Statistical Physics B

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1. **Exercise 1:** (1.9 from Pathria) Using II law of thermodynamics and the fact that entropy is extensive quantity derive the formula :
$$TS = U + pV - n\mu$$
Hint: *Think about how to formally write down the extensive function. Connect proper partial derivatives of entropy expressed as a function of its natural variables (U, V, n) with temperature T, pressure p and chemical potential μ*

2. **Exercise 2:** Volume of N dimensional sphere (e.g. see appendix C in Pathria).
Main steps of the derivation presented during the classes are written down below.

Our task is to calculate such an integral:

$$V_N(R) = \int \cdots \int_{\sum_{i=1}^N x_i^2 \leq R^2} \mathrm{d}x_1 \ldots \mathrm{d}x_N$$

Start from examples: N = 2.
We will introduce polar coordinate: $x = r\cos\phi$ and $y = r\sin\phi$, because then we can separate integration over this area into two integrals.

$$V_2(R) = \iint_{x^2+y^2 \leq R^2} \mathrm{d}x\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\phi \int_0^R \mathrm{d}r\ r = \pi^2$$

We will do the same for N = 3.

$$V_3(R) = \iiint_{x^2+y^2+z^2 \leq R^2} \mathrm{d}x\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^R \mathrm{d}r\ r^2 = \frac{4\pi}{3} r^3$$

We can see that for general N, we can separate the integral over angles from the integral over radius r

$$V_N(R) = \int \mathrm{d}\Omega \int_0^R \mathrm{d}r\ r^{N-1}$$

One can evaluate the integral over angles using the well known Gaussian integrals:

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_N\ e^{-x_1^2 - x_2^2 - \ldots - x_N^2} = \int \mathrm{d}\Omega \int_0^R \mathrm{d}r\ r^{N-1} e^{-r^2} = \pi^{\frac{N}{2}}$$

Finally we get that:

$$V_N(R) = \frac{R^N \, \pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$$

Question from audience: *How can we know that there exist such a system of coordinates?*

For those who are not satisfied with the explanation from classes, we will show by hand how to construct such a system of coordinates. We start form $N = 2$:

$$x_1 = r\cos\phi \qquad x_2 = r\sin\phi$$
$$x_1^2 + x_2^2 = r^2$$

We restrict r to be positive, so ϕ will belong to the interval $[0, 2\pi]$.
Now we add one more dimension. We want to find such a parametrisation that:

$$x_1^2 + x_2^2 + x_3^2 = r^2$$

Firstly, as before, let $x_1 = y_1 \cos\phi_1$ and $x_2 = y_1 \sin\phi_1$. This time y_1 will be a function of a second angle ϕ_2. To fulfill our condition:

$$\underbrace{x_1^2 + x_2^2}_{y_1^2} + x_3^2 = r^2$$

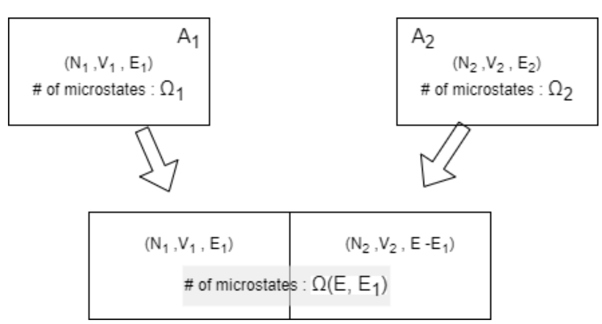
we must impose special form for $y_1(\phi_2)$: $y_1 = r\sin(\phi_2 + \phi_0)$ and $x_3 = r\cos(\phi_2 + \phi_0)$, where ϕ_2 is our new angle and ϕ_0 is constant. Now we can plug our formula for y_1 into expressions for x_1 and x_2 and we recover spherical coordinate system (we choose $\phi_0 = 0$):

$$x_1 = r\sin\phi_2\cos\phi_1$$
$$x_2 = r\sin\phi_2\sin\phi_1$$
$$x_3 = r\cos\phi_2$$

We restrict r to positive values, but x_3 can be negative. This means that $\cos\phi_2$ have to be negative for some values of ϕ_2. Then we choose that $\phi_2 \in [0, \pi]$ One can generalise this scheme to $N = 4$ and higher dimensions:

$$\underbrace{\overbrace{\underbrace{x_1^2 + x_2^2}_{y_1^2} + x_3^2}^{y_2^2} + x_4^2}_{y_3^2} + ... + x_N^2 = r^2$$

3. **Exercise 3:** (Ex 1.1 Pathria) Consider two physical systems A_1 and A_2 which are bring into a thermal contact such that they can exchange only energy.



(a) Show that the number of microstates of composed system $\Omega^{(0)}(E, E_1)$ can be expressed as a Gaussian in the variable E_1 if they are large.
Hint: *Expand the quantity* $\log \Omega^{(0)}(E, E_1)$ *around equilibrium, when* $\Omega^{(0)}(E, E_1)$ *is maximised with respect to* E_1.

(b) Determine the root-mean-square deviation of E_1 from the mean value $\overline{E}$.
Hint: *Calculate the probability for having energy* E_1. *Don't forget about the normalization*

(c) Determine the formula for a special case, when the systems are ideal gases.

4. **Exercise 4:** (Ex. 1.2 Pathria) Assuming that the entropy S and the statistical number Ω of a physical system are related through an arbitrary functional form:

$$S = f(\Omega)$$

show that the additive character of S and the multiplicative character of Ω necessarily require that the function $f(\Omega)$ be of the form $f = k \log \Omega$

5. **Exercise 5:** (Ex. 1.4 Pathria) In a classical gas of hard spheres (of diameter D), the spatial distribution of the particles is no longer uncorrelated. Roughly speaking, the presence of n particles in the system leaves only a volume $V - nv_0$ available for the $(n+1)$th particle. Assuming that $Nv_0 \ll V$, determine the dependence $\Omega(N, V, E)$ on V and show that, V gets replaced by $V - b$ in the ideal gas law. What it is the value of b