\right]}_{\frac{1}{2}\sum_z e^{i(p_1-p_2)z}W(z,\frac{p_1+p_2}{2};t)}\ . \tag{125}$$

Let us then go over to the scaling variables (121)

$$\frac{j+\ell}{2} = \Lambda\xi\ ,\quad t = \Lambda\tau\ , \tag{126}$$

and change integration variables to

$$p_+ = \frac{p_1+p_2}{2}\ ,\quad p_- = \Lambda(p_1-p_2)\ . \tag{127}$$

In the limit $\Lambda\to\infty$ we have

$$\lim_{\Lambda\to\infty}G(j,\ell,t) = \lim_{\Lambda\to\infty}\int_{-\pi}^{\pi}\frac{dp_+}{2\pi}e^{-ip_+(j-\ell)}\int_{-2\Lambda(\pi-|p_+|)}^{2\Lambda(\pi-|p_+|)}\frac{dp_-}{2\pi}\frac{1}{2\Lambda}\sum_z e^{-ip_-(\xi-\frac{z}{\Lambda})}W(z,p_+;\Lambda\tau). \tag{128}$$

We then turn the sum over z into an integral

$$\frac{1}{2\Lambda}\sum_z e^{-ip_-(\xi-\frac{z}{\Lambda})}W(z,p_+;\Lambda\tau)\approx\int_{-\infty}^{\infty}dy e^{-ip_-(\xi-y)}\rho_{y,\tau}(p_+)\ , \tag{129}$$

which gives the final result

$$\lim_{\Lambda\to\infty}G(j,\ell;t)\approx\int\frac{dp_+}{2\pi}e^{-ip_+(j-\ell)}\rho_{\xi,\tau}(p_+)\equiv g_{\xi,\tau}(j-\ell)\ . \tag{130}$$

This is precisely what our GHD "phenomenology" predicts: Interpreting $\rho_{\xi,\tau}(p)$ as a mode occupation function along the ray characterized by ξ and τ we can construct a corresponding macro-state $|\rho_{\xi,\tau}\rangle$ by a family of Fock states with momenta distributed according to $\rho_{\xi,\tau}(p)$. GHD then predicts that

$$\lim_{\Lambda\to\infty}G(j,\ell;t) = \langle\rho_{\xi,\tau}|c_{j-\ell}^\dagger c_0|\rho_{\xi,\tau}\rangle = \int\frac{dp}{2\pi}e^{-ip(j-\ell)}\ \rho_{\xi,\tau}(p)\ . \tag{131}$$

As shown below in the scaling limit the anomalous Green's function vanishes. This is expected [13] because the time evolution operator has a U(1) symmetry $c_j \to c_j e^{i\varphi}$ related to particle number conservation. The initial density matrix may break this symmetry, resulting in a non-vanishing anomalous Green's function $F(j,\ell;0)$. However, as the system relaxes locally the U(1) symmetry gets restored, which is why $F(j,\ell;t)$ must vanish at late times. This means that the expectation value of an arbitrary local operators along a given ray can be calculated by Wick's theorem using the regular Green's function only. For example, along the ray $t=\Lambda\tau, j_\alpha=\Lambda\xi+\mathcal{O}(\Lambda^0)$

$$\mathrm{Tr}\left[\hat\rho(t)c^\dagger_{j_1}c^\dagger_{j_2}c_{j_3}c_{j_4}\right]\approx g_{\xi,\tau}(j_1-j_4)g_{\xi,\tau}(j_2-j_3)-g_{\xi,\tau}(j_1-j_3)g_{\xi,\tau}(j_2-j_4)\,. \tag{132}$$

4.2.3. Anomalous Green's function

We can see that in the scaling limit the anomalous Green's function vanishes by considering

$$\begin{aligned} &W_A(z,p;t)=\mathrm{Tr}\left[\hat\rho(0)\hat W_A(z,p;t)\right]\,,\\ &\hat W_A(z,p;t)=\int\frac{dq}{2\pi}c\big(p+\frac{q}{2}\big)c\big(-p+\frac{q}{2}\big)e^{iqz-it[\epsilon(p+\frac{q}{2})+\epsilon(p-\frac{q}{2})]}\,,\quad 2z\in\mathbb{Z}. \end{aligned} \tag{133}$$

