Statistical Physics B

Final exam, 04/02/2022
09:15 — 13:00

Problem 1 (16 points):

Consider a classical gas of noninteracting particles. The particles do not have any internal degrees of freedom and their kinetic energy is given by $\epsilon(|\vec{p}|)$ where $\vec{p}$ is the momentum of a particle. The gas is contained in a volume V.

- (4 points) Compute the grand canonical partition function $Q(z, V, T)$ for this gas. Show that the answer can be presented in the form

$$Q(z, V, T) = \exp(zVf(T)), \tag{1}$$

  where z is the fugacity. Write an explicit formula for $f(T)$.
- (4 points) Write down the equation of state and a formula for the internal energy. Comment on which microscopic informations the two quantity depend on. (*Hint: in solving this part do not substitute for $f(T)$. Keep it general.*)
- (4 points) From now on we assume that $\epsilon(|\vec{p}|) = a|\vec{p}|^\alpha$ where $a > 0$ and $\alpha \geqslant 1$. In this case we can extract from $f(T)$ its dependence on T. Show that

$$f(T) = cT^\gamma, \tag{2}$$

  where c and γ are some constants. Express γ in terms of α.
- (4 points) Compute the internal energy of the gas and its heat capacity. Specialise to two cases: the nonrelativistic gas, and the ultra-relativistic gas for which $\epsilon(|\vec{p}|) = c|\vec{p}|$ where c is the speed of light.

Problem 2 (16 points):

Consider a quantum system of noninteracting, not distinguishable particles (fermions or bosons) at a fixed (but not given) temperature T and chemical potential μ. The single-particle states of given energies ϵ_j are labeled by the index j, and for any j the expectation value of the number of particles occupying the state is $\bar{n}_j$ (which is known).

- (4 points) Find the probability distribution that a single-particle state j_0 is occupied by n_{j_0} particles.
- (4 points) Specialise to fermions and express this probability distribution exclusively by $\bar{n}_{j_0}$, the average particle number in state j_0. Find the variance of the obtained probability distribution.
- (8 points) Repeat the computations from the previous point for bosons.

Problem 3 (16 points):

Consider an antiferromagnetic Ising model. Its Hamiltonian is

$$H = +J \sum_{<i,j>} \sigma_i \sigma_j + h \sum_i \sigma_i, \tag{3}$$

with $J > 0$, $\sigma_i = \pm 1$ and the first sum extends only over the nearest neighbours. Because of the positive sign in front of the interacting part, energetically favourable are configurations in which the neighbouring spins are anti-parallel.

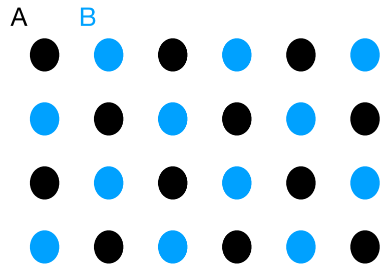

Figure 1: An example of a bi-partite lattice

- (4 points) We will assume that lattice is bi-partite. It means that it can be divided into two interpenetrating lattices such that each spin belongs to either sub-lattice A or sub-lattice B and moreover all its nearest neighbours belong to the other sub-lattice. An example of a bi-partite lattice is a square lattice. We assume that both sublattices have equal number of points $N/2$ and disregard any boundary effects. Argue that in such case

$$H = +J \sum_{\substack{<i,j> \\ i\in A, j\in B}} \sigma_i \sigma_j + h \sum_{i\in A} \sigma_i + h \sum_{i \in B} \sigma_i. \tag{4}$$

- (4 points) We utilize now the mean field approximation. Denote by $\bar{\sigma}_{A,B}$ the average value of spin in sub-lattice A and B respectively and assume that the number of nearest neighbours for each sub-lattice is q. Write down the mean-field version of the Hamiltonian.
- (4 points) Show that the partition function is

$$Q_N(h,T) = \left[Q_A(h,T)\right]^{N/2} \left[Q_B(h,T)\right]^{N/2}, \tag{5}$$

where

$$Q_A(h,T) = 2\cosh\left(-\beta(Jq\bar{\sigma}_B/2 + h)\right), \tag{6}$$
$$Q_B(h,T) = 2\cosh\left(-\beta(Jq\bar{\sigma}_A/2 + h)\right). \tag{7}$$

- (4 points) We consider now $h = 0$ and ask about possibility of a spontaneous magnetisation. On symmetry grounds we know that $\bar{\sigma}_A = -\bar{\sigma}_B$ and the total magnetisation is zero. However, this does not exclude the possibility that each of the sub-lattices is actually magnetised. Write the mean field equation for $\bar{\sigma}_A$ and determine whether there is a solution with $\bar{\sigma}_A \neq 0$. To simplify the problem work in the regime of small βJ. If you find that there is a phase transition, draw representative configurations of spins below and above the phase transition temperature (for the case of a square lattice).

Good luck!