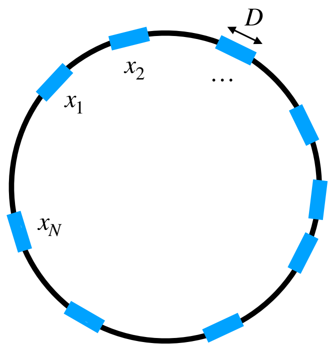
Statistical Physics B

Final exam, 01/02/2023
09:15 — 13:00

Problem 1 (20 points): A 1d hard-rod gas

Consider a gas of N hard-rods of length D on a ring of circumference L with periodic boundary conditions. The hard-rod condition can be implemented through the following pair interaction



$$u(r) = \begin{cases} \infty & \text{for } r \leqslant D, \\ 0 & \text{for } r > D. \end{cases} \tag{1}$$

- Write down the partition function $Q_N(L,T)$ and evaluate the integrals over the momenta. The remaining integrals over positions form the configurational integral $Z_N(L,T)$.
- To evaluate the configurational integral it helps to observe that in $1d$ positions of the particles can be ordered $x_1 < x_2 < \cdots < x_N$. You might want to first start with cases of small particle numbers and then generalize to arbitrary N.
- Compute the Helmholtz free energy and derive the equation of state of the gas.
- Compute the internal energy and isothermal compressibility. Discuss the resulting formulas comparing with a non-interacting case $D = 0$.

Problem 2 (20 points): BEC in a trapped gas

In this problem you will investigate the properties of the gas of bosons in an external trapping potential. This is a setup that is realized experimentally with ultra-cold atomic gases. The external potential takes a form of the harmonic trap

$$V = \frac{1}{2}m\left(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2\right). \tag{2}$$

- The energy levels of an atom in the harmonic trap are

$$\epsilon_{n_1,n_2,n_3} = \hbar\left(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3\right), \tag{3}$$

  where $n_i = 0, 1, 2, \ldots$ are the quantum numbers of the harmonic oscillator. Compute the density of states $g(\epsilon)$ which counts the number of states in the energy width $[\epsilon, \epsilon + d\epsilon]$. Assume that $\epsilon \gg \hbar\omega_i$.
- Write down an expression for the total particle number utilizing the density of states and starting from the discrete sum

$$N = \sum_{\epsilon} \frac{1}{z^{-1}e^{\beta\epsilon} - 1}. \tag{4}$$

  Remember about separating the contribution from the zero-energy state. Express your result through the Bose-Fermi function $g_\nu(z)$ (see Useful formulas below).

- The number of particles in the excited state is

$$N_{\text{exc}} = N - N_0 = \left(\frac{kT}{\hbar\omega_0}\right)^3 g_3(z). \tag{5}$$

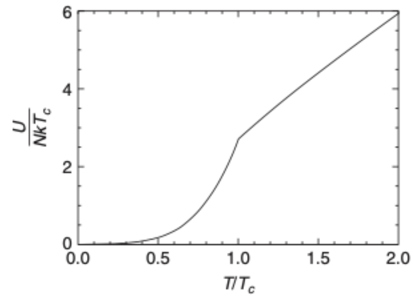
Discuss the relation between the fugacity z and temperature T for a fixed particle number N. Derive the formula for the critical temperature T_c of the Bose-Einstein condensation and confirm that the number of particles in the ground state, for $T < T_c$, follows

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^3 \tag{6}$$

- Compute the internal energy starting again from the discrete expression

$$U = \sum_{\epsilon} \frac{\epsilon}{z^{-1}e^{\beta\epsilon} - 1}. \tag{7}$$

A plot of the resulting function is shown below. Based on the plot describe qualitatively (continuous, discontinuous or diverging) the behaviour of the heat capacity in the vicinity of the critical temperature



Problem 3 (20 points): 2d ideal gases of bosons and fermions
Show that, in two dimensions, the specific heat $C_V(N,T)$ of an ideal Fermi gas is identical to the specific heat of an ideal Bose gas, for all N and T.
The equation of state for fermions is

$$N = \frac{A}{\lambda^2} f_1(z_F), \qquad U_F = \frac{AkT}{\lambda^2} f_2(z_F), \tag{8}$$

while for bosons,

$$N = \frac{A}{\lambda^2} g_1(z_B), \qquad U_B = \frac{AkT}{\lambda^2} g_2(z_B), \tag{9}$$

where A is the area to which the gas is confined and $z_{F/B}$ are fugacities for fermions and bosons respectively and λ is

- Derive the equations of state for fermions or bosons.
- Assume now that $N = N_F = N_B$ and find the relations between the fugacities z_F and z_F.
- Using the relations between the fugacities show that $U_F - U_B$ is a temperature independent constant. Is this enough to conclude that the specific heats at fixed area A are equal for the two systems?

Good luck!

Useful formulas

- Bose functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}e^x - 1}, \qquad z\partial_z g_\nu(z) = g_{\nu-1}(z).$$

- Fermi functions

$$f_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}x \frac{x^{\nu-1}}{z^{-1}e^x + 1}, \qquad z\partial_z f_\nu(z) = f_{\nu-1}(z)$$

- Special cases

$$f_1(z) = \ln(1+z), \qquad g_1(z) = -\ln(1-z).$$