We have

- The anomalous Green's function can be recovered from W_A using

$$\mathrm{Tr}\left[\hat\rho(t)c(p_1)c(p_2)\right]=\frac{1}{2}\sum_{2z\in\mathbb{Z}}e^{-iz(p_1+p_2)}W_A(z,\frac{p_1-p_2}{2};t)\,. \tag{134}$$

- The Heisenberg equations of motion give

$$i\partial_t W_A(z,p;t)=-2J\cos(p)\left[W_A(z+\frac{1}{2},p;t)+W_A(z-\frac{1}{2},p;t)\right]-2\mu W_A(z,p;t). \tag{135}$$

- Going over to scaling variables this becomes

$$\frac{1}{\Lambda}\partial_\tau W_A(\Lambda\xi,p;\Lambda\tau)\approx-2i\epsilon(p)W_A(\Lambda\xi,p;\Lambda\tau)\,, \tag{136}$$

  which shows that $W_A(\Lambda\xi,p;\Lambda\tau)$ vanishes in the scaling limit. Going beyond the scaling limit we see that the anomalous Green's function in position space in fact vanishes as a power law in t, i.e. slowly. This is important, as it imposes a restriction on the applicability of GHD: it will only work at times when the anomalous Green's function is small.[9]

4.2.4. Application of GHD away from the scaling regime

As we have seen above, in the Euler scaling limit GHD becomes exact. In practice one of course would like to apply it away from the scaling limit as well. To that end one writes the GHD equation in the original variables

$$\partial_t\rho_{x,t}(p)+\epsilon'(p)\partial_x\rho_{x,t}(p)=0\,, \tag{137}$$

and then uses that the solution of this equation is of the form

$$\rho_{x,t}(p)=\rho_{x-\epsilon'(p)t,0}(p)\,. \tag{138}$$

The "initial" value $\rho_{x,0}(p)$ is then approximated by the actual initial conditions

$$\rho_{x,0}(p)\approx W(x,p;0)\,. \tag{139}$$

Finally, the single-particle Green's function is approximated as

$$G(j,\ell;t)\approx\int\frac{dp}{2\pi}e^{ip(j-\ell)}\,\rho_{x,t}(p)\,. \tag{140}$$

Using Wick's theorem this allows us to obtain approximate results for multi-particle Green's functions as well.

We will see below how well this works in an explicit example.

4.2.5. Corrections to GHD

For free fermions it is possible to obtain a systematic expansion around the GHD limit [30,31]. In order to do so we carry out a gradient expansion of (119) beyond the leading order, i.e.

$$\partial_t W(x,p;t)+\epsilon'(p)\left[\partial_x W(x,p;t)+\frac{1}{24}\partial_x^3 W(x,p;t)+\ldots\right]=0. \tag{141}$$

Importantly, we also must account for the fact that away from the scaling limit the anomalous Wigner function does not vanish.

[9] For example there will be corrections to (132) involving the anomalous Green's function.

4.3. Example: Free expansion

Let us now consider the explicit example, where our initial density matrix correspond to a product state in position space where all sites $j \leq 0$ are occupied

$$|\Psi(0)\rangle = \prod_{j\leq 0} c_j^\dagger |0\rangle \ , \quad \hat{\rho}(0) = |\Psi(0)\rangle\langle\Psi(0)| \ . \tag{142}$$

We time-evolve this state with the tight-binding Hamiltonian (34). This problem is nice as it can be solved analytically [32,33]. Using that

$$\mathrm{Tr}[\hat{\rho}(0)c^\dagger(p)c(q)] = \sum_{n\leq 0} e^{in(p-q)} \ , \tag{143}$$

one finds

$$G(j,\ell;t) = \sum_{n=0}^{\infty} i^{\ell-j} J_{j+n}(2Jt) J_{\ell+n}(2Jt) \ , \tag{144}$$

where $J_n(z)$ are Bessel functions. Using the addition theorem for Bessel functions this can be simplified

$$G(j,j;t) = \frac{1}{2}\big[1 - J_0^2(2Jt)\big] + \begin{cases} -\sum_{m=1}^{j-1} J_m^2(2Jt) & \text{if } j > 0 \ , \\ \sum_{m=0}^{|j|} J_m^2(2Jt) & \text{if } j \leq 0 \ . \end{cases} \tag{145}$$

Similarly one has for $j > 0$ and $n \geq 1$

$$G(j,j+2n;t) = (-1)^n \Big\{ -\frac{1}{2}\sum_{k=0}^{2n} (-1)^k J_k(2Jt) J_{2n-k}(2Jt) - \sum_{k=1}^{j-1} J_k(2Jt) J_{2n+k}(2Jt) \Big\} . \tag{146}$$

The application of GHD proceeds as follows. We first determine the initial conditions for the Wigner function

$$W(z,p;0) = \sum_{j\leq 0} \frac{\sin\big(\pi(z-j)\big)}{\pi(z-j)} \approx \theta_H(-z) \ , \tag{147}$$

where $\theta_H(x)$ is the Heaviside function. Following the prescription (139) we then use this to fix

$$\rho_{x,0}(p) = \theta_H(-x) \ , \tag{148}$$

Here we have replaced the discrete lattice co-ordinate z by a continuous variable x. The rationale for doing this is that GHD is expected to work on a length scale that is large compared to the lattice spacing. The solution of the GHD equations is then

$$\rho_{x,t}(p) = \rho_{x-\epsilon'(p)t,0}(p) = \theta_H(-x+\epsilon'(p)t) \ . \tag{149}$$

Finally we use the relation between the Wigner function and the Green's function that holds in the scaling limit to obtain an approximate expression away from the scaling regime

$$G(j,\ell;t) \approx \int \frac{dp}{2\pi} e^{ip(j-\ell)} \, \rho_{\frac{j+\ell}{2},t}(p) = \int \frac{dp}{2\pi} e^{ip(j-\ell)} \, \theta_H\Big(-\frac{j+\ell}{2} + \epsilon'(p)t\Big) . \tag{150}$$

We similarly can determine the expectation values of the densities of the conserved charges (40), e.g.

$$\begin{aligned} q^{(2n,0)}(j+n,t) &\equiv \lim_{L\to\infty} \frac{1}{2}\mathrm{Tr}\big[\hat{\rho}(0)\big(c_j^\dagger c_{j+2n} + \text{h.c.}\big)\big] \\ &\approx \int \frac{dp}{2\pi} \cos(2np)\theta_H\big(-j-n+\epsilon'(p)t\big) = -\frac{\sin\big(2n \ \arcsin(\frac{j+n}{2Jt})\big)}{2\pi n}\theta_H\big(1-\frac{j+n}{2Jt}\big) . \end{aligned} \tag{151}$$

The result for the density profile in the Euler scaling limit is

$$\lim_{\substack{j,t\to\infty \\ j/t \ \text{fixed}}} \mathrm{Tr}\Big[\hat{\rho}(t)c_j^\dagger c_j\Big] = \begin{cases} 1 & \text{if } j < -v_{\max}t \ , \\ \frac{1}{\pi}\arccos\big(\frac{j}{2Jt}\big) & \text{if } -v_{\max}t < j < v_{\max}t \ , \\ 0 & \text{if } j > v_{\max}t \ , \end{cases} \tag{152}$$

where $v_{\max} = 2J$ is the maximal group velocity. In Fig. 4 we compare this asymptotic result to the density profile along rays j/t =fixed at finite times $Jt = 10$ and $Jt = 100$. We observe that the GHD result provides a rather good approximation already at times $Jt \sim 10$. To get a more precise understanding how the exact result approaches GHD in the scaling limit we show the expectation values of $q^{(2n,0)}(10,t)$ for $n = 1, 2$ in Fig. 5. Initially these expectation values are zero because

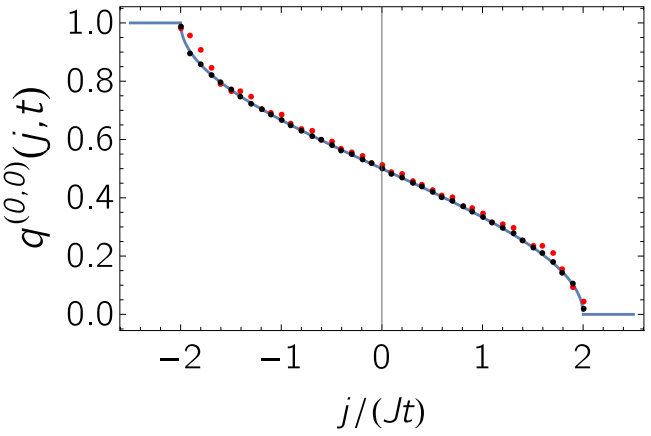

Fig. 4. Density profile along different "rays" j/Jt. Exact results at time $Jt = 10$ (red dots) and $Jt = 100$ (black dots) vs Euler scaling limit (blue line).

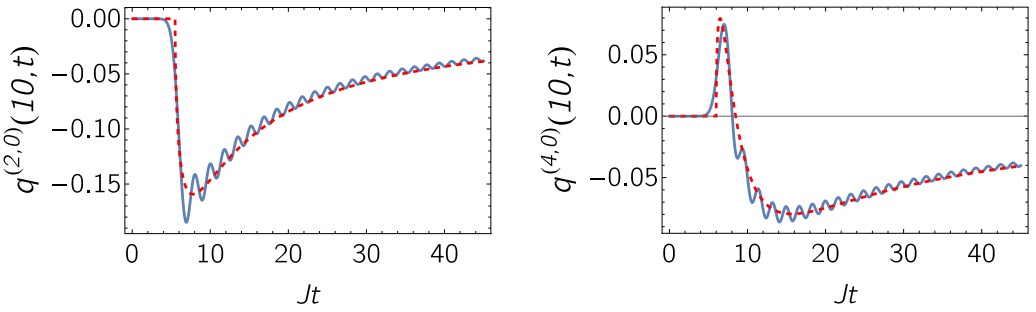

Fig. 5. Expectation values of the densities of the conserved charges $q^{(2n,0)}(10,t)$ for $n = 1, 2$ as a function of time. The solution to the GHD equation (dashed red line) is in fair agreement with the exact result (solid blue line).

there are no fermions at any of the sites $j > 0$. At a time $v_{\max} t_n^* = 10 + n$ the propagating front arrives at site $j = 10 + n$ and the expectation value $q^{(2n,0)}(10,t)$ becomes sizeable. At later times the expectation values approach the asymptotic GHD result in a power-law fashion. In particular there is no characteristic time scale for local relaxation: *the system relaxes locally in a power-law fashion*. This is expected to be a generic feature for integrable models. We observe in Fig. 5 that the exact results display high-frequency oscillatory behaviour that is not captured by GHD. This is expected to be a generic feature of GHD, which by construction is a theory designed to describe the dynamics at large scales, which correspond to small momenta and slow variations in time.

5. GHD in interacting integrable models

For interacting integrable models we proceed along the same lines as in the free fermion case:

1. Identify local conservation laws $Q^{(n)} = \sum_j Q_j^{(n)}$.
2. Construct simultaneous eigenstates of $Q^{(n)}$.
3. Construct macro-states in thermodynamic limit.
4. Work out stable excitations over macro-states and determine their group velocities.
5. Apply a "local density approximation" to the continuity equations for the densities $Q_j^{(n)}$.

We will consider the implementation of this programme for the particular example of the Lieb–Liniger model. The first four steps are discussed in Section 3. In a homogeneous macro-state with density $\rho(k)$ we have shown in section 3.2.1 that in the thermodynamic limit we have

$$\langle \boldsymbol{\rho} | Q^{(n)}(x) | \boldsymbol{\rho} \rangle = \int \frac{d\lambda}{2\pi} q^{(n)}(\lambda)\, \rho(\lambda)\,. \tag{153}$$

The expectation values of the current densities are more difficult to determine. A simple physically intuitive guess is

$$\langle \boldsymbol{\rho} | J^{(n)}(x) | \boldsymbol{\rho} \rangle = \int \frac{d\lambda}{2\pi} v_{\boldsymbol{\rho}}(\lambda)\, q^{(n)}(\lambda) \rho(\lambda)\,, \tag{154}$$

where $v_\rho(\lambda)$ is the group velocity of the stable particle excitations with momentum p over the macro-state with density $\rho(\lambda)$, *cf.* eqn (93) in Section 3.3. The main difference to the case of free fermions is that this velocity now depends on the macro-state under consideration. It turns out that the guess (154) is indeed correct [34,35]. Now we proceed as in the case of free theories and first consider the case where we prepare our system in a density matrix $\hat{\rho}(0)$ that locally corresponds to a homogeneous macro-state but varies slowly in space. Locally it will "look like" a macro-state with a position-dependent density $\rho_{x,0}(k)$, so that

$$\begin{aligned}\mathrm{Tr}\left[\hat{\rho}(0)\, Q^{(n)}(x)\right] &\approx \int \frac{d\lambda}{2\pi} q^{(n)}(\lambda)\rho_{x,0}(\lambda)\ ,\\ \mathrm{Tr}\left[\hat{\rho}(0)\, J^{(n)}(x)\right] &\approx \int \frac{d\lambda}{2\pi} v_\rho(\lambda)\, q^{(n)}(\lambda)\rho_{x,0}(\lambda)\ .\end{aligned} \tag{155}$$

Next we consider the situation at sufficiently late times after our quantum quench, where we assume that our system has relaxed locally and varies very slowly in space. Then we again can employ a description in terms of an appropriately chosen macro-state, which however now will depend on x and t

$$\begin{aligned}\mathrm{Tr}\left[\hat{\rho}(0)\, Q^{(n)}(x,t)\right] = \mathrm{Tr}\left[\hat{\rho}(t)\, Q^{(n)}(x)\right] &\approx \int \frac{d\lambda}{2\pi} q^{(n)}(\lambda)\rho_{x,t}(\lambda)\ ,\\ \mathrm{Tr}\left[\hat{\rho}(0)\, J^{(n)}(x,t)\right] = \mathrm{Tr}\left[\hat{\rho}(t)\, J^{(n)}(x)\right] &\approx \int \frac{d\lambda}{2\pi}\ q^{(n)}(\lambda) v_{\rho_{x,t}}(\lambda)\rho_{x,t}(\lambda)\ .\end{aligned} \tag{156}$$

The evolution equation for $\rho_{x,t}(k)$ then follows from combining (156) with the continuity equations (63) relating the expectation values of the charge and current densities, which gives

$$\int \frac{d\lambda}{2\pi} q^{(n)}(\lambda)\left[\partial_t \rho_{x,t}(\lambda) + \partial_x\Big(v_{\rho_{x,t}}(\lambda)\rho_{x,t}(\lambda)\Big)\right] = 0\ , \quad n = 0, 1, 2, \ldots \tag{157}$$

As the $q^{(n)}(\lambda)$ form a basis of functions on the real line these imply the following form for the GHD equations in the Lieb–Liniger model

$$\boxed{\partial_t \rho_{x,t}(\lambda) + \partial_x\Big(v_{\rho_{x,t}}(\lambda)\ \rho_{x,t}(\lambda)\Big) = 0\ .} \tag{158}$$

We stress that the group velocities $v_{\rho_{x,t}}(\lambda)$ depend on $\rho_{x,t}(\lambda)$ in a non-linear way. As we have observed before, the group velocities depend on the macro-state only through the occupation function $\vartheta_{x,t}(\lambda)$ (77). Moreover, for a given root density we can determine the corresponding occupation function from (78), and *vice versa*. Using this relation between $\rho_{x,t}$ and $\vartheta_{x,t}$ we can therefore obtain an equivalent formulation of GHD in terms of the latter

$$\boxed{\partial_t \vartheta_{x,t}(\lambda) + v_{\vartheta_{x,t}}(\lambda)\ \partial_x \vartheta_{x,t}(\lambda) = 0\ .} \tag{159}$$

5.1. Integrating the GHD equations

In order to use GHD one requires an initial value for the occupation function $\vartheta_{x,0}(\lambda)$. In practice this imposes serious restrictions on the situations that can be handled. This is easy to understand: from a physical perspective the occupation function encodes the expectation values of the densities $Q^{(n)}(x)$ of the infinite number of conserved charges. These are known in thermal equilibrium, and using a local density approximation in presence of a potential $V(x)$ can be determined in a broad class of inhomogeneous thermal equilibrium states. Certain other situations can be handled by the method introduced in Ref. [36] for homogeneous quantum quenches with lowly entangled initial states, but the problem of determining $\vartheta_{x,0}(\lambda)$ given a general initial state remains unsolved.

Given an initial value $\vartheta_{x,0}(\lambda)$ the GHD equation (158) needs to be integrated using that the group velocities are given in terms of $\vartheta_{x,t}(\lambda)$ through (93). An open source Matlab code for carrying out this programme was introduced in Ref. [37], see https://integrablefluid.github.io/iFluidDocumentation/

Once $\vartheta_{x,t}(\lambda)$ and $\rho_{x,t}(\lambda)$ have been obtained numerically they can be used to determine physical properties. The simplest ones are the expectation values of the charge and current densities (156). Extensions to certain other operators are also possible [38].

5.2. Extensions of GHD

GHD has been extended in many interesting directions, and we refer the interested reader to the reviews collected in the recent special issue Ref. [5]. One such extension that is very important for applications of GHD to cold-atom

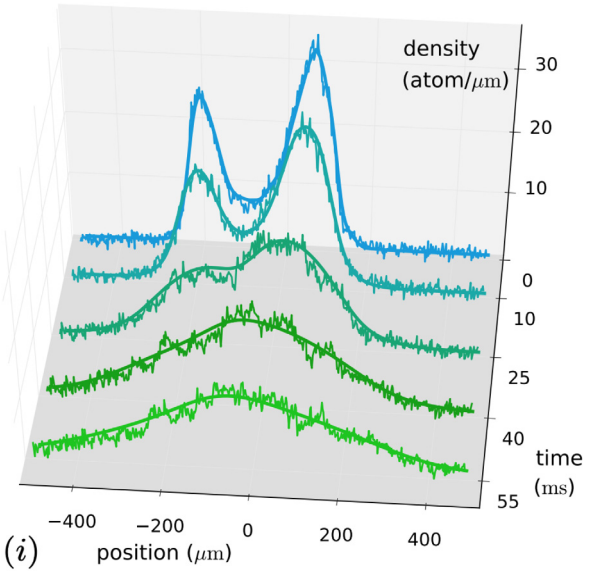

Fig. 6. Experimental results *vs* GHD prediction for atom density during expansion from an initial thermal state in a double-well potential (from Ref. [24]). The agreement is seen to be very good.

experiments is to approximately take into account the effects of an external potential $V(x)$ in the Lieb–Liniger model. This modifies the Hamiltonian to

$$H = \int dx\ \Phi^\dagger(x)\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right]\Phi(x) + c\int dx\big(\Phi^\dagger(x)\big)^2\big(\Phi(x)\big)^2\ , \tag{160}$$

and importantly breaks integrability. Hence GHD no longer provides an exact description in the Euler scaling limit. However, as long as $V(x)$ can be considered as a small perturbation, it makes sense to consider how it modifies the GHD equations. The answer was derived in Ref. [39]

$$\partial_t \rho_{x,t}(\lambda) + \partial_x\big(v_{\rho_{x,t}}(\lambda)\ \rho_{x,t}(\lambda)\big) = \big(\partial_x V(x)\big)\partial_\lambda \rho_{x,t}(\lambda). \tag{161}$$

The obvious question is on which time and length scales this evolution equation provides an accurate description of the non-integrable dynamics induced by (160). The answer is not known, but on physical grounds one would expect that if the initial state of the system is only weakly inhomogeneous and the potential $V(x)$ is weak as well, (161) may provide a good approximation at *sufficiently short times*. This is because eventually the fact that $V(x)$ breaks integrability will be felt and a description based on continuity equations for conserved quantities must fail. On the other hand, as we have seen above, in general GHD is expected to work only at *sufficiently late times* unless the initial state is judiciously chosen [40].[10] It is important to keep these considerations in mind when applying (161).

6. Applications of GHD to cold-atom experiments

GHD and its extensions has been applied to the description of cold-atom experiments that are described by the Lieb–Liniger model with confining potential (160), both in the weakly [24] and in the strongly [25] interacting regimes. A comprehensive review of these works is presented in the recent review [26]. In Fig. 6 we show some of the results obtained in Ref. [24]: a gas of very weakly interacting ^{87}Rb atoms is prepared in a thermal equilibrium state in a one-dimensional double-well potential, i.e. like in the example discussed in Section 1.1 the transverse confinement is very steep, but the potential along the y-direction in Fig. 1(a) is a double-well rather than harmonic. At time $t = 0$ the confining potential along the y-direction is changed to a shallow harmonic potential as in Fig. 1(a), and the two clouds of gases start expanding along the y-direction. The density is then measured as a function of time. The main result, and message of this section, is that the GHD prediction for this expansion is found to be in very good agreement with experiment.

[10] A similar issue is encountered after homogeneous quantum quenches in weakly perturbed integrable models, see e.g. [41,42] . Here sufficiently weak integrability breaking interactions are believed to open up an intermediate time window, in which the dynamics is characterized by the proximity to an integrable theory. At late times the system then thermalizes as it is not integrable.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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